Positron tunnelling through the Coulomb barrier of superheavy nuclei

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Abstract

We study beams of medium-energy electrons and positrons which obey the Dirac equation and scatter from nuclei with \( Z > 100 \). At small distances the potential is modelled to be that of a charged sphere. A large peak is found in the probability of positron penetration to the origin for \( Z \approx 184 \). This may be understood as an example of Klein tunnelling through the Coulomb barrier: it is the analogue of the Klein Paradox for the Coulomb potential. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Quantum tunnelling and Klein tunnelling

One of the principal characteristics of quantum mechanics as opposed to classical mechanics is that particles of energy \( E \) can pass through regions of a potential \( V(r) \) in which their kinetic energy \( T = E - V \) is negative, albeit with amplitudes which are exponentially suppressed. In two pioneering papers 70 years ago which demonstrated the importance of quantum tunnelling in the understanding of nuclear decays, the probability of an \( \alpha \)-particle to tunnel from the potential well of the nucleus through the Coulomb barrier was calculated [1]. The calculation in the quasi-classical case is straightforward [2]: the transition probability for decay \( \omega \) is given by

\[
\omega \sim \exp \left( -2 \int_0^{r_c} \sqrt{2m(V(r) - E)} \right)
\]

\[= \exp\left( -2\pi Z \alpha m/p \right) \tag{1}\]

for \( V(r) = Z \alpha/r \) where \( r_c = Z \alpha/E \) is the classical turning point, and \( m \) and \( p \) are the mass and momentum of the \( \alpha \)-particle. The integral in the exponent is over the classically forbidden region in which \( T < 0 \). Similarly a positron of energy \( E \), momentum \( p \), and mass \( m \) incident on a nucleus of charge \( Z \) can reach distances smaller than \( r_c \). If \( \rho = \left| \psi(0) \right|^2/\left| \psi(0) \right|_{\text{free}}^2 \) is the ratio of the probability of a positron penetrating a Coulomb barrier to reach the origin compared with the probability of an electron of the
same energy, then essentially the same calculation gives non-relativistically
\[ \rho = e^{-2\pi y} \]  
(2)
where \( y = Z\alpha m/\rho \) so that \( \rho \) decreases exponentially with \( Z \) for fixed \( y \). If the particles are relativistic and satisfy the Dirac equation then Eq. (2) is still obtained provided now \( y = Z\alpha E/\rho \) [4]. For relativistic problems the kinetic energy \( T = E - V - m = \sqrt{m^2 + p^2} - V - m \) so that, classically, \( T \) is still positive definite but now \( r_2 = Z\alpha/(E - m) \).

At the same time as the papers showing the effect of quantum tunnelling in \( \alpha \)-decay, Klein [5] showed that electrons in the Dirac equation could in principle tunnel through a high repulsive barrier. This is the famous Klein Paradox. Calogeracos and one of us (N.D.) have recently reassessed this phenomenon [6] and call such quantum tunnelling without the expected exponential suppression Klein tunnelling. To obtain Klein tunnelling it is essential that hole states (corresponding to negative energy states for free particles) as well as particle states are considered and allowed to propagate. For a particle of momentum \( p \) and energy \( E \) incident on a Klein step of height \( V \), a hole state of momentum \(-q\) will propagate under the barrier provided
\[ V = \sqrt{m^2 + p^2} + \sqrt{m^2 + q^2}. \]
which is possible for \( V > 2m \). In terms of the particle kinetic energy under the barrier
\[ T = E - V - m = -m - \sqrt{m^2 + q^2} \leq -2m, \]
and where \( T \leq -2m \) hole states can propagate without exponential suppression. \( T \leq -2m \) for a Coulomb potential corresponds to penetrating under the barrier to distances \( r < r_K \), where \( r_K = Z\alpha/(E + m) < r_\alpha \) is the Klein distance. While tunnelling from \( r_\alpha \) to \( r_K \) is exponentially suppressed, tunnelling from \( r_K \) to \( r = 0 \) is not; indeed the amplitude may even be enhanced. This is because the effective potential in a relativistic theory with a potential \( V \) which is the time component of a four-vector is given by
\[ 2mV_{\text{eff}}(r) = 2EV(r) - V^2(r). \]
(3)

In the region \( 0 < r < r_K \), \( V(r) > E + m > E \) and so the effective Coulomb force is attractive [3] when \( r \) is small enough [note that the force becomes attractive when \( dV_{\text{eff}}/dr \) changes sign, not when \( V_{\text{eff}}(r) \) changes sign].

When the Coulomb potential is investigated in the Dirac equation, the ground state energy \( E_1 = 0 \) at \( Z\alpha = 1 \) and becomes complex for \( Z\alpha > 1 \) [4]. This is a consequence of the \( V^2(r) \) term in Eq. (3) which leads to the “collapse” of the particle to the origin, and hence to a problem which is not well-defined [7]. So the theory breaks down at \( Z_{\text{max}} = 1/\alpha = 137 \). Since superheavy nuclei can be constructed in heavy ion collisions with values of \( Z \) larger than 137, this limitation on \( Z \) cannot be physical. A modified Coulomb potential which takes account of the finite size of the nucleus must therefore be used so that the singularity at small \( r \) is smoothed out. When this is done there seems to be no restriction on \( Z \); furthermore bound state energies become negative for sufficiently large \( Z \). These are interpreted as bound positron states [8,9]. For Klein tunnelling through the (modified) Coulomb potential it was conjectured [6] that it might occur at values of \( Z \) large enough to obtain negative energy bound states. For the simplest such potential, due to Pieper and Greiner [8], the first negative energy bound state occurs at \( Z = 147 \). Hence the prediction of Calogeracos and Dombey [6] that Klein tunnelling may occur for nuclei with \( Z = 150 \) or above. If it occurs, Klein tunnelling would lead to a breakdown of the exponential suppression given by Eq. (2).

In the Dirac equation in one-dimension, transmission resonances (with zero reflection coefficient) [10] have been demonstrated for electron scattering off square barriers [11] and off smooth potential barriers of Woods-Saxon [12] or Gaussian [13] form. These provide examples of Klein tunnelling. In each case, the transmission resonances occur when the corresponding attractive potential becomes supercritical (see next paragraph).

Since it is unlikely that superheavy nuclei with \( Z > 150 \) can be prepared and stay around for long enough for positrons to be scattered off them, we study positron scattering off nuclei with \( Z > 150 \) numerically by solving the Dirac equation for a modified Coulomb potential. We thus attempt to repeat Klein’s analysis for a modified Coulomb potential in place of a potential step. We do indeed find that positrons scattering off nuclei with \( Z > 150 \) and
especially with $Z > 170$ no longer satisfy Eq. (2). We find, in addition, an extremely sharp peak in $\rho$ near $Z = 184$. We now outline the calculation.

2. Modified Coulomb potential: bound and continuum states

Following Pieper and Greiner [8], the modified Coulomb potential experienced by an electron or positron beam is taken to be the potential arising from a homogeneously charged sphere of radius $b$. More explicitly, the time component $V(r)$ of the 4-vector potential is given by

$$V(r) = \pm \frac{Z\alpha}{b} f(r/b),$$

where the dimensionless potential shape function $f$ is defined as

$$f(r/b) = -\frac{3}{2} + \frac{1}{2} \left(\frac{r}{b}\right) r \leq b,$$

$$f(r/b) = -\frac{r}{b} r > b$$

and $b = (1.2)A^{1/3}$ fermi. We let $\psi_1$ and $\psi_2$ be the ‘large and small’ radial wave functions used to construct the Dirac spinor corresponding to a total angular momentum of $j$. We employ the variables $\tau = \pm 1$, and $k = j + \frac{1}{2}$, so that the parity $P$ of the spinor is given by $P = (-1)^{j+k} = \pm 1$. In the notation of Rose [4], with $\kappa = \tau k$, the coupled radial equations may be written

$$\psi'_1 - \frac{\tau k}{r} \psi_2 = (m + V - E) \psi_1, \quad (4)$$

$$\psi'_2 + \frac{\tau k}{r} \psi_1 = (m - V + E) \psi_2. \quad (5)$$

There are two distinct problems to be considered: (a) bound states; and (b) continuum states. For the bound states, we adopt the boundary conditions and normalization given by:

$$\psi_1(0) = \psi_2(0) = 0, \quad \int_0^\infty \left(\psi_1(r) + \psi_2(r)\right) dr = 1.$$

We now choose $j$ and $\tau$, integrate out from the origin, and search for those energies which lead to spinors that have the desired number $n_i$ of nodes in the ‘large’ radial component $\psi_1$, and vanish at large distances. If the quantity $\ell = j + \frac{1}{2} \tau$ represents the orbital angular-momentum quantum number in the first two components of the Dirac spinor, and $\nu = (n_1 + 1 + \ell)$, then the usual spectroscopic designation is written $\nu \ell_j$, where $\ell = \{0, 1, 2, \ldots\}$, $j = \frac{1}{2}, l = 0$ states, for example, correspond to $\tau = -1$. We calculate the first few energy eigenvalues as a function of $Z$ and our results agree with those given by Pieper and Greiner in their Table 3. In particular, we have found the critical values of the atomic number $Z$ that yield the energies $E = 0$ (the threshold for bound positrons) and the supercritical values of $Z$ corresponding to energy $E = -m$ (corresponding to spontaneous positron production); some results are presented in Table 1.

In the case of the continuum states, we adopt the following method. At very small distances $r \ll b$, we assume that the large and small radial functions have the asymptotic pure-power forms $Cr^\beta$ and the ratios given (after some elementary corrections) by Rose [4]. In this region there is then only one free parameter, which we take to be the amplitude $C_{1\ell}$, or, for positrons, $C_{1\ell}^\ast$, of the large component $\psi_i(r)$; the small component is smaller here by the factor $r^2$. We fix $j$, $\tau$, and the incident momentum $p$, and we integrate outwards, to a point $r_b$ well beyond the classical turning point $r_c$. At these distances the radial components have the large-$r$ Coulombic asymptotic form, which is approximately sinusoidal with $\psi_1$-amplitude, say $A_1$; in this region, the components are also (almost) exactly out of phase, so that when $\psi_1 = A_1$, we have $\psi_2 = 0$. We now search, in each case, for the value of $C_1$ so that this asymptotic amplitude has the value $A_1 = 1$. Since the numerical process is, in principle, direction invariant, this corresponds to an ‘experiment’ in which the final amplitude of an incoming beam with unit am-

Table 1

<table>
<thead>
<tr>
<th>Values of $Z$ for which $E = 0$ and $E = -m$.</th>
<th>$E = 0$</th>
<th>$E = -m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1s_{1/2}$</td>
<td>146.7</td>
<td>170.4</td>
</tr>
<tr>
<td>$2s_{1/2}$</td>
<td>195.7</td>
<td>237.0</td>
</tr>
<tr>
<td>$2p_{1/2}$</td>
<td>168.1</td>
<td>183.8</td>
</tr>
<tr>
<td>$3s_{1/2}$</td>
<td>260.1</td>
<td>316.5</td>
</tr>
</tbody>
</table>
plitude is determined near the origin. Since we are calculating wave functions at the origin, only \( l = 0 \) states contribute and thus we restrict the analysis to \( \tau = -1 \) and \( k = 1 \). For a given \( Z \) we can express the positron-electron ratio in the form \( \rho = [C_{1p}/C_{1e}]^2 \).

3. The peak

As we are not interested in the detailed behaviour of the nuclear charge distribution we require \( b \ll r_K \). We also require \( r_z \gg r_K \) so that the normal exponential suppression will be obtained for \( Z \) not too large. We ensure that this is so by choosing the momentum \( p \) in the range \( 0.1 \text{m} - 0.4 \text{m} \). For this momentum range and with \( Z < 150 \) we expect normal exponential suppression of the positrons by the modified Coulomb potential according to Eq. (2); this is demonstrated in Fig. 1 where our calculation of \( \rho \) for \( Z = 100 \) is close to \( e^{-2\pi y} \) asymptotically, showing that our method of calculation is satisfactory.

We now look at \( C_{1p}/C_{1e} \) for values of \( Z > 140 \). The principal discovery we report here is the existence of a pronounced peak in \( \rho = [C_{1p}/C_{1e}]^2 \) very near \( Z = 184 \). This peak is exhibited in Fig. 2 for the case \( p = 0.4 \text{m} \). At the peak maximum \( \rho \approx 2 \). The peak is so sharp that it seems to correspond to a singularity of some sort. It is convenient to look at the deviation from exponential suppression by introducing a Klein "logarithmic form factor" \( R \) so that

\[
\left| C_{1p}/C_{1e} \right| = e^{R} e^{-\pi y},
\]

where \( R \) is given explicitly by

\[
R(Z, p) = \ln \left| C_{1p}/C_{1e} \right| + \pi y = \frac{1}{2} \ln(\rho) + \pi y.
\]

Thus \( R \) constant corresponds to normal exponential suppression. In Fig. 3 we exhibit more detail near the peak by plotting \( R \) in terms of \( Z \) for the two cases

![Fig. 1. A comparison of \( \rho \) (solid) with \( e^{-2\pi y} \) (dashed) for \( Z = 100 \).](image1)

![Fig. 2. ln(\( \rho \)) as a function of \( Z \).](image2)

![Fig. 3. \( R \) as a function of \( Z \) for \( p = 0.4 \text{m} \) (dashed) and \( p = 0.1 \text{m} \) (solid).](image3)
The enormous peak near $Z = 184$ in the scattering of positrons by superheavy nuclei was unexpected: it involves an enhancement in $p$ by a factor $10^8$ between $Z = 179$ and $Z = 184$. While a demonstration of a deviation from the pure exponential suppression of Eq. (2) resulting from Klein tunnelling was the purpose of this calculation, we initially expected it to be a simple maximum in $p$ at the first supercritical value of $Z = 170.4$ as is the case with the one-dimensional examples referred to above, where supercriticality and the maximum of the transmission coefficient coincide. While the peak that we have found does not correspond to the first $1s_{1/2}$ supercritical state, it does correspond to the second $2p_{1/2}$ supercritical state (see Table 1). We show in a forthcoming paper [14] that this correspondence exists independently of the exact form of the potential. This is also in agreement with the result of Müller, Rafelski and Greiner [15] who predicted a resonance in the negative energy continuum in the $s_{1/2}$ state at $Z = 184$.

To explain why the peak occurs in the state which corresponds to the $2p_{1/2}$ bound state of the electron is an exercise in Dirac’s hole theory. The vacuum state for our modified Coulomb potential at supercriticality contains a vacant $2p_{1/2}$ bound state of an electron of energy $E = -m$; this in hole theory represents a positron state of zero kinetic energy. But to compare the original electron state with a positron state requires charge conjugation: Eqs. (4) and (5) are invariant under the combined operation $E ightarrow -E; V ightarrow -V; \tau ightarrow -\tau; \psi_1 \rightarrow \bar{\psi}_2$. So the vacant electron $2p_{1/2}$ bound state corresponds to a resonant $s_{1/2}$ positron scattering state since $j = \frac{1}{2}$ for both states but $\tau$ changes sign.  

The (vacant) electron $2p_{1/2}$ bound state of zero kinetic energy at supercriticality can thus be considered as a positron resonance of zero kinetic energy. In one dimension non-relativistically a zero energy resonance is a transmission resonance where the transmission coefficient is unity and the reflection coefficient is zero [16]. It seems likely that we have found a similar effect here in three dimensions since our peak occurs as close to zero kinetic energy (corresponding to the supercritical energy) as we can calculate. We should emphasise that we calculate the quantity $p$ which is not a scattering cross section. Neither is it strictly a transmission coefficient. Nevertheless our peak is more like a transmission resonance than a resonant cross section. We will examine this point in more detail in a forthcoming paper.

We note finally that our example here of positron scattering by a modified Coulomb potential shows that the phenomenon discovered by Klein of Dirac particles tunnelling through repulsive potentials is a general characteristic of the Dirac equation in the presence of strong fields.

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References


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Saturation from nuclear pion dynamics

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Abstract

We construct an equation-of-state for nuclear matter based on the chiral Lagrangian. The relevant scales are discussed and an effective chiral power expansion scheme, which is constructed to work around the nuclear saturation density, is presented. A realistic equation-of-state is obtained by adjusting one free parameter, when the leading and subleading terms in the expansion are included. The saturation mechanism is due to correlations induced by the one-pion-exchange interaction. Furthermore, we find a substantial deviation from the Fermi-gas estimate of the quark condensate in nuclear matter already at the saturation density.

For more than four decades, one of the basic problems of nuclear physics has been to find a microscopic understanding of the nuclear equation-of-state in terms of the free nucleon-nucleon interaction [1–3]. The present status is that a quantitative description of nuclear matter can be achieved in a non-relativistic approach only by invoking a three-body force [4], or alternatively with the lowest-order term in the relativistic Brueckner approach [5]. A drawback of the latter, is the lack of a systematic expansion scheme. Thus, it has so far not been possible to obtain a reliable estimate for the corrections to the leading term. Such calculations are based on the traditional semi-phenomenological approach to the nucleon-nucleon interaction, represented by e.g. the Urbana-Argonne [6] and Bonn [7] potentials.

During the last few years, a novel approach to the nucleon-nucleon interaction, based on chiral perturbation theory (χPT), has started to emerge. A key element in χPT is the power counting, which relies on a separation of scales. This allows one to organize a systematic approximation scheme. In the nuclear many-body problem, new scales enter, which requires a modification of the expansion scheme developed for the nucleon-nucleon interaction in free space. In this letter we identify the new scales and present an effective chiral perturbation theory, appropriate for the nuclear many-body problem. We employ this scheme to construct a nuclear equation-of-state and compute the quark condensate in nuclear matter.

The key element of any microscopic theory for the nuclear equation-of-state is the elementary nucleon-nucleon interaction. In the context of chiral perturbation theory this was first addressed by Weinberg who proposed to derive a chiral nucleon-nucleon potential in time ordered perturbation theory [8,9]. An alternative scheme, with chiral power counting rules applied directly to the nucleon-nucleon scattering amplitude, was subsequently proposed [10–12].
This approach relies on the crucial observation that the chiral power counting rules can be generalized for 2-nucleon reducible diagrams. Non-perturbative effects like the pseudo bound state pole in the $S_0$ channel are generated by properly renormalized local two-nucleon vertices, that carry an anomalous chiral power $Q^{-1}$.

The counting rules for the vacuum nucleon-nucleon scattering amplitude provide a suitable starting point for the construction of chiral power counting rules for the nuclear matter problem [13]. The density expansion, or equivalently the multiple scattering expansion, is readily combined with the chiral density expansion, or equivalently the multiple scattering rules for the nuclear matter problem [13]. The counting rules for the vacuum nucleon-nucleon scattering amplitude provide a suitable starting point for the construction of chiral power counting rules for the nuclear matter problem [13]. The density expansion, or equivalently the multiple scattering expansion, is readily combined with the chiral density expansion, or equivalently the multiple scattering rules for the nuclear matter problem [13].

In the generalized chiral power expansion scheme [10–12], the nucleon-nucleon scattering amplitude is expanded in the small ratio $Q/A_F$, but not in the large one $Q/A_L$, where $A_L = m_N$ and $A_S$ is a small scale, like $4 \pi f_\pi = 1$ GeV.

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The expansion coefficients $E_n$ are complicated, presently unknown, functions of the Fermi momentum $k_F$. However, by employing appropriate resum- mations, these functions can be computed from the free-space chiral Lagrangian [14].

Since the typical small scale $A_S \approx 50$ MeV is much smaller than the Fermi momentum $k_F$, one may expand the coefficients $E_n$ around $k_F^{(0)}$ in the following manner

$$E_n \left( \frac{k_F}{m_n} \right) \approx E_n \left( \frac{k_F^{(0)}}{m_n} \right) + \sum_{k=1}^{\infty} F_n^{(k)} \left( \frac{k_F}{m_n} \right) \left( \frac{k_F^{(0)}}{k_F} \right)^k$$

Note that we do not expand in the ratio $m_n/k_F$.

Clearly this scheme is constructed to work around nuclear saturation density but will fail at very small densities. However, also other approaches, like the Walecka model [17] and the Brueckner scheme, are not reliable at very low densities.

Physically the expansion (3) may be interpreted in the following way. The parquet resummation of the diagrams of the local two-nucleon interaction, as is done in lowest-order Brueckner calculations. We find that one must sum also the particle-hole and hole-hole ladder diagrams including all interference terms with self-consistently dressed nucleon propagators. According to the generalized counting rules the pions can then be evaluated perturbatively on top of the parquet resummation for the local two-nucleon interaction, described above [1]. In terms of vacuum scales this leads to an expansion of the energy per particle, $E(k_F)$, of the form

$$E(k_F) = \sum_n E_n \left( \frac{k_F}{m_n} \right) \left( \frac{k_F^{(0)}}{A_L} \right)^n$$

The expectation that the nuclear equation-of-state can be computed microscopically by summing the parquet diagrams for the local interaction and then including pions perturbatively may be too optimistic. A refined formulation, which incorporates the relevant subthreshold singularities of the vacuum nucleon-nucleon scattering amplitude (see [11]), may require a more involved treatment of pionic effects.
local nucleon-nucleon interaction gives rise to a density dependence governed by the scale \( \Lambda_s \), while the relevant scale for the pion dynamics is \( m_\pi \). If \( \Lambda_s/k_F^{(0)} \) is so small, that the parquet resummation is close to its high-density limit at normal nuclear matter density, the expansion (3) converges rapidly, and we can, as a first approximation, retain only the leading term in this expansion. In this case the density dependence of the coefficients of the expansion (2) is, to a very good approximation, given by the pion dynamics.

A systematic derivation of the expansion (2) and (3), applying suitable resummation techniques, will be presented elsewhere [14]. In this work we pursue a less microscopic approach. We assume that the coefficients \( E_a(k_F/m_\pi,k_F^{(0)}/\Lambda_s) \) can be computed in an effective theory whose unknown parameters are adjusted to the saturation properties of nuclear matter. We employ an effective Lagrangian density, where the nucleons interact through s-wave contact interactions and through the exchange of pions. Here we do not include local p-wave nor three-body interaction terms since they do not contribute to leading and subleading order in the chiral expansion. The interaction part of the Lagrangian is given by

\[
\mathcal{L}_{\text{int}}(k_F) = \frac{g_A}{2 f_{\pi}} N^\dagger (\sigma \cdot \nabla) (\pi \cdot \tau) N \quad + \frac{1}{8 f_{\pi}^2} \left( g_0(k_F) + \frac{i}{2} g_A^2 \right) \times (N^\dagger N) P_{12}^{S=1,T=0} (N^\dagger N)_2 \quad + \frac{1}{8 f_{\pi}^2} \left( g_1(k_F) + \frac{i}{2} g_A^2 \right) \times (N^\dagger N) P_{12}^{S=0,T=1} (N^\dagger N)_2 ,
\]

where \( N \) is the two component nucleon spinor field and \( P_{12}^{S,T} \) is the projection operator for a two-nucleon state with spin \( S \) and isospin \( T \). Note that in the contact interaction the spatial coordinates of the two particles are identical. The terms proportional to \( g_A^2/4 \) are counter terms, which cancel the local, high momentum, piece of the one-pion-exchange interaction. Below we discuss the role of the counter terms in more detail.

Rather than evaluating the coefficients of the chiral density expansion (2), (3), we compute the energy directly. The coefficients \( E_n \) can be extracted from the resulting expressions. In Fig. 1 we show all diagrams of chiral order \( Q^3 \) and \( Q^4 \). They correspond to the leading and subleading interaction contributions to (2). The dashed line is the pion propagator. We split the non-interacting nucleon propagator into a free-space part

\[
S_N(p_0,p) = \frac{1}{p_0 - p^2/(2m_\pi) + i\varepsilon},
\]

denoted by a directed solid line, and a density-dependent part

\[
\Delta S_N(p_0,p) = 2\pi i \delta(p_0 - p^2/(2m_\pi)) \Theta(k_F^2 - p^2)
\]

represented by a line with a ‘cross’. Thus, depending on the diagram, such a line corresponds to a hole line or the Pauli blocking of a particle line. This separation is useful for the expansion scheme we adopt.

The pion dynamics is in our scheme computed explicitly. If properly renormalized, it is perturbative like in the vacuum case. The non-perturbative short-range physics is subsumed in the local interaction. In our scheme, the power counting is much simpler than in a fully microscopic one, where one would have to sum diagrams involving the small scales \( \Lambda_s \) and \( k_F \) to all orders. This infinite set of diagrams is already included in our interaction constants \( g_a(k_F) \).

\[ \text{Fig. 1. Leading and subleading contributions to the energy of nuclear matter. The Feynman rules are explained in the text.} \]
and $g_i(k_F)$. Consequently, in order to avoid double
counting, diagrams with two or more adjacent in-
teraction vertices $g_0$ or $g_1$ are forbidden. Therefore
we introduce two types of local 2-nucleon vertices in
Fig. 1. The filled circle represents the full zero-range
vertex of (4) proportional to $g_{0,1} + g_{2}^2/4$ while
the open circle corresponds to only the counter term
proportional to $g_{2}^2/4$. Clearly there are no diagrams
of the type e-f in Fig. 1 with the full contact inter-
action (filled circles) at both vertices.

Note, that in our counting $g_0$ and $g_1$ are of order
$Q^0$ since the non-perturbative structures like the
deuteron, which give rise to the anomalously large
amplitude $\sim Q^{-1}$ in vacuum, are dissolved at densi-
ties far below the saturation density. We neglect the
density dependence of the effective coupling con-
stants $g_0$ and $g_1$. This corresponds to retaining only
the leading term in (3). We can test whether sublead-
ting terms are required, by allowing the coupling
constants to be density dependent.

The contribution of the first diagram in Fig. 1 to
the ground state energy density $\epsilon$ is proportional to
$(g_0(k_F) + g_1(k_F))k_F^3$. Since the effective vertices
$g_0(k_F), g_1(k_F) \sim Q^0$ carry chiral power zero, this
diagram is of chiral order $Q^0$. The contribution to the
energy per particle $E(k_F) = \epsilon(k_F)/\rho$ is of order $Q^3$,
since $\rho = 2k_F^3/(3\pi^2)$. In this paper, the chiral power
of a diagram is defined by its contribution to the energy
per particle. The one pion exchange contribution
(Fig. 1 b), is also of chiral order $Q^3$ since it is propor-
tional to $k_F^3$ times a dimensionless function of
$k_F/m_\pi$.

The leading interaction contributions to the en-
ergy per particle (Fig. 1 a, b) can be expressed in
terms of a momentum dependent effective scattering
amplitude $T_{\epsilon\ell}(p)$:

$$E_{\ell}(k_F) = \frac{k_F^3}{2\pi^2} \times \int_0^1 x^2 \, dx \,(1 - x)^2(2 + x) \, T_{\epsilon\ell}(k_F \, x) ,$$

(7)

where

$$T_{\epsilon\ell}(p) = \frac{3}{f^2_m} \left( g_0 + g_1 + \frac{g_2^2}{2} \right) - \frac{3}{2} \frac{g_2^2}{f^2_m} \frac{p^2}{m_\pi^2} + p^2 .$$

(8)

In (8) the cancellation between the counter terms in
the local effective interaction (4) and the high-
momentum part from the one-pion-exchange inter-
action is evident. As an example for the link between
the Lagrangian (4) and the chiral density expansion,
we consider the coefficient $E_\ell(k_F/m_\pi, k_F^0)/\Lambda_3$ in
(3). The contribution of the contact interaction is
given by $E_\ell/\Lambda_3 = -2(g_0 + g_1)/(4\pi f^2_m)^2$. We note
that the small scale $\Lambda_3$ is hidden in the coupling
constants $g_0$ and $g_1$.

Next we consider the diagrams of subleading
order, shown in Fig. 1. These diagrams fall into two
distinct classes: those with two and three crosses. We
do not show the diagrams with four crosses since
their real parts vanish and their imaginary parts
cancel that of the other diagrams so that, to a given
order, the resulting energy is real. Furthermore, we
do not include vacuum-polarization diagrams and
vacuum vertex renormalization diagrams. The former
vanish in a non-relativistic scheme $^2$, while the latter
are suppressed by two chiral powers $Q^2$.

We note that after a proper renormalization, dia-
grams with only one cross are implicitly included in
the kinetic energy term

$$E_{\text{kin}}(k_F) = \frac{3}{10} \frac{k_F^2}{m_\pi} .$$

(9)

The contribution of diagrams d, f, h, j and l
with two crosses can again be expressed in terms of
an effective scattering amplitude $T_{\ell\ell}(p)$ with

$$T_{\ell\ell}^{(d\ell)}(p) = 6 \frac{g_2^2}{f^2_m} \left( g_0 + g_1 + \frac{g_2^2}{4} \right) \frac{m_\pi^2}{8\pi p} I(p) ,$$

$$T_{\ell\ell}^{(f\ell)}(p) = 3 \left( g_0 + g_1 + \frac{g_2^2}{2} \right) \frac{g_2^2}{f^2_m} \frac{m_\pi^2}{8\pi p} \times i \log \left( 1 - 2i \frac{p}{m_\pi} \right) - 6 \left( g_0 + g_1 + \frac{g_2^2}{2} \right) \times \frac{g_2^2}{f^2_m} \frac{m_\pi^2}{8\pi} I(p) ,$$

(10)

$^2$The non-relativistic scheme presented here is identical with
the non-relativistic reduction of a relativistic scheme order by
order in the chiral expansion [11].
The loop integral $I(p)$ is divergent and must be regularized:

$$I(p) = \frac{1}{8\pi^2 m_N} \int_0^\Lambda \frac{l^2 dl}{l^2 - p^2 - i\epsilon} = \frac{1}{16\pi m_N} \left( \frac{2}{\pi} \Lambda + ip + o\left(\frac{p^2}{\Lambda^2}\right) \right).$$  \hspace{1cm} (11)

Here $\Lambda$ is a cutoff parameter.

In (10) the appropriate symmetry factors are included. We note that the diagrams d), f), h), j) and l) in Fig. 1 are all divergent. However, the sum of all diagrams is finite once we include a further counter term in the spin triplet channel:

$$g_0 \rightarrow g_0 - \frac{m_N}{4\pi^2} \frac{g_A^4}{f_a^2}.$$ \hspace{1cm} (12)

This counter term, which is formally of order $Q$, cancels the divergence due to the second-order tensor interaction. Collecting all real terms from (8) and (10) we find

$$T_{el}(p) = \frac{3}{f_a^2} (g_0 + g_1) + \frac{3}{2} \frac{g_A^2}{f_a^2} \frac{m_N^2}{m^2 + 4p^2} + \frac{3}{2} \frac{g_A^2}{f_a^2} \frac{m_N^2}{m^2 + 4p^2} \frac{m_N}{p} \arctan\left( \frac{2p}{m_N} \right) \frac{1}{16\pi f_a^2}

\times \left( \frac{m_N}{m^2 + 4p^2} - \frac{3}{2} \frac{m_N}{p} \arctan\left( \frac{2p}{m_N} \right) - 1 \right)

\times \left( 3 + 4 \frac{p^2}{m_N^2} \right) - \arctan\left( \frac{p}{m_N} \right) \right).$$ \hspace{1cm} (13)

The corresponding contribution to the energy per particle is again given by Eq. (7).

We now turn to the remaining diagrams c), e), g), i) and k). Their contribution to the energy per particle reads

$$E_p(k_F) = \frac{m_N}{4\pi f_a^2} \left( \frac{3}{2} g_A^2 \left( g_0 + g_1 + \frac{g_A^2}{4} \right) J_0 \right.

- \frac{1}{2} g_A^2 \left( g_0 + g_1 + \frac{g_A^2}{2} \right) J_2(k_F)

+ \frac{1}{2} g_A^2 J_2(k_F) - \frac{1}{2} g_A^2 J_2(k_F) \right) \hspace{1cm} (14)
where

\[ J_{\omega<3}(k_F) = \frac{24}{k_F^2} \int_{0}^{k_F} q^2 \, dq \int_{-1}^{1} x_q \, dx_q \left( k_F | q| x_q \right) \]

\[ + \frac{k_F^2 - q^2 x_q^2}{2} \delta \left( k_F - |q| x_q \right) \left( F_\omega(0) \right) \]

\[ - F_\omega(\lambda(q, x_q)) ) , \]

\[ J_3(k_F) = \frac{24}{k_F^2} \int_{0}^{k_F} p^2 \, dp \int_{-1}^{1} x_p \, dx_p \int_{-1}^{1} x_q \, dx_q \]

\[ \times \left( \Theta \left( x_q^2 + x_p^2 - 1 \right) \right) \]

\[ \times \left( F_1(\lambda(p, x_q)) - F_1(0) \right) \]

\[ \times \left( F_1(\lambda(p, x_p)) - F_1(0) \right) \]

(15)

and

\[ \lambda(q, x_q) = \sqrt{k_F^2 - q^2 (1 - x_q^2)} - |q| x_q \]

\[ F_0(l) = \frac{t^2}{2} , \quad F_1(l) = \frac{t^2}{2} - \frac{m^2}{2} \log(t^2 + m^2) , \]

\[ F_2(l) = \frac{t^2}{2} - \frac{m^3}{2} \left( t^2 + m^2 \right) - m^3 \log(t^2 + m^2) . \]

(16)

The integral \( J_0 \) can be performed analytically with the result \( J_0 = 16(11 - \ln4)/35 \).

The resulting equation-of-state is given by \( E(k_F) = E_{\text{kin}} + E_r + E_p \). We note that in symmetric nuclear matter there is effectively only one free parameter, namely the effective coupling constant \( g = g_0 + g_1 \), since we neglect the density dependence of \( g_0 \) and \( g_1 \), as discussed above. In Fig. 2 we show the energy per nucleon for \( g = 2.8, 3.0, 3.2, 3.4 \) and 3.6. We emphasize that the coupling functions \( g_0(k_F), g_1(k_F) \) are to be determined from the nuclear equation-of-state. Thus, by allowing \( g_0(k_F) \) and \( g_1(k_F) \) to be arbitrary functions of \( k_F \) we could have trivially obtained a realistic equation-of-state. However there is a strong consistency constraint: according to our scale argument discussed above, the density dependence of the coupling functions must be weak for \( k_F \) larger than the small scales, which formally have been integrated out. Thus, our scheme would have to be rejected, had we found a strong density dependence of \( g_0 \) and \( g_1 \). In fact, the density independent set of parameters \( g = 3.23 \), \( g_0 = 1.26 \), \( m_\pi = 140 \text{ MeV} \) and \( f_p = 93 \text{ MeV} \) yields an excellent equation-of-state. We obtain saturation at the Fermi momentum \( k_F^{(0)} = 265 \text{ MeV} \), which corresponds to a density \( \rho_0 = 0.16 \text{ fm}^{-3} \), in agreement with the empirical value. Furthermore, the empirical binding energy of 16 MeV is reproduced and we obtain an incompressibility of \( \kappa = 218 \text{ MeV} \), compatible with the empirical value (210 ± 30 MeV) of Ref. [15].

It is interesting to explore the convergence properties of the expansion (2). At the saturation density, the leading term of chiral order \( Q^1 \) (the kinetic energy) contributes 22.5 MeV to the energy per particle \( E(k_F) \), the terms of order \( Q^2 \) (Fig. 1 a, b) contribute −93.5 MeV and the terms of order \( Q^4 \) (Fig. 1 c−l) +55.0 MeV. We observe a partial cancellation between the \( Q^3 \) and \( Q^4 \) terms. However, based on these terms alone, one can draw only qualitative conclusions on the convergence properties of the expansion. Quantitative results have to await the evaluation of terms of higher order in \( Q \).

As one finds e.g. for the hole-line expansion in Brueckner theory [2], it may well be that the convergence is in fact much better than suggested by the first few (fairly large) terms.

The saturation mechanism of our model is quite different from that of popular effective models, like the Skyrme and Walecka models. In the Skyrme model the saturation is due to the density and velo-
ity dependence of the zero-range effective interaction [15,16], while in the Walecka model it is a relativistic effect [17]. In neither of these models are pionic degrees of freedom explicitly included. In our model, on the other hand, saturation is obviously due to pion dynamics, since for \( g_s = 0 \) only the diagram in Fig. 1a survives and \( g_0 \) and \( g_1 \) are independent of density. This is reminiscent of the saturation mechanism in the Brueckner approach, which, to a large extent, is due to the second-order tensor contribution [1]. We note however that there is no one-to-one correspondence between the pion-exchange diagrams in the two approaches. For instance, the short-range part of the iterated one-pion-exchange is in our approach, through the renormalization procedure, subsumed in the zero-range interaction. Since the coupling constants \( g_0 \) and \( g_1 \) are adjusted to reproduce the properties of nuclear matter, this part of the one-pion-exchange interaction is effectively treated non-perturbatively. In the Brueckner approach, on the other hand, the divergencies are regularized by form factors. Consequently, there the corresponding finite contributions to the energy are associated with the original pion-exchange diagrams. Since the one-pion-exchange is perturbative only once the short-range part has been removed, it must be iterated to all orders in the Brueckner approach.

Our equation-of-state is useful for applications, where the pion dynamics plays an important role. As an example, we consider the quark condensate in nuclear matter. The quark condensate, \( \langle \bar{q}q \rangle \), is an order parameter for the spontaneously broken chiral symmetry. Thus, its dependence on baryon density, indicates to what extent chiral symmetry is restored in nuclear matter. Furthermore, it is an important ingredient in the QCD sum rules [18] and in the Brown-Rho scaling approach [19]. According to the Feynman-Hellman theorem the quark condensate can be extracted unambiguously from the total energy per particle, \( m_N + E(\rho) \), of nuclear matter, once the dependence on the current quark mass is known:

\[
\langle \bar{q}q \rangle(\rho) - \langle \bar{q}q \rangle(0) = \rho \frac{\partial}{\partial m_Q} \left( m_N + E(\rho;m_Q) \right).
\]

(17)

Here \( m_Q = (m_u + m_d)/2 \) is the average of the \( u \) and \( d \) quark masses. In our approach the contribution of the pion degrees of freedom to the energy is included in a chirally consistent manner to order \( Q^4 \). This allows us to estimate the effect of correlations on the quark condensate in nuclear matter to leading and subleading order.

It is convenient to consider the relative change of the quark condensate

\[
\frac{\langle \bar{q}q \rangle(\rho)}{\langle \bar{q}q \rangle(0)} = 1 - \frac{\Sigma_{\sigma N}}{m_s f_s^2} - \frac{\alpha_\sigma(\rho) \rho}{2 m_s f_s^2}.
\]

(18)

The penultimate term in (18) is the modification of the quark condensate in a free Fermi gas of nucleons, where

\[
\Sigma_{\sigma N} = m_Q \langle N|\bar{q}q|N \rangle = m_Q \frac{d m_N}{d m_Q}
\]

(19)

is the pion-nucleon sigma term. Finally, the last term in (18) is due to the dependence of the interaction energy on the pion mass

\[
\alpha_\sigma(\rho) = -\frac{2 m_s f_s^2}{\langle \bar{q}q \rangle(0)} \frac{\partial}{\partial m_Q} E(\rho;m_Q)
\]

\[
= \left(1 + \rho \left( m_s^2 \right) \right) \frac{\partial}{\partial m_s} E(\rho;m_s),
\]

(20)

where we use the Gell-Mann–Oakes–Renner relation to convert the dependence on the current quark mass \( m_Q \) to a dependence on the pion mass. We emphasize that the second term on the right-hand side of (18), which was first written down in Refs. [20–22], does not probe the interactions in nuclear matter. Therefore the linear density dependence of

![Fig. 3. The quark condensate in isospin symmetric nuclear matter.](image-url)
the quark condensate, which results when only this term is retained, should be considered with caution. It is a priori not clear that this term is the most important one at the saturation density and beyond.

Note that here we neglect any implicit $m_q$ dependence of the effective couplings $g_0$ and $g_1$, since naive counting arguments suggest that the corresponding contribution to the energy per particle would be of order $Q^5$. In Fig. 3 we show the resulting quark condensate in nuclear matter. We confront the ‘leading’ term, which is proportional to the sigma term $\Sigma_{\pi N} = 45 \text{ MeV}$, with the full result given by (18). The inclusion of pionic interaction effects counteracts the reduction of the condensate due to the leading term. Our results confirm calculations performed within the Brueckner [23] and Dirac-Brueckner [24] approach qualitatively insofar that the nuclear many-body system reacts against chiral symmetry restoration. This result has important implications for the restoration of chiral symmetry in matter at finite baryon density.

References

Angular momentum projected analysis of quadrupole collectivity in $^{30,32,34}$Mg and $^{32,34,36,38}$Si with the Gogny interaction

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Abstract

A microscopic angular momentum projection after variation is used to describe quadrupole collectivity in $^{30,32,34}$Mg and $^{32,34,36,38}$Si. The Hartree–Fock–Bogoliubov states obtained in the quadrupole constrained mean field approach are taken as intrinsic states for the projection. Excitation energies of the first $2^+$ states and the $B(E2;0^+\rightarrow 2^+)$ transition probabilities are given. A reasonable agreement with available experimental data is obtained. It is also shown that the mean field picture of those nuclei is strongly modified by the projection.

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Neutron-rich nuclei with $N \approx 20$ are spectacular examples of shape coexistence between spherical and deformed states. Experimental evidence for an island of deformed nuclei near $N = 20$ has been found in the fact that $^{31}$Na and $^{32}$Na are more tightly bound than could be explained with spherical shapes [1]. Additional support comes from the unusually low excitation energy of the $2^+$ state in $^{32}$Mg [2]. A large ground state deformation has also been inferred from intermediate energy Coulomb excitation studies [3] in $^{32}$Mg. Quadrupole collectivity of $^{32–38}$Si has also been studied in [4]. Very recently this region has been the subject of detailed experimental spectroscopic studies at ISOLDE [5]. From a theoretical point of view, deformed ground states have been predicted for nuclei with $N \approx 20$ [6–8]. In those calculations the rotational energy correction is the essential ingredient for the stabilization of the deformed configuration. On the other hand, some calculations have predicted [9–12] a spherical ground state for $^{32}$Mg but it has also been found [12] that deformation effects may appear as a result of dynamical correlations. Some shell model calculations, even with restricted configuration spaces, have been able to explain the increased quadrupole collectivity at $N = 20$ as a result of neutron $2p–2h$ excitations into the $fp$ shell, see for example [13,14]. Recently, a mean field study has explored the suitability of several Skyrme parameterizations [15] in the description of this and other regions of shape coexistence.

The mean field description of nuclei is usually a good starting point as it provides a qualitative, and in many cases quantitative, understanding of the nuclear properties. This is the case when the mean field solution corresponds to a well defined minimum. However, in regions of shape coexistence where two minima are found at a comparable energy, the correlation effects stemming from the restoration of bro-
ken symmetries and/or collective motion can dramatically alter the energy landscape thus changing the mean field picture. For this reason, we have included in our mean field calculations the effects related to the restoration of the broken rotational symmetry by performing, for the first time with the Gogny force, angular momentum projected calculations of the energies and other relevant quantities. The reason for choosing to restore rotational symmetry is that the zero point energy associated with this restoration is somehow proportional to deformation and ranges, in this region, from a few keV for nearly spherical configurations to several MeV for well deformed ones. This energy range is comparable to the energy differences found between different shapes in nuclei of this region. Therefore, in addition to the mean field results, both angular momentum projected \( I = 0 \) and \( I = 2 \) surfaces were computed for the nuclei \(^{30,32,34}\text{Mg}\) and \(^{32,34,36,38}\text{Si}\) and angular momentum projected transition probabilities \( B(E2,0^+ \rightarrow 2^+) \) among different configurations.

The calculation proceeds in two steps: in the first one we perform a set of constrained Hartree–Fock–Bogoliubov (HFB) calculations using the D1S parameterization [16] of the Gogny force [17] and the mass quadrupole operator \( Q_{20} = z^2 - \frac{1}{2}(x^2 + y^2) \) as the constraining operator in order to obtain a set of “intrinsic” wave functions \( |\phi(q_{20})\rangle \). The self-consistent symmetries imposed in the calculation were axial symmetry, parity and time reversal. The two body kinetic energy correction was fully taken into account in the minimization process. On the other hand, the Coulomb exchange term was replaced by the local Slater approximation and neglected in the variational process. The Coulomb pairing term as well as the contribution to the pairing field from the spin-orbit interaction were neglected. A harmonic oscillator (HO) basis of 10 major shells was used to expand the quasi-particle operators and the two oscillator lengths defining the axially symmetric HO basis were kept equal for all the values of the quadrupole moment. The reason for choosing the basis this way was that we wanted a basis closed under rotations (i.e. an arbitrary rotation of the basis elements always yields wave functions that can be solely expressed as linear combinations of the elements of the basis) in order to avoid the technical difficulties discussed in [18] when a non-closed basis is used. In the second step we compute the angular momentum projected energy for each intrinsic wave function \( |\phi(q_{20})\rangle \) obtaining in this way a set of energy curves \( E_I(q_{20}) \) for each value of \( I = 0,2,... \). The minima of each curve provide us with the energies and wave functions of the \( I = 0^+,2^+,... \) yrast and isomeric states.

The theoretical background for angular momentum projection is very well described in [19,20] and therefore we will not dwell on the details here. However, a few remarks concerning the peculiarities of our calculation are in order: first, and due to the axial symmetry imposed in the HFB wave functions, the angular momentum projected energy is given by

\[
E_I(q_{20}) = \int_0^\pi d\beta s e^{\beta e^{\beta e^\beta}} (\langle \phi(q_{20}) | \hat{H} | \phi(q_{20}) \rangle) \int_0^\pi d\beta s e^{\beta e^{\beta e^\beta}} (\langle \phi(q_{20}) | e^{-i \beta I} | \phi(q_{20}) \rangle)
\]

(1)

with \( \hat{H}[\rho_0(r)] = \hat{H}[\rho_0(r)] - \lambda_x(N_e - Z) - \lambda_y(N_e - N) \). The term \( -\lambda_x(N_e - Z) - \lambda_y(N_e - N) \) is included to account for the fact that the projected wave function does not have the right number of particles on the average. The previous term would correspond to the application of first order perturbation theory if the chemical potentials used were the derivatives of the projected energy with respect to the number of particles. In our calculations we have simply used the chemical potentials obtained in the HFB theory \(^1\). This simplification is justified by the fact that the deviations induced in the number of particles due to the angular momentum projection are always small and so are their effects on the projected energies.

For the computation of the matrix elements of the rotation operator in a HO basis we have used the results of Ref. [21].

Another relevant point to be discussed is the prescription to use for the density dependent part of the Gogny force. In the calculation of the energy

\(^1\)This recipe has been previously used [22] in the context of angular momentum projection and in Generator Coordinate Method (GCM) calculations [23]. In both cases it has been found that the present recipe works very well.
functional $E[\phi'] = \langle \phi' | \hat{H} | \phi' \rangle$ the density appearing in the density dependent part of the force is simply $\rho(r) = \langle \phi | \hat{p} | \phi' \rangle$ rendering the energy a functional of the density and the pairing tensor but with a dependence on the density different from the canonical quadratic one of the standard HFB theory. On the other hand, the energy overlap $E[I, \phi, \phi'] = \langle \phi | \hat{H} | \phi' \rangle / \langle \phi | \phi' \rangle$ can be evaluated using the extended Wick theorem. The final expression is the same as the HFB functional $E[I, \phi]$ but replacing the density matrix by $\tilde{\rho}_{ij} = \langle \phi | e^{\beta c^\dagger} c | \phi' \rangle / \langle \phi | \phi' \rangle$ and the pairing tensor by $\tilde{\kappa}_{ij} = \langle \phi | c_j c_i | \phi' \rangle / \langle \phi | \phi' \rangle$ and $\tilde{\kappa}_{ji} = \langle \phi | c_i c^\dagger_j | \phi' \rangle / \langle \phi | \phi' \rangle$. As a consequence, it seems rather natural to use the density $\tilde{\rho}(r) = \langle \phi | \hat{p} | \phi' \rangle / \langle \phi | \phi' \rangle$ in the evaluation of the density dependent term of the force in $E[I, \phi, \phi']$. In our case, this leads to the introduction of a density dependent term depending on $\tilde{\rho}(r, \beta) = \langle \phi | \hat{p} e^{-i \beta J} | \phi \rangle / \langle \phi | e^{-i \beta J} | \phi \rangle$. This density dependence seems to yield to bizarre consequences like having a non-hermitian and non rotationally invariant hamiltonian. These apparent inconsistencies can be overcome if we think of a density dependent force, not as an operator to be added to the kinetic energy in order to obtain a hamiltonian, but rather as a device to get energy functionals like $E[I, \phi]$ and $E[I, \phi, \phi']$ with the property of yielding an energy that is a real quantity and independent of the orientation of the reference frame. The density dependence just mentioned fulfills these two requirements as can be readily checked. In addition, when the intrinsic wave function is strongly deformed and the Kamlah expansion can be used to obtain an approximate expression for the projected energy (the cranking model) the above density dependence yields the correct expression for the angular velocity $\omega$ including the "rearrangement" term [24]. A more elaborated argumentation in favor of the density dependence just mentioned will be given elsewhere.

As an example of the results obtained we show in Fig. 1 the HFB and projected energies as a function of $q_{20}$ for the nucleus $^{34}\text{Mg}$. In contrast with the HFB result, the $I = 0$ energy surface shows two pronounced minima in the prolate and oblate side which are rather close to each other in energy being the prolate minimum slightly deeper than the oblate one. Therefore, it is difficult to assign a given character to the $I = 0$ state until a configuration mixing calculation is performed, although it is very likely that the predominant configuration for the $I = 0$ state is going to be the prolate one. For $I = 2$ there is a well developed prolate minimum. Let us also mention that for configurations with a $q_{20}$ value close to zero (i.e. close to the spherical configuration $q_{20} = 0$ which is a pure $I = 0$ state) it is very difficult to compute the $I = 2$ projected energy due to numerical instabilities related to the smallness of $\langle \phi(q_{20}) \mid \hat{p} \rangle \langle \phi(q_{20}) \rangle$.

In the inset of Fig. 1 we have plotted the energy difference $E_{\text{ROT}}(I) = \Delta E_{\text{FHB}} - E_0$ as a function of $q_{20}$ for $I = 0$ (full line) in order to compare it with the rotational energy correction $E_{\text{App}}(I) = J_0^2 / \mathcal{F}$ often used in mean field calculations (dashed line). The Yoccoz moment of inertia $\mathcal{F}$ has been computed, as it is usually done, in an approximate way by neglecting the two body quasiparticle interaction term of the hamiltonian (the same kind of approximation yields to the Inglis–Belyaev moment of inertia instead of the Thouless–Valatin one). We notice that $E_{\text{ROT}}^{\text{App}}$ agrees qualitatively well with $E_{\text{ROT}}(0)$ for $q_{20}$ values greater than 100 fm$^2$ and smaller than −50 fm$^2$ as expected: these are regions of strong deformation where the validity conditions for $E_{\text{ROT}}^{\text{App}}$ to be a good approximation to $E_{\text{ROT}}$ (Kamlah expansion) are satisfied. On the other hand, the behavior of $E_{\text{ROT}}^{\text{App}}$ is completely wrong in the inner region. One prescription to extend the rotational formula to
weakly deformed states is the one of [25] based on results with the Nilsson model. The prescription multiplies $E_{\text{App}}$ by a function of $\langle J^2 \rangle$ with the property of going to zero (one) for $\langle J^2 \rangle$ going to zero (infinity). The resulting rotational energy is also depicted in the inset of Fig. 1 (dotted line) and, although the qualitative agreement with the exact result improves somewhat, the quantitative one is far from satisfactory in the nucleus considered. The rotational energy correction formula is based on the assumption that the quantity $h(b) = \langle \hat{H} e^{-i\beta J} \rangle / \langle e^{-i\beta J} \rangle$ can be very well approximated by a quadratic function $h(b) = h(0) + \frac{1}{2} h''(0) b^2$ where $h''(0)$ is related to the exact Yoccoz moment of inertia by the expression $\mathcal{J} = -\langle \hat{J}^2 \rangle / h''(0)$. It is well known that this assumption is justified for deformed heavy nuclei. However, we have checked that it is not the case for the nuclei studied here even for the largest deformations considered. Therefore, we conclude that the exact restoration of the rotational symmetry is fundamental for a qualitative and quantitative description of the rotational energies in these light nuclei.

The main outcomes of the calculation are summarized in Fig. 2 where we show, on the left hand side panel, the HFB potential energy surfaces for Mg and Si isotopes as a function of the mass quadrupole moment. These surfaces have been shifted accordingly to fit them in the plot. We observe that only in the nuclei $^{34}$Mg and $^{38}$Si we obtain a prolate minimum at $\beta_2$ deformations of 0.4 and 0.35 respectively. For the other nuclei, the minimum corresponds to the spherical configuration. For all the nuclei considered the energy curves are very flat around the corresponding minimum indicating that further correlations can substantially modify the energy landscape and therefore the conclusions obtained from the raw HFB results.

On the middle and right hand side panels of Fig. 2 we show the angular momentum projected $I = 0$ and $I = 2$ potential energy surfaces for all the nuclei considered. These surfaces have also been shifted to fit them in the plot. For $I = 0$, apart from the nucleus $^{34}$Mg that shows a rather clear prolate minimum, the general trend for the ground state is to show shape coexistence. For $I = 2$ we have prolate minima for $^{32,34}$Mg and oblate minima for $^{32,34}$Si whereas the other nuclei are examples of shape coexistent structures. The results just shown indicate that, for a quantitative description of the ground and $2^+$ states in all these nuclei, a configuration mixing calculation (GCM) using the mass quadrupole moment as generating coordinate is needed. In spite of this, we present in Table 1 the $0^+ - 2^+$ energy differences for the four possible configurations with the $0^+$ in the prolate (P) or oblate (O) minimum and the $2^+$
yield to a reduction of the $B(E2)$ values (see column two for the $B(E2, 0^+_y \rightarrow 2^+_x)$) that will bring the theoretical predictions in closer agreement with the experimental data. For the $^{32}$Si, $^{34}$Si and $^{36}$Si isotopes we underestimate the $B(E2)$ values but, presumably, admixtures of the $0^+_y \rightarrow 2^+_x$ transition will help to bring the theoretical results in closer agreement with the experiment, specially for the $^{38}$Si nucleus. Concerning $^{36}$Si we can only conclude that a strong $0^+_y \rightarrow 2^+_x$ component has to be present in the evaluation of the $B(E2)$.

In Table 3 the HFB and projected ground state energies for the nuclei under consideration are shown and compared to the experimental data taken from [26]. The inclusion of the zero point energy stemming from the restoration of the rotational symmetry clearly improves the theoretical description of the binding energies.
In conclusion, we have computed several properties of neutron rich Mg and Si isotopes using the HFB theory and exact angular momentum projection. In the calculations the finite range density dependent Gogny force has been used. The results for the excitation energies $0^+ - 2^+$ and $B(E2,0^+ \rightarrow 2^+)$ transition probabilities obtained from the angular momentum projected wave functions are in reasonable agreement with the experiment. The analysis of the projected energy surfaces and also the discrepancies found between theory and experiment indicate that configuration mixing is an important ingredient in these nuclei. Work is in progress in order to incorporate such configuration mixing.

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References

Analytic solution for relativistic transverse flow at the softest point

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Abstract

We obtain an extension of Bjorken’s $1+1$ dimensional scaling relativistic flow solution to relativistic transverse velocities with cylindrical symmetry in $1+3$ dimension at constant, homogeneous pressure (vanishing sound velocity). This can be the situation during a first order phase transition converting quark matter into hadron matter in relativistic heavy ion collisions. © 2000 Published by Elsevier Science B.V. All rights reserved.

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Hydrodynamics often allow for nonrelativistic scaling solutions. Relativistic flow, however, seems to be an exception: besides Bjorken’s $1+1$ dimensional ansatz and the spherically symmetric relativistic expansion, no analytical solution is known [1].

In this paper we present an extension of Bjorken’s ansatz [2] for longitudinally and transversally relativistic flow patterns with cylindrical symmetry in $1+3$ dimensions. This is an analytical solution of the flow equations of a perfect fluid for physical situations when the sound velocity is zero, $c_s^2 = dp/de = 0$, with energy density $e$ and pressure $p(e)$. In particular this happens during a first order phase transition, the pressure is constant while the energy density changes (in heavy ion collisions increasing and drops again). This should, in principle, be signalled by a vanishing sound velocity. A remnant of this effect in finite size, finite time transitions might be a softest point of the equation of state, where $c_s^2$ is minimal. In fact, this has been suggested as a signal of phase transition by Shuryak [3], and investigated numerically in several recent works [4,5].

In the light of this research, the presentation of an analytical solution including relativistic transverse flow is worthwhile. For a general equation of state $p(e)$ the analytical solution given in this paper cannot be extended, only a perturbative expansion in terms of mild ($v \ll 1$) transverse velocities can be established. Such an approximation has been recently presented in [6]. Nonrelativistic analytic solution has been also given several times, with respect to heavy ions see [7–9].

The $1+1$ dimensional Bjorken flow four-velocity is a normalized, timelike vector. It is natural to
choose this as the first of our comoving frame basis vectors (vierbein). The three further, spacelike vectors will be constructed orthogonal to this, separating the two transverse directions. This basis fits excellently to a cylindrical symmetry and to longitudinally extreme relativistic flow.

\[
e^0_\mu = \left( \frac{t}{\tau}, \frac{z}{\tau} \right), \quad e^1_\mu = \left( -\frac{z}{\tau}, \frac{t}{\tau} \right),
\]

\[
e^2_\mu = \left( 0,0, -\frac{x}{r}, \frac{y}{r} \right), \quad e^3_\mu = \left( 0,0, -\frac{y}{r}, \frac{x}{r} \right), \quad (1)
\]

with \( t \) coordinate time and \( z \) longitudinal (beam-two transverse, cartesian coordinates. The cylindrical radius is given by, \( r = \sqrt{x^2 + y^2} \), and \( \tau \) is the longitudinal proper time: \( \tau = \sqrt{t^2 - z^2} \).

This way the basis can be re-written in terms of a hyperbolic angle (coordinate-rapidity) \( \eta \) and a polar angle \( \phi \)

\[
e^0_\mu = (\cosh \eta, \sinh \eta, 0, 0), \quad e^1_\mu = (\sinh \eta, \cosh \eta, 0, 0), \quad e^2_\mu = (0,0, \cos \phi, \sin \phi),
\]

\[
e^3_\mu = (0,0, -\sin \phi, \cos \phi).
\]

(2)

This basis is orthonormal, \( e^a_\mu \cdot e^{\mu b} = g^{ab} \), and satisfies the differential relations:

\[
de e^0_\mu = e^1_\mu d\eta, \quad de^1_\mu = e^0_\mu d\eta,
\]

\[
de e^2_\mu = e^3_\mu d\phi, \quad de^3_\mu = -e^2_\mu d\phi.
\]

(3)

The space-time coordinate differential and the partial derivative are in our basis given by,

\[
dx_\mu = e^0_\mu d\tau + e^1_\mu d\eta + e^2_\mu dr + e^3_\mu rd\phi,
\]

\[
\partial_\mu = e^0_\mu \frac{\partial}{\partial \tau} - e^1_\mu \frac{1}{\tau} \frac{\partial}{\partial \eta} - e^2_\mu \frac{1}{r} \frac{\partial}{\partial r} - e^3_\mu \frac{1}{r} \frac{\partial}{\partial \phi}.
\]

(4)

We consider ideal fluids ("dry water"). where the energy momentum tensor is given by

\[
T_{\mu \nu} = (\epsilon + p) u_\mu u_\nu - pg_{\mu \nu},
\]

(5)

and the equation of state is given in the form of \( p(\epsilon) \). The ansatz for an almost boost invariant flow with some transverse, cylindrically symmetric component is then given by

\[
u_\mu = \gamma (e^0_\mu + \nu e^2_\mu), \quad (6)
\]

using the Lorentz factor \( \gamma = (1 - v^2)^{-1/2} \). This four-velocity is normalized to one:

\[
u_\mu u^\mu = \gamma^2 - \gamma^2 v^2 = 1.
\]

(7)

For the four divergence of the flow we obtain

\[
\gamma \partial^\mu u_\mu = \gamma \left( \frac{\partial}{\partial \tau} + \frac{\gamma}{r} \frac{\partial}{\partial \phi} \right).
\]

(8)

The co-moving flow derivative is

\[
u_\mu \partial^\mu = \gamma \left( \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \phi} \right).
\]

(9)

In order to formulate the Euler equation describing a possible acceleration or deceleration of the flow, one also needs a projection orthogonal to \( u_\mu \). We use

\[
\gamma = (g_{\lambda \mu} - u_\lambda u_\mu) \partial^\mu,
\]

(10)

which is given by

\[
\gamma = -\gamma \left( e^0_\lambda + e^2_\lambda \right) \left( \frac{\partial}{\partial \tau} + \frac{\gamma}{r} \frac{\partial}{\partial \phi} \right)
\]

\[
- e^1_\lambda \frac{1}{\tau} \frac{\partial}{\partial \eta} - e^3_\lambda \frac{1}{r} \frac{\partial}{\partial \phi}.
\]

(11)

The relativistic equation of local energy and momentum conservation, \( \partial^\mu T_{\mu \nu} = 0 \), can then be projected to a component parallel to \( u_\mu \) and components orthogonal to that. Denoting the co-moving derivative (9) by \( \gamma \times \) an overdot, i.e. \( u_\mu \partial^\mu f = \gamma f \) for any \( f \), we arrive at

\[
\gamma \dot{\epsilon} + w \gamma \dot{u}_\mu u^\mu = 0, \quad w \gamma \dot{u}_\lambda - \gamma \lambda p = 0.
\]

(12)

Here we introduced the enthalpy density \( w = \epsilon + p \). The first equation of (12) is the local form of the \( dE + pdV = 0 \) adiabatic flow condition; in fact this is equivalent to the conservation of the entropy flow in a one-component matter. The second equation resembles the familiar form of the nonrelativistic Euler equation.
Summarizing we have altogether four independent equations:
\[
\left( \frac{\partial}{\partial \tau} + v \frac{\partial}{\partial r} \right) \epsilon + w D = 0,
\]
\[
\gamma^2 w \left( \frac{\partial}{\partial \tau} + v \frac{\partial}{\partial r} \right) v + \left( \frac{\partial}{\partial r} + v \frac{\partial}{\partial \tau} \right) p = 0,
\]
\[
1 \frac{1}{\tau} \frac{\partial}{\partial \eta} p = 0, \quad 1 \frac{\partial}{r \partial \phi} p = 0.
\]

(13)

Here the four divergence (8) without the Lorentz factor \( \gamma \) is denoted by \( D = \gamma \hat{k}_a u^a \), which can be expanded to
\[
D = \gamma^2 \left( \frac{\partial}{\partial r} + v \frac{\partial}{\partial \tau} \right) v + \frac{1}{\tau} \frac{\partial}{\partial r} v.
\]

(14)

Let us restrict ourselves now to \( p = p_0 = \) constant situations. In this case \( c_4^2 = 0 \) and \( w = \epsilon + p_0 \). From (13) we arrive at two independent equations only, the first and the second one. Since derivatives of the pressure vanish, we obtain an equation for the transverse flow component alone:
\[
\left( \frac{\partial}{\partial \tau} + v \frac{\partial}{\partial r} \right) v = 0.
\]

(15)

This equation is valid even for relativistic transverse flow, it is a quasilinear partial differential equation. A general solution can be obtained by the factorizing ansatz \( v = a(\tau) \cdot b(r) \). The above Eq. (15) leads to
\[
\frac{a'(\tau)}{a^2(\tau)} + b'(r) = 0.
\]

(16)

Here both terms are constant, balancing each other to zero. The solution, which is regular at the cylindrical axis \( r = 0 \) is given by
\[
v = \frac{\alpha r}{1 + \alpha \tau}.
\]

(17)

The first equation of the system (13) is the cooling equation. It reduces to
\[
\left( \frac{\partial}{\partial \tau} + v \frac{\partial}{\partial r} \right) \epsilon + (\epsilon + p_0) D = 0,
\]

(18)

with \( D \) given by Eq. (14). Utilizing the solution (17) for \( v(\tau, r) \) we obtain
\[
D = \frac{1}{\tau} + \frac{2\alpha}{1 + \alpha \tau}.
\]

(19)

which can also be written as a comoving derivative,
\[
D = \left( \frac{\partial}{\partial \tau} + v \frac{\partial}{\partial r} \right) \log(e(\tau)),
\]

(20)

with
\[
E(\tau) = \tau(1 + \alpha \tau)^2.
\]

(21)

Using \( D \) in this form the cooling Eq. (18) can be rewritten as
\[
\left( \frac{\partial}{\partial \tau} + v \frac{\partial}{\partial r} \right) \log(e(\tau)(\epsilon + p_0)) = 0.
\]

(22)

A particular solution of this equation is, when the quantity on which \( (\frac{\partial}{\partial \tau} + \frac{\partial}{\partial r}) \) operates is constant,
\[
\epsilon + p_0 = \text{const} \frac{\tau}{(1 + \alpha \tau)^2}.
\]

(23)

This solution interpolates between the one-dimensional cooling law \( \epsilon \propto 1/\tau \) and a three-dimensional one for large longitudinal proper time, \( \epsilon \propto 1/\tau^3 \).

There is, however, a more general solution to the "cooling law". First inserting the analytic solution (17) for \( v \) in the co-moving derivative operation we get
\[
\left( \frac{\partial}{\partial \tau} + \frac{\alpha}{1 + \alpha \tau} \frac{\partial}{\partial \tau} + \frac{1}{1 + \alpha \tau} \frac{\partial}{\partial r} \right) \log w + \log E(\tau) = 0.
\]

(24)

Multiplying this by \( (1 + \alpha \tau)/\alpha \) we obtain
\[
\left( \frac{\partial}{\partial \log(1 + \alpha \tau)} + \frac{\partial}{\partial \log r} \right) \log w + \log E(\tau) = 0.
\]

(25)

This equation is of type
\[
\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F(x, y) = 0,
\]

(26)

whose solution is simply a function of \( x - y \), \( F(x, y)_{\text{solution}} = F(x - y) \). Henceforth we obtain
\[
\log w + \log E(\tau) = F(\log r - \log(1 + \alpha \tau)).
\]

(27)

In this result one easily recognizes the analytic form of the transverse flow velocity \( v \) (17) after collecting the terms in the argument of the general function \( F \) inside one logarithm. Finally we arrive at
\[
\log w + \log E(\tau) = F(v/\alpha) = \log f(v)
\]

(28)
as an analytic solution of the cooling law with relativistic cylindrical transverse flow:

$$\epsilon (r, \tau) = -p_0 + \frac{f(v)}{\tau (1 + \alpha \tau)^2}, \quad v = \frac{\alpha}{1 + \alpha \tau}r.$$  \hfill (29)

The unknown function of one variable \( f(v) \) can be used to match the initial profile of the radial distribution of the energy density at \( \tau = \tau_0 \),

$$\epsilon (r, \tau_0) = -p_0 + \frac{1}{\tau_0 (1 + \alpha \tau_0)^2} f \left( \frac{\alpha r}{1 + \alpha \tau_0} \right).$$  \hfill (30)

Let us finally discuss some properties of this analytic solution. First one would like to be convinced that the Bjorken scaling limit is contained in Eq. (29). (The \( 1 + 1 \) dimensional solution was presented by Gyulassy and Matsui in 1984 [10].) This is indeed the case, for \( v \to 0 \) one obtains \( \alpha = 0 \) and

$$\epsilon (r, \tau) = -p_0 + \frac{f(0)}{\tau},$$  \hfill (31)

leading to

$$\left( \epsilon (\tau) + p_0 \right) = \left( \epsilon_0 + p_0 \right) \frac{\tau_0}{\tau}. \hfill (32)$$

As an example of the \( p = \text{const.} \) situation let us consider an oversimplified system: massless quark-gluon plasma described by the bag equation of state,

$$\epsilon_0 = \sigma T^4 + B, \quad p_0 = \frac{1}{3} \sigma T^4 - B,$$  \hfill (33)

keeps a Gibbs equilibrium with a light, relativistic pion gas, described by

$$\epsilon_\pi = hT^4, \quad p_\pi = \frac{1}{3} hT^4.$$  \hfill (34)

During a first order phase transition, for simplicity assumed to take place in the total volume, the pressure is constant and homogeneous (Gibbs criterion). Actually from this requirement \( p_\pi = p_\pi \) one usually obtains the transition temperature due to

$$\frac{1}{2} (\sigma - h) T_c^4 = B.$$ \hfill (35)

The energy density is that of a mixture, containing \( x \) part quark matter and \((1 - x)\) part hadron matter,

$$\epsilon = x \epsilon_0 (T_c) + (1 - x) \epsilon_\pi (T_c). \hfill (36)$$

This partition \( x \) is what changes according to the cooling law, respectively its solution (29). Utilizing Eqs. (33)–(35) we arrive at

$$\epsilon = 4Bx(\tau) + \epsilon_\pi = -p_\pi + (\epsilon_0 + p_0) \frac{\tau_0}{\tau}. \hfill (37)$$

Assuming at \( \tau = \tau_0 \) pure quark matter \( x(\tau_0) = 1 \), the time of the total conversion \( \tau_1 \) when \( x(\tau_1) = 0 \) is given by

$$\tau_1 = \frac{\epsilon_0 + p_0}{\epsilon_{\pi} + p_\pi} \frac{\tau_0}{\tau} = \frac{\sigma}{h} \tau_0. \hfill (38)$$

It is determined by the relative number of degrees of freedom in the two phases in this simple scenario.

The same discussion is somewhat more complex using the analytic solution with a cylindrical flow. At the beginning of the phase transition \( x(\tau_0) = 1 \) let be a transverse velocity,

$$v (\tau_0, r) = \frac{v_0}{R_0} r.$$ \hfill (39)

The analytic solution (17) leads to

$$v (\tau, r) = \frac{v_0}{R_0 + v_0 (\tau - \tau_0)} r,$$ \hfill (40)

featuring a slowing down of the transverse flow. From the initial condition for the transverse flow we also obtained the parameter \( \alpha \)

$$\alpha = \frac{v_0}{R_0 - v_0 \tau_0}. \hfill (41)$$

Since the four-flow is given by

$$u_\mu = \gamma (\epsilon^{\mu}_\mu + \epsilon_\mu \gamma),$$

a slowing down of the transverse component implies a slowing down of the original Bjorken component as well (\( \gamma \) is decreasing to one, when \( v \) is decreasing to 0).

The cooling of the energy density can be obtained now by fitting the profile at \( \tau = \tau_0 \) to \( \epsilon (r, \tau_0) = \epsilon (r) \),

$$\epsilon (r, \tau) + p_0 = \frac{\tau_0}{\tau} \frac{\tau_0}{R_0^2} \left[ \left( \epsilon \left( \frac{R_0}{R} r \right) + p_0 \right) \right. \hfill (42)$$

with

$$R (r) = R_0 + v_0 (\tau - \tau_0). \hfill (43)$$
Fig. 1. Contour lines of several QGP – hadronic mixtures from 0 to 1 by 0.1 steps in the $t-t_R$ plane. The initial profile at $\tau = \tau_0$ was assumed to be linear giving zero percent quark matter exactly at $R = R_0$. The initial transverse flow is $v_0 = 0.6$ at the edge $R = R_0$.

This solution follows the Bjorken cooling law at the beginning $\tau \approx \tau_0$,

$$\epsilon(r, \tau) + p_0 = \frac{\tau_0}{\tau} \left( \epsilon(r) + p_0 \right), \quad (44)$$

but for long times $\tau \gg \tau_0$ turns over to a three-dimensional scaling

$$\epsilon(r, \tau) + p_0 = \frac{\tau_0 R_0^2}{\tau t} \left( \epsilon \left( \frac{R_0 \tau}{v_0 \tau} \right) + p_0 \right). \quad (45)$$

Here for large times, the driving force besides the constant pressure will be $\epsilon(0)$, the original energy density at the axis.

Finally we repeat the calculation in the simplified scenario assuming a first order phase transition between ideal quark gluon plasma and ideal pion gas. The general solution for the quark matter part, $x$ is given by

$$x(\tau, r) = \frac{h}{\sigma - h} + \frac{\tau_0 R_0^2}{\tau R^2} \left( \frac{h}{\sigma - h} + x \left( \frac{R_0}{R} \right) \right). \quad (46)$$

Assuming a linearly decreasing transverse profile of the QGP part initially (at the beginning of the phase transition),

$$x(\tau_0, r) = 1 - r/R_1, \quad (47)$$

we obtain the transverse radius of the mixed phase as

$$r_b = \frac{R_1}{R_0} \left( 1 - \frac{h}{\sigma - h} \left( \frac{\tau R^2}{\tau_0 R_0^2} - 1 \right) \right). \quad (48)$$

In the simple case, when $R_0 = v_0 \tau_0$, the scaling transverse radius $R$ is proportional to the longitudinal proper time $\tau$: $R = v_0 \tau$. The transverse velocity field is also particularly simple: $v = r/\tau$. The above result simplifies to

$$r_b = R_1 \frac{\tau}{\tau_0} \left( \frac{\sigma}{\sigma - h} - \frac{h}{\sigma - h} \frac{\tau^3}{\tau_0} \right). \quad (49)$$

This expression initially grows, achieves a maximum and then decreases towards zero. The maximum,

$$r_b^{\text{max}} = \frac{2}{\pi} R_1 \left( \frac{\sigma}{4h} \right)^{1/3} \frac{\sigma}{\sigma - h}. \quad (50)$$
is achieved at time
\[ \tau_{\text{max}}^{\text{r}} = \tau_{0} \left( \frac{\sigma}{4h} \right)^{1/3}, \] (51)
the \( x = 0 \) for the whole space (equivalent to \( r = 0 \)) at
\[ \tau_{t} = \tau_{0} \left( \frac{\sigma}{h} \right)^{1/3}. \] (52)
Realistic estimates use \( \sigma = 37 \) for the quark gluon plasma and \( h = 3 \) for the pion gas. This leads to \( \tau_{0}^{\text{r}} = 1.14 \tau_{0}, \tau_{\text{max}}^{\text{r}} = 1.45 \tau_{0}, \) and \( \tau_{t} = 2.31 \tau_{0}. \) It is realistic to assume \( \tau_{0} = 5 \text{ fm/c} \) for a CERN SPS Pb + Pb experiment. The radial extension of the quark matter can grow with about 14 per cent and then the conversion into hadrons eats it up, reaching the zero radius in about 11.5 fm/c. This is a fast hadronization even in this simple scenario.

For a smaller initial radial flow, realistically \( v_{0} = 0.6 \) at the radial edge of the cylinder \( r = R_{0}, \) one obtains \( v_{0} \tau_{0}/R_{0} = 0.428. \) Fig. 1 shows the radial space - time evolution profiles of the mixture: the outmost curve corresponds to \( x = 0 \) (pure hadron matter) and the steps are 0.1. The innermost curve belonging to \( x = 1.0 \) shrinks to a point, since the phase conversion starts immediately at \( \tau = \tau_{0}. \) Here we assumed a linear initial profile \( x(r, \tau_{0}) = 1 - r/R_{0}. \) With these parameters the mixed phase of \( R_{0} = 7 \text{ fm} \) lasts for about \( \tau_{0} = 17.5 \text{ fm/c}. \)

In conclusion we have presented an analytical solution for relativistic transverse flow of a perfect fluid at the softest point. This can be realized during a first order phase transition, when the pressure is homogeneous and constant in a large volume. The phase conversion during this stage of expansion is given by this analytical solution which scales with the longitudinal proper time initially like the Bjorken flow \( 1/\tau, \) but eventually like a spherical flow \( 1/\tau^{3}. \)

For simple equations of state for the quark and hadronic matter side the estimated longitudinal proper time spent in the mixed phase turns out to be more than twice the time when the phase transition began.

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References

The physics of the centrality dependence of elliptic flow

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Abstract

The centrality dependence of elliptic flow and how it is related to the physics of expansion of the system created in high energy nuclear collisions is discussed. Since in the hydro limit the centrality dependence of elliptic flow is mostly defined by the elliptic anisotropy of the overlapping region of the colliding nuclei, and in the low density limit by the product of the elliptic anisotropy and the multiplicity, we argue that the centrality dependence of elliptic flow should be a good indicator of the degree of equilibration reached in the reaction. Then we analyze experimental data obtained at AGS and SPS energies. The observed difference in the centrality dependence of elliptic flow could imply a transition from a hadronic to a partonic nature of the system evolution. Finally we exploit the multiplicity dependence of elliptic flow to make qualitative predictions for RHIC and LHC.

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1. Introduction

The goal of the ultrarelativistic nuclear collision program is the creation of the QGP – quark-gluon plasma – the state of deconfined quarks and gluons. It is understood that such a state requires (local) thermalization of the system brought about by many rescatterings per particle during the system evolution. It is not clear when and if such a dynamical thermalization can really occur. An understanding of these phenomena can be achieved by considering elliptic flow [1] recently studied at AGS [2] and SPS [3] energies. It will be shown how the centrality dependence of the strength of elliptic flow, \( v_2 \), defined as the second coefficient in the Fourier decomposition of the particle azimuthal distribution [4], is an indicator of the degree of equilibration (thermalization) achieved in the system.

Our qualitative conclusions are based on the observation, that in the hydro limit (which we equate in our discussion to complete thermalization) and in the opposite limiting case, the low density limit (where dynamical thermalization is not expected), the centrality dependence of elliptic flow is different. In the hydro limit, the mean free path is much less than the geometrical size of the system. The centrality dependence of flow is totally governed in this case by the initial geometry (eccentricity), the latter being roughly proportional to the impact parameter. In the low density limit, the mean free path is comparable to or larger than the system size. The final anisotropy in this case should be proportional to the ratio of the system size to the mean free path (the number of collision). The anisotropy vanishes in the limit of infinite mean free path. The latter in its turn depends on the particle density, which is largest for central
collisions and vanishes for very peripheral collisions. Note that the factors involved change drastically with centrality. One could imagine other reasons for centrality dependence of elliptic flow in the hydro model, such as the initial conditions, viscous corrections, resonances, or effective volume corrections, but we expect that all these other factors have a much weaker dependence on the impact parameter. By considering the two limiting cases we hope to highlight qualitative considerations important for understanding the degree of thermalization and the partonic or hadronic nature of the collisions. In essence we present a framework for examining these questions experimentally, however at the moment, it is mainly the experimental data which are not adequate to answer these questions convincingly.

We will often use the term “physics of the collision”. By this we mean both the degree of equilibration and whether the hadronic picture in terms of nucleons, pions, etc., or the partonic picture in terms of deconfined quarks and gluons, is more applicable to the evolution of the system. The partonic picture in our view is similar to a QGP but the system is not necessarily thermalized.

2. Low density limit

To discuss the centrality dependence of $v_2$ more quantitatively, we start from the hypothesis that the system is not dense and its evolution can be described by the first correction to the collisionless limit [5]. Physically this means that the rescattering occurring during the system evolution changes the particle momenta very little on the average and the corresponding change in the distribution functions can be treated in first order as perturbations. Under this assumption the final elliptic flow, $v_2$, is proportional to the initial overlapping region elliptic anisotropy, $e$ (introduced in flow analyses in [1] and in its present form in [5,6]), and to the initial particle space density which defines the probability of particles to rescatter [5].

The initial geometry of the overlapping zone can be evaluated in a simple Glauber type model with a Woods–Saxon nuclear density. The results are weakly dependent on the weights used [7]. What is important is that if one wants to compare different energies, e.g. AGS, SPS and RHIC, the nuclear geometry cancels out, and only the dependence on multiplicity is left. This is true provided that the “physics” of the system evolution stays the same. If it changes then the scaling with multiplicity will be violated. This is a very important point if one reads it the other way around: if scaling is not observed then probably the physics has changed.

Under the assumption that the system is relatively dilute the momentum anisotropy is proportional to the spatial anisotropy, but also the particles must scatter to probe that anisotropy. Thus, the spectra distortion is directly proportional to the spatial anisotropy and the number of rescatterings, or the particle density in the transverse plane. In this limit the final elliptic flow (see a more detailed formula in [5])

$$v_2 \propto \frac{1}{S} \frac{dN}{dy}$$

where $S = \pi R_x R_y$ is the area of the overlapping zone, with $R_x^2 \equiv \langle x^2 \rangle$ and $R_y^2 \equiv \langle y^2 \rangle$ describing the initial geometrical sizes of the system in $x$ and $y$ directions, respectively. (The $x$-z axes lie in the reaction plane.) The averages include a weighting with the number of collisions along the beam axis. The initial space elliptic anisotropy is defined as

$$e = \frac{R_x^2 - R_y^2}{R_x^2 + R_y^2}.$$  

In our calculation we use a Woods–Saxon parameterization of the nuclear density with parameters $R_A = 1.12 \cdot A^{1/3}$, and $a = 0.547 \text{ fm}$. More information on the effect of different weights and the values of $R_x^2, R_y^2, S$ and $e$ as a function of impact parameter can be found in [7]. The proportionality coefficient in Eq. (1) is defined by the “physics” of the rescattering. If the physics is the same in central and peripheral collisions then Eq. (1) yields the centrality dependence of $v_2$.

3. Hydro limit

As follows from Eq. (1) the elliptic flow increases with the particle density. Eventually it will saturate...

Fig. 1. Top: comparison of elliptic flow, $v_2$, for pions from RQMD ver. 2.3 (filled circles) with the dependence expected for the low-density limit (solid line) and that expected for the hydro limit (dashed line). Bottom: ratios of $v_2/v_2^{LDL}$ and $v_2/v_2^{HYDRO}$.

[8] at the hydro limit, which would mean complete thermalization of the system. In this regime the centrality dependence of elliptic flow is mainly determined by the initial elliptic anisotropy of the overlapping zone in the transverse plane [8], and the ratio of the two should be approximately constant as shown in the first such calculations done by Ollitrault [1]. From his results it follows that $(v_2/e)_\text{hydro} \approx 0.27–0.35$, depending on the equation of state used (with or without QGP) [1]. The calculations [9] give a somewhat smaller flow, resulting in $(v_2/e)_\text{hydro} = 0.21–0.23$ (partly due to the realistic treatment of resonances which decrease the pion flow by about 15%). Note that in both calculations, [1] and [9], the longitudinal expansion of the system is treated analytically assuming Bjorken scaling. Real 3D hydro calculations would be very useful, although we do not expect that they would greatly change the centrality dependence.

4. RQMD

Before discussing the experimental data we will first consider a realistic model. We take RQMD v2.3 [10] for our calculations. Fig. 1 top shows the comparison of the directly calculated $v_2$ of pions in Pb + Pb collisions at 158 GeV · A collisions at midrapidity ($-1 < y < 1$) with the expectation from the low density limit, $v_2^{LDL}$ (Eq. (1) normalized to the same area under the curve in order to illustrate just the centrality dependence.) One can see rather good agreement, which suggests that RQMD is close to the low density limit even as one scans the centrality from peripheral to central collisions. In this version of RQMD no QGP is simulated. This is not that striking a conclusion, considering that no hydro-type behavior has ever been observed in RQMD. Note that the low density limit does not mean a low number of total rescatterings. The number of rescatterings can be large provided all of them are relatively soft and the particle momentum changes little.

To avoid confusion, note the difference in definitions of $e$ used in Eq. (2) of this paper and $\alpha_s$ from [1]. For Pb + Pb collisions the maximal value of $e \approx 0.44$ compared to $\alpha_s \approx 0.3$. Then, the results [1] yield $e(p^2)/e \approx 0.55–0.7$, where $e(p^2)$ means the elliptic flow weighted with $p^2$. Recent calculations [9] show that the particle elliptic flow is related to this quantity as $v_2 \approx 0.5 \cdot e[p^2]$.

Fig. 2. Elliptic flow at the AGS (open circles) and the SPS (filled squares).
compared to the initial momentum. The cross section which enters the equations is the transport (not total) cross section (see [5]). The centrality dependence expected for the hydro limit is shown on the same plot by a dashed line also normalized to the same area under the curve \( \langle v_2^{\text{HYDRO}} \rangle = 0.059 \). Note the large difference between the two curves, which was not noted in [5]. Fig. 1 bottom shows that the ratio of \( v_2 \) to the expected functional form is flat for the low density limit but not for the hydro limit. A centrality dependence similar to the low density limit was also observed in [11] where a computer simulation of a pion gas expansion was studied.

5. Data

Now let us turn to the experimental data. At AGS energies the elliptic flow of charged particles and of transverse energy was measured by the E877 Collaboration. Unfortunately, the publication [2] containing the detailed pseudorapidity dependence for each centrality lacks a figure showing just the centrality dependence. Our estimates based on their data [2] of charged particle flow at midrapidity are presented in Fig. 2.

The data indicate that at AGS the flow peaks at mid-centralit\(y \), consistent with the low density limit prediction and no change in physics with centrality. At this energy some decrease of elliptic flow in peripheral collisions can be also attributed to shadowing by spectator matter. At SPS [13], preliminary data indicate that the elliptic flow peak moves towards peripheral collisions. This fact itself would hint at the hydro-dynamical picture of the system evolution. A more detailed look at the data shows that this is unlikely. First, the maximal value of elliptic flow \( \langle v_2 \rangle \leq 0.04 \) is significantly less than predicted by hydro calculations [1,9] (about 0.09–0.1) \(^2\). Second, in the hydro limit elliptic flow should depend only on the initial space elliptic anisotropy, \( \varepsilon \). The preliminary NA49 data indicate that the ratio \( \langle v_2 \rangle / \varepsilon \), at least for semi-central collisions, is likely increasing with centrality [13] (see the data presented in Fig. 3 below). This centrality dependence (natural for the low density limit) implies that we still could be far from the hydro regime \(^3\).

Assuming that at SPS the hydro regime is not reached yet, the observed centrality dependence of elliptic flow would indicate that the physics of the system evolution is different in central and peripheral collisions. Elliptic flow peaks at more peripheral collisions because the central collisions exhibit too little flow compared to that expected from the AGS data scaled with multiplicity. A natural explanation for this would be that peripheral collisions are described by hadronic (re)scatterings (the same as at the AGS in both peripheral and central collisions)

\(^2\)A similar centrality dependence of transverse energy flow (from the same data [2]) can be found in the thesis of Chang [12].

\(^3\)In [9] agreement was claimed between hydro and the NA49 mid-central data [3] leading to their conclusion of complete equilibration. However, this comparison was done for \( p_t < 0.3 \) GeV/c and it could be that the \( p_t \) dependence of \( v_2 \) in the hydro model does not agree with experiment.

A one can argue that, taking into account systematic uncertainties, the preliminary SPS data for \( v_2 / \varepsilon \) are consistent with being constant as a function of centrality. In this case it would indeed mean that the system has equilibrated and the hydro regime has been reached. The low absolute strength of the elliptic flow in this case would indicate that the equilibration happens at a rather late time when the spatial anisotropy \( \varepsilon \) has decreased due to initial “free streaming”. We do not exclude this possibility but must wait for the final SPS data and the coming RHIC data to answer the question.
while in central collisions partonic physics becomes important. One of the possible mechanisms responsible for the change could be a color percolation occurring at high parton densities in the central collisions and discussed in more detail below.

6. Discussion

Summarizing, our view of the overall picture is: at AGS energies, the physics of rescattering which defines the system evolution is hadronic in nature, while at SPS it is the same for peripheral collisions, but for central collisions the physics is likely to be partonic. The partonic picture will remain at RHIC energies, with some extension toward more peripheral collisions. At RHIC equilibration becomes more important, but it is not clear if complete thermalization will be reached. At LHC energies the parton densities could become so high that (partonic) rescattering would lead to dynamical equilibration of the (partonic) system (creation of regions of real QGP) and consequently to a hydro-dynamical type of system evolution.

The above picture for collisions of heavy nuclei implies that the shape of the centrality dependence of elliptic flow would change continuously with beam energy. At AGS, the elliptic flow is peaked at an impact parameter value slightly higher than \( R_A \), just as prescribed by the low density limit. At SPS energies the peak moves toward more peripheral collisions because possibly the physics of relatively central collisions may have changed from hadronic to partonic, which leads to weaker flow than one would expect taking into account the increased multiplicity. If thermalization is not reached at RHIC, the elliptic flow peak could move back toward mid-central collisions because the physics of the peripheral and central collisions will be the same – partonic rescattering, unlike the situation at SPS when peripheral collisions are driven by the hadronic rescatterings resulting in relatively large flow signal. At even higher energies at LHC, the elliptic flow should peak at more peripheral collisions just as predicted by hydrodynamic calculations.

The schematic overall picture based on these observations is presented in Fig. 3, where the ratio of elliptic flow to the initial space elliptic anisotropy is presented as a function of initial particle density. In this plot we use the experimental charged particle multiplicity, assuming that it is proportional to the total particle multiplicity and also to the initial particle multiplicity. For the experimental values we use \( dN_{ch} / dy \) at mid-rapidity from [14,15].

In the limit of very low density the objects which rescatter must be hadrons. At some critical density a partial deconfinement happens. Parton density becomes high enough such that the color parton can propagate in the perpendicular plane without hadronization. Each parton is always close enough to other partons which screen its color. Once the motion in the perpendicular plane becomes easier (there is no need for hadronization), the elliptic flow decreases. Note that the system still can be far from being dynamically thermalized, which would occur only at even higher particle densities. Even more important, such a significant change in the behavior of \( v_2 / \rho \) can only happen if the system is not thermalized. See also the discussion of this question in [6,9] along with the discussion of the possibility of observation of the QGP to hadron gas phase transition.

To prove or disapprove the picture described above one needs more accurate data on the centrality dependence of elliptic flow. We would like to emphasize the importance of flow measurements not only at medium impact parameters but in the full

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5 At the moment this plot is qualitative as many things shown have large uncertainties. The hydro limits can depend slightly on the initial particle density [1,9] and, more importantly, on the time of thermalization of the system. The values shown are an average of the results of [1,9]. The predictions for the case without QGP are only for the EoS of a massless pion gas. Resonances can soften the EoS and lead to weaker flow. The uncertainty in the experimental points is mainly from the determination of the collision centrality required for calculation of the initial space elliptic anisotropy and the area of the overlapping region. The data points correspond to the centrality determined from the fraction of the total cross section corresponding to each centrality bin. Higher centralities were estimated from experimental measurement of the number of participants [14]. Finally, the smooth dashed curves are just schematic illustrations for hadronic and partonic scenarios and the solid curve includes a transition between the two.

6 This picture is very close to the deconfinement (color percolation) model discussed by Satz [16] for \( J/\Psi \) suppression.
range of centrality including rather central collisions where the anisotropic flow is small. The measurement of elliptic flow and its centrality dependence at RHIC thus becomes very important. Different models predict different rapidity densities for RHIC and LHC. Assuming that they are higher than at SPS by factors of 2 and 8, respectively, we have indicated the regions expected for Au + Au (Pb + Pb) collisions in Fig. 3. The measurements of elliptic flow in collisions of lighter systems (e.g. Cu + Cu) are also very important since they would cover the region of the SPS Pb + Pb data and would be useful in testing the above picture. The new SPS data taken at 40 GeV · A energy are also of great interest since they would bridge the two other sets of data and may scan the onset of deconfinement from hadronic to partonic physics.

Note that our picture of nuclear collisions and QGP production is different from what is usually discussed, which assumes thermal equilibrium even at rather low beam energies, when QGP is not expected, and then with an increase in collision energy, formation of regions of QGP. We believe that what could happen is that the deconfinement can occur before dynamical thermalization is achieved [17] and that the centrality dependence of elliptic flow would be a good indicator of this.

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Evidence for short-range correlations in $^{16}$O

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Abstract

The reaction $^{16}\text{O}(e,e'p)p^{14}\text{C}$ has been investigated at three values of the transferred energy $\omega$. The differential cross sections were determined as a function of the missing energy and the missing momentum. Evidence for short-range correlations in $^{16}$O has been obtained from the transition to the ground state of $^{14}$C. The cross sections for this transition are well reproduced by two independent parameter-free microscopic calculations. The results of both calculations show that the reaction is dominated by knockout of a proton pair in a $1^1S_0$ state, driven by short-range-correlations.

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1. Introduction

Recently, it was shown that the two-proton knockout reaction $(e,e'p)$ is a suitable probe for studying short-range correlations (SRC) in nuclei [1–3]. A comparison of the cross sections measured for the reaction $^{16}\text{O}(e,e'p)p^{14}\text{C}$ with the results of calculations, performed within a factorization approximation of the cross section, led to the conclusion that direct knockout of two protons in a $1^1S_0$ state dominates the reaction [1]. In Ref. [4], the measured
cross sections were compared to microscopic calculations performed by Giusti et al. [5]. In particular, the data for the transition to the ground state are well reproduced by the calculations. A similar result was obtained by Rosner et al. [6], who investigated the reaction $^{16}\text{O}(e,e'pp)^{14}\text{C}$ at comparable energy and momentum transfer, but different kinematical conditions for the ejectiles. Both studies provided clear signatures for a reaction driven by SRC, but other competing processes are known to contribute to the cross section as well. In particular, a sizeable contribution from the excitation and subsequent decay of the $\Delta$-isobar via a $\Delta N \rightarrow NN$ reaction cannot be excluded. Though this reaction is suppressed for knockout of a proton pair in a $S = 0$, $T = 1$ state, a thorough investigation of the reaction mechanism is required before strong statements about the role of SRC can be made. Therefore, we have studied the $^{16}\text{O}(e,e'pp)^{14}\text{C}$ reaction at three values of the transferred energy ($\omega = 180$, 210 and 240 MeV), keeping the momentum transfer fixed at $q = 300$ MeV/c. This implies that the invariant energy of the virtual photon and the two protons in the initial state gradually increases. Consequently, the contribution of intermediate $\Delta$-excitation to the reaction will also increase with increasing energy transfer, which allows to experimentally estimate its size.

Nevertheless, to disentangle the contribution of the various processes contributing to the $(e,e'pp)$ reaction the data have to be compared to model predictions. In these models approximations are made. Exact many-body calculations starting from realistic NN interactions are, at this time, only feasible for $A \leq 7$ [7]. For more complex nuclei, the dynamics of nucleons at close proximity is incorporated in nuclear structure calculations by means of state-dependent correlation functions [8] or defect functions [9]. The experimental data, presented in this paper, are compared with the results of two calculations performed with the models of Ref. [5] and Ref. [10], respectively. Both models treat the dynamics of the the proton pair in the initial state, the medium effects of the $\Delta$, and the final-state interaction of the ejectiles in a somewhat different way. By comparing the experimental cross sections with two independent calculations we aim at minimizing the model dependence in the interpretation of data.

The paper is organized as follows. In Section 2, some theoretical aspects and the kinematics of electron-induced two-nucleon knockout reactions are discussed. In Section 3, the experiments are briefly described, and the data are presented and compared to the results of theoretical calculations. Finally, conclusions are drawn and an outlook is given in Section 4.

2. Electron-induced two-nucleon knockout

The missing energy ($E_m$) and the missing momentum ($p_m$) for the $(e,e'NN)$ reaction are defined as:

$$E_m = \omega - T_1 - T_2 - T_{\Delta-2} \quad (1)$$

$$p_m = q - p_1 - p_2 \quad (2)$$

where $T_{1,2}$ ($p_{1,2}$) and $T_{\Delta-2}$ ($p_{\Delta-2}$) are the kinetic energies (momenta) of the knocked-out nucleons and the $A - 2$ nucleus. The excitation energy of the $A - 2$ nucleus is given by $E_i = E_m - S_{NN}$, where $S_{NN}$ is the two-nucleon separation energy. Furthermore, the missing momentum $p_m$ is equal to the momentum of the recoiling nucleus $p_{A-2}$. In a quasi-free knockout process, where the $A - 2$ nucleus acts as spectator, $p_{A-2}$ is equal in size and opposite in direction to the momentum of the nucleon pair in the initial state: $p_m = p_{A-2} = -P$. Hence, in quasi-free two-nucleon knockout, discarding the interactions in the final state, the cross section as a function of $p_m$ reflects the centre-of-mass motion of the pair in the initial state. This motion is characterized by an angular momentum $L$. Contrary to single-nucleon knockout from a target nucleus with $J = 0$, this angular momentum is not uniquely determined for a transition to a selected state in the residual nucleus; the total angular momentum of the two protons in the initial state $J = |j_1 + j_2|$ can generally be composed in more than one way from the angular momenta associated with the relative motion $I$ and the center of mass motion of the pair $L$, i.e. $J = I + L + S$. Nevertheless, the cross section as a function of $p_m$ already contains valuable information on the reaction, as is discussed in Section 3.

More interesting information on the processes contributing to the knockout reaction is contained in the distribution of the cross section as a function of
Fig. 1. Examples of the momentum transfer in an $e,e'p$ reaction; in the left figure $q$ is transferred to $p_2$, the proton emitted in the backward direction, and in the right figure to $p_1$, the proton that is emitted in the forward direction.

the relative momentum of the nucleons in the initial state, which is defined as $p = \frac{1}{2}(p_1 - p_2)$. $p_1$ and $p_2$ are the nucleon momenta in the initial state. The part of the cross section stemming from SRC is directly related to this momentum. This is illustrated in a simple factorization approximation of the cross section \[d\sigma = K \sigma_{pp}(p) F(E,P)\] (3)

with $dV = dE_t d\Omega_t dE_{p_1} d\Omega_{p_1} dE_{p_2} d\Omega_{p_2}$. In this expression $F(E,P)$ represents the probability of finding a proton pair in the target nucleus, with energy $E$ and center of mass momentum $P = p_1 + p_2$, and $K$ is a kinematical factor. The dynamics of the knockout process is contained in $\sigma_{pp}(p)$, which accounts for the probability that the virtual photon is absorbed by a proton pair with relative momentum $p$. Unfortunately, the cross section as a function of $p$ cannot be determined unambiguously from the measured proton momenta, because $q$ can either be transferred to one of the nucleons, or shared by both. The former preferably occurs when the reaction is driven by SRC. If the reaction proceeds by meson exchange (MEC) or by excitation of the $\Delta$-isobar, $q$ is shared by both nucleons. For example, if the virtual photon is absorbed by the proton emitted in the forward direction, the momenta of both protons in the final state are $p'_1 = p_1 + q$ and $p'_2 = p_2$, respectively.

By exchanging the indices 1 and 2 one obtains the momenta in the final state for a reaction, in which $q$ is transferred to the proton that is emitted in the backward direction. Fig. 1 shows that, for given values of $\omega$ and $q$, the relative momenta of the two protons in the initial state are much larger in the second case than in the first one. Reversibly, the initial momenta $p_1$ and $p_2$ can be deduced from the measured ones $p'_1$ and $p'_2$ only if assumptions about the momentum transfer are made. Note that the relative momentum in the initial state probed in the $(e,e'NN)$ reaction increases at increasing value of $\omega$ (cf. Fig. 2). Moreover, the range for $|p|$ covered in an $(e,e'pp)$ experiment depends on the ranges in $p'_1$ and $p'_2$ spanned by the proton detectors. Assuming $P = 0$, and taking constant values of $\omega$ and $|q|$, the relative momenta in the initial state probed in the reaction increase at increasing value of $\gamma_1$, i.e. the angle of $p'_1$ with respect to $|q|$.

Theoretically, the dependence of the cross section on the momenta of the nucleons in the initial state is expressed in the transition matrix elements for the nuclear charge-current density operators. The one-body part of this operator describes the coupling of the virtual photon to either of the two nucleons. In the Pavia-model [5], the two-proton transition ampli-

Fig. 2. The solid curve shows the defect function for a $1^1S_0$ proton pair in $^{16}$O according to Ref. [9], whereas the dot-dashed curve is obtained from a Fourier transform of the correlation function of Ref. [14]. The arrows indicate the ranges in relative momenta, probed at the three values of the transferred energy. Details are given in the text.
tude, linking the initial state to a specific final state, is obtained through the calculation of the spectral density function in a harmonic oscillator basis, and within a large configuration space [9]. This wave function is converted to wave functions for the center-of-mass motion and the relative motion of the pair. SRC are implemented by adding the defect functions \( D_{\text{S}}(r) \) [13] to the wave functions of the relative motion, according to:

\[
\Psi_{\text{rel}}(r) = \Phi_{\text{rel}}(r) + D_{\text{S}}(r)
\]  

(4)

where \( r \) is the inter-nucleon distance, and \( \Phi_{\text{rel}}(r) \) are harmonic oscillator wave functions.

These defect functions depend only on the quantum state of the relative motion. SRC are strongest for a pair in a \( ^1S_0 \) state. In Fig. 2, the defect function for a proton pair in a \( ^1S_0 \) state calculated with the Bonn-A potential is shown for a relative momentum \( p \) in the range 200–600 MeV/c. This defect function and the one for the \( ^3P_1 \) state are used for the calculations presented here.

In the Gent-model [10] single-particle wave functions obtained from Hartree–Fock calculations are used. Therefore, no formal separation into relative and center-of-mass coordinates can be made. In this model the many-body wave function, approximated to first order in the correlation function, is expressed as:

\[
\Psi(r_1, \ldots, r_A) = \prod_{i<j} (1 - g(r_{ij})) \Phi(r_1, \ldots, r_A) / \sqrt{N}.
\]  

(5)

SRC are contained in the factor \((1 - g(r_{ij}))\). In the absence of correlations \( g(r_{ij}) = 0 \). \( N \) is a normalisation factor. The results presented in this paper are obtained using the correlation function \( g(r_{ij}) \) of Ref. [14]. Fig. 2 shows the defect function obtained by performing a Fourier transform of this correlation function.

In both calculations, the excitation of the \( \Delta \)-resonance and its subsequent decay by exchange of a pion, which is the dominant process contributing to the two-body hadronic currents, is derived from an effective Lagrangian. Medium effects of the \( \Delta \)-isobar are, however, included in a different way. This also holds for the distortion effects in the final state. In the Pavia-model, the interaction of the outgoing particles with the residual nucleus is accounted for by means of an optical potential for each of the outgoing nucleons, whereas in the Gent-model an \( A \)-body wave function for the two ejectiles and the residual nucleus in a two-hole state is obtained by a partial wave expansion in terms of the \( 2p-2h \) eigenstates of a mean-field Hamiltonian.

As mentioned before, information on the mechanism of the \((e,e' pp)\) reaction can be obtained by measuring the differential cross section at various values of the transferred energy and momentum. In particular, the contribution to the cross section of the excitation of the \( \Delta \)-isobar and its subsequent decay via a \( \Delta N \rightarrow NN \) reaction strongly depends on the invariant energy of the virtual photon and the two protons in the initial state \( W_{\gamma p_1 p_2} \). In a quasi-free knockout reaction this quantity is equal to the invariant energy of the two ejectiles \( W_{\gamma p_1 p_2} \). In Fig. 3 the distribution of \( W_{\gamma p_1 p_2} \) calculated within the detection volume for the transition to the ground state, is depicted for the three values of \( v \). This figure illustrates that the differences in \( W_{\gamma p_1 p_2} \) are sufficient to expect a systematic increase of the strength of the two-body currents in the three measurements. Furthermore, the invariant energy corresponding to the largest value of \( v \) is still well below the mass of the \( \Delta N \) system, which is 2170 MeV/c².

3. The reaction \( ^{16}O(e,e' pp)^{14}C_{gs} \)

The experiments were performed with electrons extracted from the Amsterdam Pulse Stretcher (AmPS). The energy of the incident electrons was in the range 580–585 MeV, and the average beam current in the range 2–3 µA, with a macroscopic
duty factor of approximately 70%. A waterfall target, with a thickness of 210 mg/cm² was used [15]. The scattered electrons were detected in a magnetic spectrometer of the QDQ type, and the knocked-out protons in two plastic scintillator arrays, HADRON3 and HADRON4 [16]. In the forward hemisphere, protons were detected at angles in the range $3^\circ \leq \gamma_1 \leq 42^\circ$, and in the backward direction at angles in the range $-114^\circ \geq \gamma_2 \geq -173^\circ$. The detection volume, determined by these angular ranges and the energy acceptances of the HADRON detectors, includes for the transition to the ground state $p_n$ values as low as $50$ MeV/c at $\omega = 180$ MeV, and $25$ MeV/c at $\omega = 240$ MeV. Details of the experiments and the data analysis can be found in Ref. [17]. In Fig. 4 the low energy part of the excitation-energy spectrum ($E_x \leq 10$ MeV), measured for $\omega = 180$ MeV, is shown. This spectrum includes the transitions to the ground state and the two lowest $2^+$ states of $^{14}$C. The strength in the range $-4 \leq E_x \leq 4$ MeV can be almost exclusively attributed to the ground-state transition. The average background in this region, determined from the yield in the range $-100 \leq E_x \leq -10$ MeV is $(4 \pm 6 \%)$ of the peak intensity. Due to the achieved energy resolution of 4.5 MeV (FWHM), the contribution of the first $2^+$ to the ground-state is $\leq 5\%$. This contribution has been deduced from a fit of three Gaussian functions, corresponding to the transitions to the ground state and the two $2^+$ states at excitation energies of 7.0 and 8.3 MeV, to the data.

Fig. 5 shows the cross sections as a function of the missing momentum for the interval $-4 \leq E_x \leq 4$ MeV at the three values of $\omega$. In Section 2, it is argued that the angular momenta of the two protons can couple to the angular momentum for the relative and the center-of-mass motion in various ways. The scheme for coupling a $(p_{1/2})^2$ or $(p_{3/2})^2$ proton-pair to total angular momentum $J = 0$, leaving the $^{14}$C nucleus in the ground state, is as follows. The two protons can be either in a $^1S_0$ or in a $^3P_1$ state. With a $^1S_0$ state always $L = 0$ for the angular momentum of the center-of-mass motion is associated, and with a $^3P_1$ state always $L = 1$. From Fig. 5 it is clear that for all three values of $\omega$ the missing-momentum dependence of the measured cross sections is similar. In Ref. [1,4] it has been pointed out that such a momentum distribution reflects an angular momentum $L = 0$ for the center-of-mass motion of the pair, and thus suggests a dominant role for the knockout of a $^1S_0$ pair driven by SRC. Two-body hadronic...
currents (intermediate $\Delta$-excitation) contribute mainly to the $^3P_1$ wave [4,5].

A prominent role of two-step processes in the $(e,e'pp)$ reaction, including an $(e,e'pn)$ reaction followed by a charge exchange $(pn,pp)$ reaction, can be excluded. A calculation performed within the Lane model [18], implying charge exchange between isobaric analog states shows that the contribution due to the absorption of a virtual photon by a $pn$ pair in $S = 0, T = 1$ state, followed by a charge exchange $(n,p)$ reaction is small. The other contribution stems from charge exchange after absorption of the virtual photon by a $S = 1, T = 0$ $pn$ pair. The strength for the first step in this process is predicted to be about a factor 5–10 larger than that for knockout of a $S = 0, T = 1$ pair. However, in this case the second step is a spin-flip transition. The cross section for such transition is known to be much smaller than that for a transition between analog states. Hence, also this process is not expected to contribute strongly to the $(e,e'pp)$ reaction. These theoretical predictions are confirmed by the ratios for the $(\gamma,pp)$ and $(\gamma,pn)$ cross sections at small missing energy, measured for absorption of real photons by the nucleus $^{12}$C [19]. These ratios vary strongly with the proton emission angle. A strong feeding of the $^{12}$C$(\gamma,pp)$ reaction to the $^{12}$C$(\gamma,pp)$ reaction, for which the measured cross section is about a factor six smaller than that for the $(e,e'pn)$ reaction, would result in comparable angular distributions for both reactions. This has led to the conclusion that both reactions likely proceed via different mechanisms.

This qualitative interpretation of the data is supported by the results of the microscopic calculations performed with the models developed by the Pavia group [5] and Gent group [10], which are also shown in Fig. 5. In these calculations, the transition matrix elements contain contributions from one-body and two-body hadronic currents. The two-proton overlap amplitudes for the transition $^{16}$O $\rightarrow$ $^{14}$C_{gs}, used in the Pavia-model, are taken from DRPA (Dressed-Random-Phase-Approximation) calculations, performed within a large configuration space [9]. In the calculations performed with the Gent-model the "two-nucleon coefficients of fractional parentage" (cf.p.) of Cohen and Kurath [20] are used. These coefficients are renormalised by a factor 0.85² to account for the depletion of the calculated spectroscopic strength in the $1p_{1/2}$ shell due to long-range correlations, according to Ref. [9]. This means that the spectral amplitude for knockout of two protons from the $1p_{1/2}$ shell is equal in both calculations, but that the amplitude for knockout of a $1p_{3/2}$ pair is a factor 1.8 larger in the calculations with the Gent-model than in those performed with the Pavia-model. However, this difference has only a small effect on the cross section for knockout of a $1S_0$ pair with angular momentum $L = 0$ for the centre-of-mass motion, because this cross section is largely determined by knockout of a proton pair from the $1p_{1/2}$ shell.

The theoretical cross sections are represented in Fig. 5 by the solid curves; the contributions of the one- and two-body currents are given by the dashed and dotted curves, respectively. The calculated cross sections agree well with the data at all three values of $\omega$. The curves, representing the contributions of the one- and two-body currents to the ground-state transition, indicate that at $\omega = 180$ and 210 MeV the reaction is dominated by one-body currents, and that the contribution of two-body currents increases with increasing energy transfer. Conceptually both models are quite similar, and neither of the two contains free parameters. As discussed in Section 2, SRC are accounted for in a different way, e.g. by defect functions in the Pavia-model and a correlation function in the Gent-model. The Bonn-A and Reid Soft Core potentials, adopted in the calculations, are both realistic NN-potentials and successfully used in many nuclear-structure calculations. Hence, the data agree with the theoretical results obtained independently with the two models. Furthermore, both models predict that the largest contribution to the cross section stems from one-body hadronic currents driven by SRC. This justifies the conclusion that evidence is obtained for SRC.

Note that the experimental as well as the theoretical cross sections are rather constant as a function of $\omega$. Apparently the increase of the two-body currents, due to intermediate $\Delta$-excitation, is compensated by a decrease of the one-body currents. As pointed out in Section 2, the contribution of the two-body currents to the cross section, originating from the excitation and subsequent decay of the $\Delta$-isobar, increases with the invariant energy $W_{pp'}$. The observed differences between the theoretical predictions are likely...
due to the assumptions with respect to the $\Delta$-propagators. The Gent-model accounts explicitly for medium effects in the $\Delta$-propagator. Therefore, the peak of the $\Delta$-resonance appears at a slightly lower energy, and its width is somewhat larger in this model than in the Pavia-model. This explains the larger contribution of the two-body currents in the calculations performed with the Gent-model at the lowest value of $\omega$.

Though the relative momenta involved in the knock-out process cannot be determined unambiguously from the data, the defect functions shown in Fig. 2 determine the ranges in relative momentum probed in the reaction and the dependence of the cross sections on the transferred energy. For example, in the calculations performed with the Pavia-model the strength generated by the one-body currents decreases with increasing $\omega$. This is due to decrease of the the defect function with an increasing relative momentum of the protons in the initial state in the domain $200 \leq p \leq 400$ MeV/$c$. The increase of the relative momentum probed in the $(e,e'pp)$ reaction at increasing $\omega$ is a kinematic effect, discussed in Section 2. Furthermore, the defect function for the Bonn-A potential, displayed in Fig. 2 is, within the phase space of this experiment, largest for $\omega = 180$ and 210 MeV when $q$ is transferred to $p_1$ (cf. Fig. 1). In this case, the ranges in relative momenta spanned are $220$–$270$ MeV/$c$ and $270$–$330$ MeV/$c$, respectively. When $q$ is transferred to $p_2$, these relative momenta are $480$–$525$ MeV/$c$ ($\omega = 180$ MeV) and $515$–$540$ MeV/$c$ ($\omega = 210$ MeV). Hence, for these values of the transferred energy, the transition amplitude is largest when the virtual photon is absorbed by the proton that is emitted in the forward direction. At $\omega = 240$ MeV, the contributions of both possible absorption processes to the one-body currents, including relative momenta in the ranges $300$–$360$ MeV/$c$ and $540$–$590$ MeV/$c$, respectively, are comparable in size. The situation is different for the calculations performed with the Gent-model. The correlation function adopted in this model is of shorter range than the defect function used in the Pavia-model. Correspondingly, the absolute value of the defect function is largest in the domain of relative momenta $400$–$600$ MeV/$c$ (cf. Fig. 2). This implies that the contribution to the cross section stemming from the transfer of the virtual photon to the proton that is emitted in the backward direction is the largest one at all three values of $\omega$. Hence, the momentum dependence of the employed defect/correlation function indicates that the contribution of SRC to the cross section in the Gent-model is dominated by relative momenta in the range $480$–$600$ MeV/$c$, whereas the Pavia-model predicts that the cross section is largely determined by relative momenta in the range $220$–$370$ MeV/$c$. Though the presented data nicely illustrate the potential of the $(e,e'pp)$ reaction to probe specific ranges of relative momenta, no formal discrimination between the results of both calculations can be made yet, and neither the defect function obtained with the Bonn-A potential nor the correlation function calculated with the Reid Soft Core potential can be excluded. Measurements at complementary values of $(\omega,q)$, thus probing other domains of relative momenta, could resolve the observed ambiguity. Furthermore, a more consistent description of the dynamics of nucleons in the nuclear medium at small inter-nucleon distances could reduce the uncertainties in the theoretical results.

4. Conclusions and outlook

The reaction $^{16}$O$(e,e'pp)^{14}$C$_{gs}$ has been studied at three values of the transferred energy. The dependences of the cross sections on the missing momenta reflect the characteristic features for knockout of two protons with an angular momentum of the center-of-mass motion $L = 0$. The cross sections are well reproduced by the results of two independent calculations. Both calculations predict a relatively small contribution from intermediate $\Delta$-resonance excitation to the cross section; the major part is due to SRC. In that respect we are confident to claim direct evidence for SRC in the nucleus $^{16}$O.

Within the phase space covered by the experiment, relative momenta between the two protons in the initial state in the range $220$–$600$ MeV/$c$ are involved. The data do not allow to determine which relative momenta predominantly contribute to the cross sections measured at the three values of $\omega$. They can be either in the range $220$–$370$ MeV/$c$ or in the range $480$–$600$ MeV/$c$, depending on the coupling of the virtual photon to either of the two
jectiles. Also the calculations cannot discriminate between those two possibilities. Due to the differences in both models with respect to the short-range part of the relative wave function of the proton-pairs in the initial state, both models give different predictions with respect to relative momenta that predominantly determine the cross sections. An improved description of the nuclear structure and a more detailed study of the final state interaction may provide insight in this intriguing ambiguity. In addition, a study of the $q$ dependence of the cross section may shed light on this issue, because the relative momenta in the initial state, probed experimentally, depend on $q$ and $\omega$ in a different way.

Acknowledgements

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References

Extra dimensions in the early universe

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Abstract

We investigate the possible occurrence of extra spatial dimensions \((D = 3 + \epsilon)\) in the early universe. A detailed calculation is presented which shows that the crucial signal is the apparent inequality of the cosmological \(Z\)-term between matching Lyman alpha \((\text{Ly}_\alpha)\) and Lyman beta \((\text{Ly}_\beta)\) spectral lines, both emission and absorption, when using the present epoch (laboratory) wavelengths. We present preliminary upper limits to the value of epsilon, to be improved by direct, more careful analysis of the spectra. We take catalogued quasar Ly forest data and perform Student’s t-test to determine whether we should reject the null hypothesis (no fractal dimensions). Finally, a \(\chi^2\) analysis is done for fitting \(\epsilon\) in the early universe. The statistical tests and experimental data are all consistent with \(\epsilon = 0\) for \(Z \leq 3.3\), but the experimental data support non-zero \(\epsilon\) values for \(Z \geq 4\). However, it should be emphasized that the non-zero values of epsilon found for \(Z \geq 4\) may be due to undiscovered systematic errors in the original data. © 2000 Published by Elsevier Science B.V. All rights reserved.

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Many modern physical theories predict [1] that space has more than 3 spatial dimensions, some of which would reveal themselves only at small distances. This means that because of an expanding universe, the effective dimension of space may be a time-dependent parameter. In fact, the dimension of space is an experimental quantity [2], whose present epoch value and past distant value may be different. Since we are interested in the possibility that the early universe had fractal \((D = 3 + \epsilon\) with \(\epsilon \ll 1)\) spatial dimensions, the natural place for investigation is in the spectra of distant \((Z = \frac{\Delta a}{a} > 2.5)\) quasars. In this regard, the quasar’s Lyman spectra and the existence of spectroscopic \(\text{Ly}_\alpha\) forests [3] provide an ideal opportunity to probe the fractal nature of early universe space on an atomic length scale, provided we can identify a suitable experimental signal.

Past researchers have investigated the possibility of a time-varying fine structure constant \((\alpha)\), using the hyperfine multiplets of absorption lines in quasar ionized iron and magnesium spectra, and obtaining an upper bound of \(\delta \alpha/\alpha < 1.1 \times 10^{-3}\) [4]. In this paper we use ancient quasar light to probe the dimension of space in the early universe, specifically 0370-2693/00/$ - see front matter © 2000 Published by Elsevier Science B.V. All rights reserved. PII: 0370-2693(00)00004-6
matching (same Z) Ly\textsubscript{\alpha} and Ly\textsubscript{\beta} hydrogen lines. As far as we know, both the main idea of this paper and the specific matching lines technique have not been considered before. Therefore we discuss in detail the errors involved, which are both experimental and theoretical.

To lowest order in quantum corrections, the atomic electromagnetic potential is the solution of the Poisson equation, whose form depends on the spatial dimension. In [5] the D(spatial)-dimensional Schrödinger equation is derived. From a mathematical point-of-view [6], expectation values in quantum mechanics can be analytically continued into continuous dimensions and it becomes meaningful to construct a Taylor’s expansion for \( D = 3 + \epsilon \) with \( \epsilon \ll 1 \)

\[
\langle \langle H \rangle \rangle_{D=3+\epsilon} = \langle \langle H \rangle \rangle_{D=3} + \frac{d\langle \langle H \rangle \rangle}{dD} \bigg|_{D=3} \epsilon + \cdots .
\]  

(1)

Ref. [5] has proven the generalized Hellmann-Feynman theorem

\[
d\langle \langle H \rangle \rangle \bigg|_{D=3} = \langle \langle \frac{\partial H}{\partial D} \rangle \rangle_{D=3},
\]  

(2)

where \( H \) is the \( D \)-dimensional Hamiltonian.

An interesting aspect of fractal dimensions is that the electric charge \( e \) becomes a dimensional constant. The scaling is \( e \sim r_\text{D}^{-(D-3)/2} \). Thus a length parameter \( r_\text{v} \) enters into the problem. In discussing the present-epoch Lamb shift, Ref. [5] has shown that if this length parameter is not very much smaller than the Planck length \((10^{-33} \text{ cm})\), then it makes a negligible contribution to atomic energy levels.

Using Eq. (2) we have calculated the shift in energy levels due to the first order \( \epsilon \) contributions to atomic hydrogen, and these are given in Table 1. The atomic energy levels are

\[
E(nl) = E(nl)\bigg|_{\text{lab}} + \Delta E(nl) \times \epsilon.
\]  

(3)

From these energy levels we obtain the Ly\textsubscript{\alpha} (\( \alpha \)) and Ly\textsubscript{\beta} (\( \beta \)) rest frame transition wavelengths

\[
\lambda_\alpha = \lambda_\text{lab}^{\text{obs}} + a \epsilon, \quad \lambda_\beta = \lambda_\text{lab}^{\text{obs}} + b \epsilon,
\]  

(4)

where \( \lambda_\text{lab}^{\text{obs}} = 1215.67 \text{ Å} \) and \( \lambda_\text{lab}^{\text{obs}} = 1025.72 \text{ Å} \) are the laboratory (present epoch) wavelengths, and \( a = 1418.27 \text{ Å} \) and \( b = 1111.18 \text{ Å} \) are the corresponding shifts due to the extra dimensions. Eq. (4) demonstrates that a fractal dimension of space affects the two primary cosmological transitions differently. From the physics viewpoint, the kinetic energy and centripetal terms in the \( D \)-dimensional Schrödinger equation are most sensitive to the presence of fractal dimensions and make the atomic energy levels ideal indicators of additional space-time dimensions.

The effect of a non-zero fractal dimension is as follows. When measurements are taken of matching (same hydrogen cloud) Ly\textsubscript{\alpha} and Ly\textsubscript{\beta} lines (both absorption and emission), it will not be possible to obtain the same \( Z \) shift using present epoch (i.e. laboratory) rest wavelengths. Conversely, since matching Ly\textsubscript{\alpha} and Ly\textsubscript{\beta} lines must have the same cosmological \( Z \), the measurement of the two redshifted lines allows us to obtain the original early universe rest frame frequencies, and determine statistically whether early universe fractal dimensions exist. By equating \( Z \) and \( \epsilon \) for the two transitions, we obtain two equations in the two unknowns \( \lambda_\alpha, \lambda_\beta \) which are the original early universe rest frame transitions. From either solution, using Eq. (4), \( \epsilon \) can be deduced:

\[
\epsilon = \frac{\lambda_\text{obs}^{\alpha}}{a} \left[ \frac{\lambda_\text{lab}^{\text{obs}}}{a} - \frac{\lambda_\text{lab}^{\text{obs}}}{b} \right] - \frac{\lambda_\text{obs}^{\beta}}{a},
\]

(5)

where \( \lambda_\text{obs}^{\alpha}, \lambda_\text{obs}^{\beta} \) are the respective Ly\textsubscript{\alpha}, Ly\textsubscript{\beta} measured red-shifted transition wavelengths, and \( \lambda_\text{lab}^{\alpha}, \lambda_\text{lab}^{\beta} \) are the laboratory measured wavelengths.

In order to obtain the unknown quasar rest frame transitions from the measured cosmological red shifted lines, one must have strict equality of the \( Z \) factors. By choosing matching Ly\textsubscript{\alpha} and Ly\textsubscript{\beta} lines,

| state vector \( |n\ell(j)\rangle \) | \( \Delta E(nl) \) |
|-----------------|-------|
| \( |1s1(1/2)\rangle \) | 1/2   |
| \( |2s1(1/2)\rangle \) | 1/16  |
| \( |2p1(1/2)\rangle \) and \( |2p3(3/2)\rangle \) | 1/16  |
| \( |3s1(1/2)\rangle \) and \( |3p3(3/2)\rangle \) | 1/54  |
| Lyman alpha    | -7/16 |
| Lyman beta     | -13/27 |

(For example, the 2-dimensional Coulomb solution is \( V = \ln(\rho/\rho_0) \).

Table 1
Table 2

QSOs with matching Lyα and Lyβ. Here $\lambda_{\alpha}^{\text{obs}}$ and $\lambda_{\beta}^{\text{obs}}$ are the measured redshifted transition wavelengths, FWHM is the spectral resolution of the data, and $\sigma_{\alpha}$ and $\sigma_{\beta}$ are the standard deviations.

<table>
<thead>
<tr>
<th>QSO</th>
<th>$\lambda_{\alpha}^{\text{obs}}$ (Å)</th>
<th>$\lambda_{\beta}^{\text{obs}}$ (Å)</th>
<th>FWHM (Å)</th>
<th>$\sigma_{\alpha}$</th>
<th>$\sigma_{\beta}$</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1206+119</td>
<td>4890.79 4126.63</td>
<td>50 km/s 0.3</td>
<td>0.2 [7]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q1315+472</td>
<td>4342.76 3664.28</td>
<td>50 km/s 0.3</td>
<td>0.2 [7]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q1315+472</td>
<td>4322.87 3647.27</td>
<td>50 km/s 0.3</td>
<td>0.2 [7]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q1315+472</td>
<td>4220.69 3561.08</td>
<td>50 km/s 0.3</td>
<td>0.2 [7]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q1623+268</td>
<td>4289.70 3619.42</td>
<td>50 km/s 0.3</td>
<td>0.2 [7]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BR 0401-1711</td>
<td>6383.5 5396.3</td>
<td>5 Å 1.8</td>
<td>1.8 [8]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BRI 1108-0747</td>
<td>6000.5 5050.7</td>
<td>5 Å 1.8</td>
<td>1.8 [8]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BRI 1114-0822</td>
<td>6755.5 5659.7</td>
<td>5 Å 1.8</td>
<td>1.8 [8]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BR 1117-1329</td>
<td>6130.6 5111.2</td>
<td>5 Å 1.8</td>
<td>1.8 [8]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q0010+008</td>
<td>4963.7 4190.2</td>
<td>3.6 Å 1.3</td>
<td>1.3 [9]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q0014+813</td>
<td>5253.29 4432.44</td>
<td>0.06 Å 0.02</td>
<td>0.04 [10]</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This is assured. Other spectra, such as metal ions, have narrower lines than the hydrogen Lyman, but finding two matching transitions in the same element is difficult. Using two different elements will introduce uncertainty as to the equality of the two Z. Finally, computing the expansion contraction coefficients for elements other than hydrogen involves a considerable effort, with greater error in the theoretical $a$ and $b$ values compared to hydrogen.

Using standard error propagation, the error associated with $\epsilon$, $\delta \epsilon$, can be calculated. Thus

$$
\delta \epsilon^2 = \left( \frac{1}{a^2} \lambda_{\alpha}^{\text{obs}} \frac{\delta \lambda_{\alpha}^{\text{lab}}}{a \lambda_{\beta}^{\text{obs}} / a - \lambda_{\beta}^{\text{obs}} / b} \right)^2 + \left( \frac{\lambda_{\alpha}^{\text{obs}}}{a \lambda_{\beta}^{\text{obs}} / a - \lambda_{\beta}^{\text{obs}} / b} \right)^2 \delta \lambda_{\beta}^{\text{lab}}
$$

Here $\delta \lambda_{\alpha}^{\text{lab}}$ and $\delta \lambda_{\beta}^{\text{lab}}$ are the uncertainties in the lab (present epoch) wavelengths (taken to be 0.01 Å, since we average over the hyperfine Lyman doublet). $\sigma_{\alpha}^{\text{obs}}$ and $\sigma_{\beta}^{\text{obs}}$ are the standard deviations of the measured red-shifted lines, and $\delta a, \delta b$ are the theoretical uncertainties in the expansion contraction values (taken as $\delta a / a = \epsilon = \delta b / b$). This equation includes all the errors associated with the quantities determining the value of $\epsilon$, for each matching Lyα and Lyβ data pair. For example, the absorption widths of Lyβ are larger than Lyα due to contamination by lower redshift Lyα lines. Thus the centroid of the line has an ambiguity, resulting in a possible non-zero $\epsilon$ for that pair, having nothing to do with $D \neq 3$.

Table 3

$\epsilon$ with its uncertainty $\delta \epsilon$ for each mean QSO Z.

<table>
<thead>
<tr>
<th>QSO</th>
<th>$\langle Z \rangle$</th>
<th>$\epsilon$</th>
<th>$\delta \epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1206+119</td>
<td>3.0222</td>
<td>-0.000059</td>
<td>0.00095</td>
</tr>
<tr>
<td>Q1315+472</td>
<td>2.5723</td>
<td>-0.00023</td>
<td>0.0011</td>
</tr>
<tr>
<td>Q1315+472</td>
<td>2.556</td>
<td>0.00052</td>
<td>0.0011</td>
</tr>
<tr>
<td>Q1315+472</td>
<td>2.48</td>
<td>0.00045</td>
<td>0.0011</td>
</tr>
<tr>
<td>Q1623+268</td>
<td>2.5287</td>
<td>0.000064</td>
<td>0.0011</td>
</tr>
<tr>
<td>BR 0401-1711</td>
<td>4.236</td>
<td>-0.022</td>
<td>0.0058</td>
</tr>
<tr>
<td>BRI 1108-0747</td>
<td>3.9222</td>
<td>0.030</td>
<td>0.0074</td>
</tr>
<tr>
<td>BRI 1114-0822</td>
<td>4.495</td>
<td>0.094</td>
<td>0.016</td>
</tr>
<tr>
<td>BR 1117-1329</td>
<td>3.958</td>
<td>0.17</td>
<td>0.030</td>
</tr>
<tr>
<td>Q0010+008</td>
<td>3.076</td>
<td>-0.0059</td>
<td>0.0049</td>
</tr>
<tr>
<td>Q0014+813</td>
<td>3.32</td>
<td>0.00008</td>
<td>0.00019</td>
</tr>
</tbody>
</table>
This spectral contamination leads to an increase in the line width, so even though $\epsilon$ picks up a non-zero contribution, so too does $\delta\epsilon$, and it is the relative magnitude of these two quantities which determines the significance of the data pair.

The SIMBAD database was searched for papers having either emission or absorption spectra containing matching Ly$_\alpha$ and Ly$_\beta$ lines. Those spectra which have multiple very closely grouped Ly$_\alpha$ and Ly$_\beta$ lines, indicating the presence of closely spaced hydrogen cloud groupings, were not used due to the possible ambiguity of assigning which Ly$_\alpha$ line goes with which Ly$_\beta$ line. This non-ambiguity filter and the fact that of the many references identified in the database, very few contained tabular identification of both Ly$_\alpha$ and Ly$_\beta$ lines, restricted the number of matching pairs to 11, given in Table 2. In all cases, the original authors (astronomers) identified the Lyman lines.

We then transferred each authors’ data into Mathematica for analysis, and $\epsilon$ was computed for each matched pair using (5). We used weighted averages based on the quantum mechanical intensities for the laboratory wavelengths $\lambda_{\alpha}^{lab}$, $\lambda_{\beta}^{lab}$ from the (1/2-1/2) and (1/2-3/2) transitions. Next, we computed the uncertainty for each $\epsilon$, $\delta\epsilon$, based on the listed spectral resolution from each reference and the order uncertainty due to higher order quantum corrections. Table 3 lists these results.

These data have a mean for $\epsilon$ of 0.02436, with a standard deviation of 0.057. Fig. 1 shows $\epsilon$ for each of the emission and absorption line pairs, against the mean QSO $Z$ value for the pair. We next employed Student’s t-test (10 degrees of freedom) to check the null hypothesis $\epsilon = 0$. Computing the value for $t$ gave 1.4056. Statistically, this means that to the level of significance of 0.05, we cannot reject $\epsilon = 0$, but to the level of significance of 0.10, it can be rejected [11]. Finally a $\chi^2$ fitting gave $\epsilon = 0$ with a goodness-of-fit probability of 0.2867.

Concerning the data, [10] ($Z = 3.32$) taken with the Keck 10 m instrument, is the highest quality. The uncertainty in $\epsilon$, $\delta\epsilon$, due to the error measurements in the red-shifted lines is considerably smaller than epsilon itself for the data of Ref. [8]. This can be an indicator that $\epsilon$ really is non-zero for high $Z \geq 4$ objects, or that the $Z \geq 4$ data are contaminated with some undetermined systematic error.

In conclusion, the statistical tests and experimental data are all consistent with $\epsilon = 0$ for $Z \leq 3.3$, but the experimental data support non zero $\epsilon$ values for $Z \geq 4$. High spectral resolution data for $Z \geq 4$ would allow $\epsilon$ in the early universe to be better determined.

Acknowledgements

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References


Duality of coordinates and matter fields in curved spacetime

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Abstract

We show that there exists a duality between the local coordinates and the solutions of the Klein-Gordon equation in curved spacetime in the same sense as in the Minkowski spacetime. However, the duality in curved spacetime does not have the same generality as in flat spacetime and it holds only if the system satisfies certain constraints. We derive these constraints and the basic equations of duality and discuss the implications in the quantum theory.

According to [1], there exists a duality between the Cartesian coordinates of flat spacetime and the solutions of the equation of motion of a certain physical systems. In the nonrelativistic case, this duality allows to express the coordinates as functionals on the wave functions of an one particle quantum system and some other functionals called prepotentials introduced in [2] (see also [3,4]). In Minkowski spacetime all the Cartesian coordinates are functionals on the solutions of the classical Klein-Gordon equation and prepotentials.

The results from [1] suggest that the spacetime measurements can be represented in terms of either coordinates or scalar fields, maybe even quantum fields. This interpretation might be particularly useful at the Planck scale where the classical rulers and clocks can be at most idealized while the fields are physical. Also, the Heisenberg inequalities introduce a strong indeterminacy in the spacetime measurements performed by the inertial observers [5,6]. In particular, this might be a hint that the microscopic structure of spacetime is actually operatorial and noncommutative [7,8]. However, at distances of the order of Planck length, the effects of gravity are rather strong. According to the principles of general relativity, in the presence of gravity the coordinates cease to have the same important role as in the flat spacetime physics [9]. Nevertheless, they are still essential to perform local physical measurements. Therefore, it is important to understand whether the coordinate-field duality described in [1] has a counterpart in curved spacetime.

The purpose of this letter is to investigate the circumstances which allow a duality between the...
local coordinates and the solutions of the Klein-Gordon equation in curved spacetime in both classical and quantum theories (for recent attempts to formulate the gravity in terms of quantum quantities see [10–14].) The sought for duality relations should have the same interpretation as in the flat spacetime and should include it as a particular case.

Let us consider a curved spacetime manifold $M$ of dimension $n$, endowed with a Lorentzian metric tensor field $g$ and a scalar field $\phi$ of mass $m$. In a local coordinate system $(x^0, x^1, \ldots, x^{n-1})$ the equation of motion of $\phi$ is given by

$$\left(\Box + m^2 + \xi R(x)\right)\phi(x) = 0,$$

(1)

where $\Box$ is the d’Alambertian, $\xi$ is a real parameter and $R(x)$ is the scalar curvature of $M$ at $x$. In order to find a duality of the same type as in [1] one has to express the coordinate along a direction $x^\nu$ as a function of some solution of (1) for which $x^\nu$ (with $\nu \neq \mu$) represent independent parameters. These ‘one dimensional’ solutions do not exist in general unless some constraints are imposed on the system. In order to find them we put (1) under the form

$$M^{(\mu\nu)} \partial_\mu \phi(x^\mu) + N^{(\mu)} \partial_\mu \phi(x^\mu) + P^{(\mu)} \phi(x^\mu) + Q^{(\mu)} = 0$$

(2)

for $\mu = 0, 1, \ldots, n-1$. The coefficients $M^{(\mu\nu)}$, $N^{(\mu)}$, $P^{(\mu)}$ and $Q^{(\mu)}$ are specific functions on $g$, $\xi$ and $R$, but it is convenient to treat (2) in its full generality and to imposed the dependence later as a constraint. Since the variable in (2) is $x^\mu$ while $x^\nu$ are treated as parameters for $\nu \neq \mu$ and since we are seeking for independent dualities along each $x^\nu$ we have to make a first technical assumption that $\phi$ and $\partial_\mu \phi$ can be treated as independent functions. As a consequence, the coefficients in (2) can be considered as continuous functions on $x^\mu$ depending on the parameters $x^\nu$. The second nontrivial requirement that should be satisfied by the system is that (2) admits two linearly independent solutions. This condition can be translated into a statement about a second order differential equation obtained by changing the coordinates in (2) to local Cartesian coordinates in $\mathbb{R}^n$ [16]. A general solution of the later can be constructed out of a particular integral of (2) and a complementary function that is a solution of the corresponding homogeneous equation. For the later a fundamental solution always exist [15].

The two conditions above refer to the possibility of decomposing (2) along the directions $x^\mu$ as in the flat case. In general, it is known that such of decomposition is not possible due to the presence of the off-diagonal elements of the metric in the d’Alambertian. Moreover, even the notion of direction is of little use since there is no invariance of the metric along any vector field [9]. These facts question the existence of any solution to the conditions stated above. However, a large class of manifolds should meet these requirements, namely the manifolds that present at least a number of local Killing vectors equal to the dimension of the manifold. In this case it is known that the Klein-Gordon equation admits wave-function solutions along the integral lines of the Killing vectors [9]. Then the fundamental theorem for linear differential equations [15] guarantees that the homogeneous one-variable equation (2) has two linearly independent solutions and that the homogeneous equation has solution over the interval where the coefficients $M^{(\mu\nu)}$, $N^{(\mu)}$, $P^{(\mu)}$ and $Q^{(\mu)}$ are continuous and defined and $M^{(\mu\mu)} \neq 0$. The existence of the second linearly independent solution depends on the specific form of $Q^{(\mu)}$ which at its turn depends on the particular metric. The fact that $\phi$ and $\partial_\mu \phi$ are treated as independent functions implies that any duality relation that can be derived under this assumption is strictly local. Otherwise, a relationship between the two functions can be shown to arise.

Let us assume that the above requirements are satisfied for our system. We denote by $\phi^\mu$ and $\ddot{\phi}^\mu$ two linearly independent solutions of (2). Following [1] we introduce the prepotentials $\mathcal{F}^{(\mu)}[\phi^\mu]$ by

$$\ddot{\phi}^{(\mu)} = \frac{\partial \mathcal{F}^{(\mu)}[\phi^\mu]}{\partial \phi^\mu}.$$  

(3)

The variation of $\mathcal{F}^{(\mu)}$ with respect to $x^\mu$ is given by

$$\partial_\mu \mathcal{F}^{(\mu)} = \frac{1}{2} \partial_\mu \left( \ddot{\phi}^{(\mu)} \ddot{\phi}^{(\mu)} \right) + \frac{1}{2} W^{(\mu)},$$

(4)
where

\[ W^{(\nu)}(s) = \overline{\delta}^{(\nu)\mu}_{\nu} \phi^{(\nu)} - \overline{\partial}_{\mu} \phi^{(\nu)} \phi^{(\nu)} \]  

(5)

is the Wronskian of \( \phi^{(\nu)}(s) \) and \( \overline{\partial}^{(\nu)}(s) \). In general, \( W^{(\nu)}(s) \) is a continuous function on \( x^{\nu} \) and depends on the parameters \( x^{\nu} \) with \( \nu \neq \mu \). However, since we want to express \( x^{\mu} \) as an explicit function on \( \phi^{(\nu)}(s) \) and \( \overline{\partial}^{(\nu)}(s) \) from (4) \( W^{(\nu)}(s) \) should be a nonvanishing constant function with respect to \( x^{\mu} \). This imposes the following supplementary constraint on the system

\[ N^{(\nu)\mu}(x^{\mu}) + M^{(\nu)\mu}(x^{\mu})Q^{(\nu)}(s)\left( \overline{\partial}^{(\nu)}(s) - \phi^{(\nu)}(s) \right) = 0, \]  

(6)

for \( \mu = 0, 1, \ldots, n - 1 \). Now integrating (4) with respect to \( x^{\mu} \) we obtain the following relation

\[ x^{\mu} = \frac{2}{W^{(\nu)}(s)} \left( \overline{\partial}^{(\nu)}(s) - \frac{1}{2} \overline{\partial}^{(\nu)}(s) \phi^{(\nu)}(s) - C^{(\nu)}(s) \right), \]  

(7)

for \( \mu = 0, 1, \ldots, n - 1 \). Here, \( C^{(\nu)}(s) \) is an integration constant with respect to \( x^{\mu} \) and an arbitrary function on the parameters. Eq. (7) expresses the duality between the coordinate \( x^{\mu} \) on one hand, and \( \phi^{(\nu)}(s) \), \( \overline{\partial}^{(\nu)}(s) \) and \( \partial^{(\nu)}(s) \) on the other hand. We note that it has the same form as in flat spacetime, due to the imposed requirements. By setting \( W^{(\nu)}(s) \) and \( C^{(\nu)}(s) \) constants with respect to \( x^{\mu} \) as in the flat spacetime and by using (3) one can see that \( x^{\mu} \) depends explicitly only on \( \phi^{(\nu)}(s) \) and \( \partial^{(\nu)}(s) \) which is unknown.

In order to derive the equation satisfied by \( \partial^{(\nu)}(s) \) we introduce the functional \( x^{\mu} = \frac{\partial^{(\nu)}(s)}{\partial \phi^{(\nu)}(s)} \). Following [4] we express the derivatives with respect to \( x^{\mu} \) in terms of the derivatives with respect to \( \phi^{(\nu)}(s) \). We see that

\[ \frac{\partial}{\partial x^{\mu}} = \left( \partial^{(\nu)}(s) \right)^{-1} \frac{\partial}{\partial \phi^{(\nu)}(s)} \]  

(8)

and

\[ \frac{\partial^2}{\partial x^{\mu} \partial x^{\nu}} = \left( \partial^{(\nu)}(s) \right)^{-1} \left( \frac{\partial^2}{\partial \phi^{(\nu)}(s)} \right) \frac{\partial}{\partial \phi^{(\nu)}(s)} \]  

(9)

where \( \partial = \partial/\partial \phi^{(\nu)}(s) \). Using (7) we calculate the first two derivatives of \( \partial^{(\nu)}(s) \) and find the following relations

\[ \partial \partial^{(\nu)}(s) = \frac{1}{W^{(\nu)}(s)} \left( \partial \partial^{(\nu)}(s) - \partial^2 \partial^{(\nu)}(s) \phi^{(\nu)}(s) \right), \]  

(10)

Next, by using (3), (8), (9) and (10) in (2) we obtain after some simple computations the following equation

\[ M^{(\mu)\nu}(s) \partial \partial^{(\mu)}(s) \partial^{(\nu)}(s) + \frac{1}{W^{(\nu)}(s)} N^{(\nu)\mu}(s) \left( \partial \partial^{(\nu)}(s) - \partial^2 \partial^{(\nu)}(s) \phi^{(\nu)}(s) \right)^2 \partial^2 \partial^{(\nu)}(s) \]  

\[ + 1 \frac{1}{W^{(\nu)}(s)} \left( P^{(\mu)\nu}(s) \partial \partial^{(\mu)}(s) + Q^{(\mu)\nu}(s) \right) \]  

\[ \times \left( \partial \partial^{(\nu)}(s) - \partial^2 \partial^{(\nu)}(s) \phi^{(\nu)}(s) \right)^3 = 0 \]  

(11)

for \( \mu = 0, 1, \ldots, n - 1 \). Eq. (11) is the equation of motion of \( \partial^{(\nu)}(s) \) in the space of solutions of (2). Together with (7) it represents the generalization of the duality relations obtained in [1] to a curved manifold. However, (7) and (11) do not have the same generality as in the Minkowski space since we have assumed that the system satisfies three strong mathematical requirements. Actually, in order to make contact with physics, the general coefficients of (2) are connected to the geometrical objects defined on \( M \) through the following relations

\[ M^{(\mu)\nu}(s) = g^{(\mu)(\nu)} \sqrt{g}, \]  

\[ N^{(\nu)}(s) = g^{(\mu)}(s) \sum V_{\nu} g^{(\nu)\nu} \sqrt{g}, \]  

\[ P^{(\mu)}(s) = \sqrt{g} \left( m^2 + \xi \right), \]  

\[ Q^{(\mu)}(s) = 2 \sum_{\nu \neq \mu} \left( g^{(\mu)(\nu)} \sqrt{g} \right) f_{\nu}, \]  

\[ f_{\nu} = \partial \phi \]  

(12)

for \( \mu = 0, 1, \ldots, n - 1 \). Thus we see that (6) represents a constraint on the geometry as well as on the derivatives of \( \phi \) with respect to the parameters \( x^{\nu} \) with \( \nu \neq \mu \).

Our technical assumptions on the non-homogeneous second order differential equation (2) are quite restrictive. Nevertheless, they include the homogeneous equation \( Q^{(\mu)}(s) = 0 \). The relations describing the duality can be directly obtained from the general
case. Indeed, from (12) we see that the homogeneous case is obtained for the metrics on \( M \) that satisfy the following relation
\[
\sum_{\nu \neq \mu} (g^{\mu \nu}) \partial_{\mu} \partial_{\nu} \phi = 0,
\]
which represent just the homogeneity condition. Also, the condition (6) of having a constant Wronskian becomes in this case
\[
\sum_{\nu = 0}^{n-1} \partial_{\mu} (g^{\mu \nu}) = 0,
\]
which can be obtained from (6) and (12) with the homogeneity condition (13). Eq. (14) fixes the metric on \( M \). When used together with (13) in the Klein-Gordon equation it leads to the following equation for \( \phi \) along \( x^\mu \)
\[
\frac{1}{\mu}(g^{\mu \nu}) \partial_{\nu} \phi + g^{\mu \nu} \partial_{\mu} \partial_{\nu} \phi + (m^2 + \xi R(\phi)) \phi = 0.
\]
Eq. (15) is a homogeneous differential equation of rank two and it admits always two linearly independent solutions if the coefficients are continuous and nonsingular. The duality relation (7) has the same nonsingular. The duality relation (7) has the same
satisfies the following conditions
\[
\phi(\mu) = \sum_{\alpha} (a_\alpha f_\alpha(\phi) + \partial_\alpha f_\alpha(\phi))
\]
where \( \phi(\mu) \) is a local orthonormal and complete set of solutions of (1) and \( a_\alpha \) and \( \partial_\alpha \) are the (covariant) annihilation and creation operators, respectively. Since the Poincaré group is not a symmetry group of the system, the particle interpretation is problematic and the theory suffers from the known inconsistencies [9]. Nevertheless, let us examine the circumstances under which this theory can display a coordinate-field duality of the same kind as the one discussed in [1]. As in the classical case, in general there is no such of duality in the system. However, one can see that it exists if the Klein-Gordon field admits a local decomposition in independent modes along each of the directions \( \mu = 0, 1, \ldots, n-1 \). In particular, this implies that Eq. (2) has an orthonormal and complete set of solutions \( \{f_\alpha(\phi)\} \), where \( \alpha \) represent all the necessary indices to label an independent mode. This linearizing condition strongly constraints the possible types of coefficients to be used in the quantum equation (2) and consequently it constrains the spacetime geometries.

Let us assume that a \( \{f_\alpha(\phi)\} \) exists such that it satisfies the following conditions
\[
f_\alpha(\phi) \cdot f_\beta(\phi) = -\delta_\alpha^\beta, \quad f_\alpha(\phi) \cdot \bar{f}_\beta(\phi) = 0
\]
where \( \cdot \) is the scalar product in the space of solutions defined by an integral over a spacelike hypersurface, then we can construct two linearly independent solutions of (2) of the following form
\[
\phi_\alpha(\phi) = a_\alpha f_\alpha(\phi), \quad \bar{\phi}_\alpha(\phi) = a_\alpha \bar{f}_\alpha(\phi),
\]
which correspond to operators that locally annihilate and create, respectively, the quantum mode \( \alpha \) of the field along the direction \( \mu \). Using the same definitions (3) as in the classical case, one can associate to each mode \( \alpha \) the quantum prepotential \( \mathcal{F}_\alpha(\phi) \) which is an operator depending on \( \phi(\phi) \).

The existence of a basis (19) restricts the possible manifolds that can support a quantum duality of the type presented in [1]. In particular, let us suppose that \( M \) and its geometric structure satisfies all the requirements necessary for a classical duality between coordinates and fields to hold. As we argued previously, depending on \( Q(\phi) \), this can take place if there are local Killing vector fields that define the local directions around \( (x^\mu) \) on \( M \). In this case, the modes \( f_\alpha(\phi) \) can be taken as the eigenvalues of the translation operator along the Killing vector that defines the direction \( x^\mu \). In particular this means
that \( \partial_{\mu} f_{\alpha}^{(\mu)} = k_{\mu} f_{\alpha}^{(\mu)} \) and that Eq. (2) is also satisfied by the elements of the basis (for certain functions \( Q^{(\mu)} \) the solutions of the inhomogeneous differential equation (2) can be decomposed in the modes of the translation operator, the trivial case being of the homogenous equations.) Now if we introduce a solution of the form (20) in (2) and expand the function \( Q^{(\mu)} \) in the basis (19) with the coefficients \( q_{a}^{\mu} \) then we obtain the following equation

\[
g_{a}^{\mu} = -a_{\alpha} \left( k_{\alpha} k_{\mu} M^{(\mu)\alpha} + k_{\mu} N^{(\mu)} + p^{(\mu)} \right). \tag{21}
\]

Thus, a basis that satisfies (19) exists on \( M \) if the local directions are defined in terms of local Killing vectors and if the classical duality coordinate-fields holds. Also, it is necessary that the coefficients of (2) that depend on the geometry of \( M \) satisfy (6) beside (6). The resolution of the basis depends on \( Q^{(\mu)} \). Eq. (21) admits in general solutions, but these cannot be uniquely determined only from it. However, beside this equation, the functions \( M^{(\mu)\alpha}, N^{(\mu)} \) and \( Q^{(\mu)} \) should also satisfy (6) but even in this case the solutions are not completely determined even if their existence can be inferred from a simple analysis of the rank of the system. It is possible that a basis of the form (19) exists in a more general or different case, but this problem is not clear to us at present. The basis could be in principle continuous or discrete (for example in the case of a compact Killing vector, the momentum eigenvalues are quantized.)

It is easy to verify that the rest of conditions that should be imposed on the system actually refer only to the modes \( f_{\alpha}^{(\mu)} \) and they reduce to the ones discussed in the case of classical fields. This is a consequence of the fact that the modes are assumed to be independent. In particular, the coefficients \( M_{\alpha}^{(\mu)\alpha}, N_{\alpha}^{(\mu)} \) and \( Q_{\alpha}^{(\mu)} \) that enter the corresponding equation of the form (2) written for the mode \( \alpha \) along the direction \( \mu \) should satisfy the constraint (6) with \( W_{\alpha}^{(\mu)} \) the Wronskians of \( f_{\alpha}^{(\mu)} \) and \( f_{\alpha}^{(\mu)} \). Proceeding along the same line as in the classical case, it turns out that the duality between coordinates and fields is expressed by the following relation

\[
X_{\alpha}^{(\mu)} = \frac{2}{W_{\alpha}^{(\mu)}} \left( (W_{\alpha}^{(\mu)})^{-1} \right) \left( \Phi_{\alpha}^{(\mu)} - \frac{1}{2} \partial_{\alpha}^{(\mu)} f_{\alpha}^{(\mu)} a_{\alpha} - C_{\alpha}^{(\mu)} a_{\alpha} \right),
\]

\[
X_{\alpha}^{(\mu)} = x^{\mu} a_{\alpha} a_{\alpha} \tag{22}
\]

for \( \mu = 0, 1, \ldots, n - 1 \). Here, \( W_{\alpha}^{(\mu)} \) and \( C_{\alpha}^{(\mu)} \) are constant functions with respect to \( x^{\mu} \) and may depend arbitrarily on the parameters \( x^{\nu} \) with \( \nu \neq \mu \). The operators \( \Phi_{\alpha}^{(\mu)} \) are hermitian only if both of \( W_{\alpha}^{(\mu)} \) and \( C_{\alpha}^{(\mu)} \) are real. There is an orthonormal and complete set of states which are eigenstates of the operators \( \Phi_{\alpha}^{(\mu)}, N_{\alpha} \) and \( X_{\alpha}^{(\mu)} \) since

\[
\begin{align*}
\left[ X_{\alpha}^{(\mu)}, X_{\beta}^{(\nu)} \right] &= \left[ \Phi_{\alpha}^{(\mu)}, \Phi_{\beta}^{(\nu)} \right] = \left[ X_{\alpha}^{(\mu)}, N_{\alpha} \right] \\
&= \left[ \Phi_{\alpha}^{(\mu)}, X_{\beta}^{(\nu)} \right] = \left[ \Phi_{\alpha}^{(\mu)}, N_{\alpha} \right] = 0
\end{align*}
\]

as a consequence of the independence of the modes.

If the operator \( \partial X_{\alpha}^{(\mu)} \) is invertible it is possible to obtain the equation of motion for \( \Phi_{\alpha}^{(\mu)} \) in the space of modes. Like in the classical case we first determine \( \partial X_{\alpha}^{(\mu)} \) and \( \partial^{2} X_{\alpha}^{(\mu)} \) and we obtain the following relations

\[
\partial X_{\alpha}^{(\mu)} = - \frac{2}{W_{\alpha}^{(\mu)}} \left[ \left( 1 + \frac{C_{\alpha}^{(\mu)}}{f_{\alpha}^{(\mu)} f_{\alpha}^{(\mu)}} \right) \partial_{\alpha}^{2} \Phi_{\alpha}^{(\mu)} \right] + \left( 1 - \frac{C_{\alpha}^{(\mu)}}{f_{\alpha}^{(\mu)} f_{\alpha}^{(\mu)}} \right) \partial_{\alpha}^{2} \Phi_{\alpha}^{(\mu)} \tag{24}
\]

and

\[
\partial^{2} X_{\alpha}^{(\mu)} = - \frac{2}{W_{\alpha}^{(\mu)}} \left( 1 + \frac{C_{\alpha}^{(\mu)}}{f_{\alpha}^{(\mu)} f_{\alpha}^{(\mu)}} \right) \partial_{\alpha}^{3} \Phi_{\alpha}^{(\mu)} - \frac{4}{W_{\alpha}^{(\mu)^{2}}} \left( \frac{C_{\alpha}^{(\mu)}}{f_{\alpha}^{(\mu)} f_{\alpha}^{(\mu)}} \right) \partial_{\alpha}^{2} \Phi_{\alpha}^{(\mu)}. \tag{25}
\]

Next introduce (24) and (25) in the operatorial equation corresponding to (2) and express the derivatives \( \partial_{\alpha} \) in terms of \( \partial / \partial X_{\alpha}^{(\mu)} \). For the sake of clarity let us introduce the following notations for the functions that enter the final result

\[
\begin{align*}
\Sigma &= \left( f_{\alpha}^{(\mu)} f_{\alpha}^{(\mu)} \right)^{-1}, \\
\Omega_{1} &= - \frac{2}{W_{\alpha}^{(\mu)}} \left( 1 + \frac{C_{\alpha}^{(\mu)}}{f_{\alpha}^{(\mu)} f_{\alpha}^{(\mu)}} \right), \\
\Omega_{2} &= - \frac{4}{W_{\alpha}^{(\mu)^{2}}} \left( \frac{C_{\alpha}^{(\mu)}}{f_{\alpha}^{(\mu)} f_{\alpha}^{(\mu)}} \right). \tag{26}
\end{align*}
\]

Next we drop the indices \( \mu \) and \( \alpha \) and put a hat on
operators. With these notations the equation satisfied by the quantum prepotentials has the following form

\begin{equation}
M \Sigma \partial \hat{\phi} \hat{\gamma} \left[ \left( \Sigma \partial^2 \hat{\phi} \hat{\gamma} - \Sigma \partial \hat{\phi} \hat{\gamma} \left( \Omega, \partial^2 \hat{\phi} \hat{\gamma} + \Omega, \partial^2 \hat{\phi} \hat{\gamma} + \Omega, \partial^2 \hat{\phi} \hat{\gamma} + \Sigma \partial \hat{\phi} \hat{\gamma} \right) \right) \hat{\gamma} + 1 \right) \partial^2 \hat{\phi} \hat{\gamma} + \Sigma \partial \hat{\phi} \hat{\gamma} \hat{\gamma} + \frac{\partial^2 \hat{\phi} \hat{\gamma}}{\partial \gamma^2} + \tau = 0,
\end{equation}

where \( \hat{\gamma} = (\partial \hat{\gamma})^{-1} \). Eq. (27) is nonlinear and nontrivial, however some simplifications are possible in the homogeneous case and by taking the constant \( C_{\mu} = 0 \). Nevertheless, even in the simplest case it is a very difficult task to find a nontrivial solution of it.

Eqs. (22) and (27) represent the content of the duality between the coordinates and fields in the quantum theory. This duality was obtained by imposing strong constraints on the system. The reason for these constraints is the requirement that the duality has the same interpretation as in the classical theory in the Minkowski spacetime [1] (see also [16]).

In conclusion, we have generalized the classical duality between coordinates and matter fields to curved spacetime. In the classical case, in order to obtain a theory that reduces to the one discussed in [1], the constraints expressed by (6) and (12) should be imposed on the system. Moreover, in order to apply the method used to derive the flat spacetime duality, one should assume that the system satisfies some restrictive mathematical conditions. In the quantum case, the duality exists on those manifolds on which the local decomposition of the scalar fields in independent modes is possible along each coordinate. We obtained in particular the relations derived in the case of Minkowski spacetime in [1]. Our analysis shows that the duality coordinate-fields should also hold locally in the case of other flat manifolds. However, we emphasise that this is not a property of general spacetime manifolds, at least not in the form of [1]. We have shown that, in principle, this type of duality should hold for some nontrivial manifolds but the complexity of the mathematical restrictions that should be imposed to the system prevents us of giving examples at present.

Despite the technical difficulties, the theory presented here might be useful in studying the general relativity at Planck scale. One interesting way to apply the coordinate-field duality to gravity would be to parametrize locally the spacetime manifold in terms of matter fields in a consistent way with the principles of general relativity.

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Kasner asymptotics of mixmaster Hořava–Witten cosmology

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Abstract

Bianchi type I and type IX ('Mixmaster') geometries are investigated within the framework of Hořava–Witten cosmology. We consider the models for which the fifth coordinate is a $S^1/Z_2$ orbifold while the four coordinates are such that the 3-space is homogeneous and has geometry of Bianchi type I or IX while the rest six dimensions have already been compactified on a Calabi–Yau space. In particular, we study Kasner-type solutions of the Bianchi I field equations and discuss Kasner asymptotics of Bianchi IX field equations. We are able to recover the isotropic 3-space solutions found by Lukas et al. [10]. Finally, we discuss if such Bianchi IX configuration can result in chaotic behaviour of these Hořava–Witten cosmologies. © 2000 Published by Elsevier Science B.V. All rights reserved.

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String cosmology has attracted a lot of interest recently (for a review see [1]), especially in the context of duality symmetry, which is a striking feature of the underlying string theory and justifies a kinetic-energy-driven inflation, known as pre-big-bang inflation [2]. Pre-big-bang inflation is the result of the admission of the cosmological solutions for bosonic low-energy-effective-action for strings [3]. Bosonic action is the simplest stringy action and in view of duality symmetry this action together with other superstring actions are not necessarily the right description of physics at strong coupling - M-theory.

Hořava and Witten [4] proposed that the right candidate for M-theory is strongly coupled limit of $E_8 \times E_8$ heterotic superstring theory compactified on a $S^1/Z_2$ orbifold with $E_8$ gauge fields on each orbifold fixed plane. This means the gauge fields live on 10-dimensional planes while gravity can propagate in the whole 11-dimensional bulk. The idea of having extra dimensions in which only gravity can propagate has been under intensive studies recently and different scenarios (including extra time dimensions [9]) have been considered [5–8].

In this paper we will consider the models in which M-theory is compactified on an orbifold and then reduced to four dimensions using Calabi–Yau manifold [10,11]. Since the size of the orbifold is bigger than the size of the Calabi–Yau space then there was a period in the history of the universe...
during which the universe was five-dimensional. That means we can consider the cosmological models for which the fifth coordinate is an orbifold while the remaining four coordinates are such that the three-space is homogeneous with Bianchi type I or IX geometry. The main objective is to study Kasner-type solutions of Bianchi type I field equations and Kasner asymptotic states as a result of Kasner-to-Kasner transitions) of Bianchi type IX field equations. The form of these solutions allows us to find out whether there is a possibility for chaotic behaviour in these Hořava–Witten cosmologies. Similar question was addressed in [12] for pre-big-bang cosmology with the answer that only finite number of chaotic oscillations are possible.

It is well known that the vacuum BIX homogeneous cosmology in general relativity is chaotic [13]. An infinite number of oscillations of the orthogonal scale factors occurs in general on any finite interval of proper time including the singularity at $t = 0$.

If a minimally coupled, massless scalar field (e.g. the inflaton) is admitted, the situation changes. Only a finite number of spacetime oscillations can occur before the evolution is changed into a state in which all directions shrink monotonically to zero as the curvature singularity is reached and the oscillatory behaviour ceases [14]. This is also the case in 4-dimensional pre-big-bang cosmology where the role of a scalar field is played by the dilaton [12]. On the other hand, 5-dimensional vacuum Einstein solutions of Bianchi type IX do not allow chaos to occur either [15–17]. The point is that the fifth dimension plays effectively the role of a scalar field in scalar field cosmologies and stops chaotic oscillations. In 5-dimensional Hořava–Witten cosmology the situation is in some ways analogous to both of the above cases which gives some new interesting points to be made and this is the task of our paper.

The Hořava–Witten field equations are given by [4,10,11]

$$R^\mu_\nu = \nabla^\mu \nabla_\nu \phi + \frac{\alpha_0^2}{6} g^\mu_{\rho} g^\nu_{\sigma} e^{-2\sqrt{2} \phi} + \sqrt{2} \alpha_0 e^{-\sqrt{2} \phi} \sqrt{\frac{g}{g'}} \frac{g'}{g}\left[ g_{\mu\rho} g^\rho_{\sigma} g^\sigma_{\nu} \right] [ \delta(y) - \delta(y - \pi\lambda) ] \right].$$

where

$$\phi = 1 / \sqrt{2} \ln V$$

and $V$ is a scalar field measuring the deformation of the Calabi–Yau space, $g_{\mu\nu}$ is the five dimensional metric tensor while $g^\mu_{\rho}$ is the four dimensional metric which denotes the pull-back of the metric on five-dimensional manifold $M_5$ onto the orbifold fixed four-dimensional manifolds $M_4^{(i)}$ and $M_4^{(j)}$. In (1), (2) we have neglected the terms which come from the three-form on the Calabi–Yau space. Actually, they will not make any qualitative change in our discussion - this is on the same footing as it was the case in pre-big-bang models [12]. In (1), (2) $y \in [-\pi\lambda, \pi\lambda]$ is a coordinate in the orbifold direction and the orbifold fixed planes are at $y = 0, \pi\lambda$. $Z_2$ acts on $S^1$ by $y \rightarrow -y$. The terms involving delta functions arise from the stress energy on the boundary planes.

Following [10] we consider cosmological models of the form

$$ds^2 = -N^2(\tau, y) d\tau^2 + ds_5^2 + d^2(\tau, y) dy^2,$$

where

$$ds_5^2 = a^2(\tau, y)(\sigma^1)^2 + b^2(\tau, y)(\sigma^2)^2 + c^2(\tau, y)(\sigma^3)^2,$$

is a homogeneous Bianchi type IX 3-metric and the orthonormal forms $\sigma^1, \sigma^2, \sigma^3$ are given by

$$\sigma^1 = \cos \psi d\theta + \sin \psi \sin \theta d\varphi,$$

$$\sigma^2 = \sin \psi d\theta - \cos \psi \sin \theta d\varphi,$$

$$\sigma^3 = d\psi + \cos \theta d\varphi,$$

and the angular coordinates $\psi, \theta, \varphi$ span the following ranges,

$$0 \leq \psi \leq 4\pi, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi.$$
Similarly as in [10,11] we will look for separable solutions of the form
\[ N(\tau, y) = n(\tau) \tilde{a}(y), \]
\[ a(\tau, y) = \alpha(\tau) \tilde{a}(y), \]
\[ b(\tau, y) = \beta(\tau) \tilde{a}(y), \]
\[ c(\tau, y) = \gamma(\tau) \tilde{a}(y), \]
\[ d(\tau, y) = \delta(\tau) \tilde{a}(y). \]
\[ V(\tau, y) = \epsilon(\tau) \tilde{a}(y). \tag{9} \]

The nonzero components of the field equations read as (an overdot means a derivative with respect to time \( \tau \) and a prime means a derivative with respect to an orbifold coordinate \( y \))
\[
\frac{\tilde{a}^2}{\tilde{d}^2} \left[ - \frac{\tilde{d}}{\tilde{a}} - \frac{\tilde{d}}{\tilde{a}} \left( \frac{\tilde{d}}{\tilde{a}} - \frac{\tilde{d}}{\tilde{a}} \right) - \frac{\tilde{d}^2}{\tilde{d}^2} \right] - n^2 \left[ \frac{\tilde{d}}{\tilde{a}} \left( \frac{\tilde{d}}{\tilde{a}} + \frac{\tilde{d}}{\tilde{a}} \right) - \frac{\tilde{d}^2}{\tilde{d}^2} \right] 
= \frac{\delta^2}{n^2} \left[ \frac{\tilde{d}}{\tilde{a}} \left( \frac{\tilde{d}}{\tilde{a}} + \frac{\tilde{d}}{\tilde{a}} \right) - \frac{\tilde{d}^2}{\tilde{d}^2} \right], \tag{10} \]
\[
\frac{\tilde{a}^2}{\tilde{d}^2} \left[ - \frac{\tilde{d}}{\tilde{a}} - \frac{\tilde{d}}{\tilde{a}} \left( \frac{\tilde{d}}{\tilde{a}} - \frac{\tilde{d}}{\tilde{a}} \right) - \frac{\tilde{d}^2}{\tilde{d}^2} \right] 
- n^2 \left[ \frac{\tilde{d}}{\tilde{a}} \left( \frac{\tilde{d}}{\tilde{a}} + \frac{\tilde{d}}{\tilde{a}} \right) - \frac{\tilde{d}^2}{\tilde{d}^2} \right] 
= \frac{\delta^2}{n^2} \left[ \frac{\tilde{d}}{\tilde{a}} \left( \frac{\tilde{d}}{\tilde{a}} + \frac{\tilde{d}}{\tilde{a}} \right) - \frac{\tilde{d}^2}{\tilde{d}^2} \right]. \tag{11} \]

The equation of motion (2) for the scalar field \( V \) is
\[
\frac{\tilde{a}^2}{\tilde{d}^2} \left[ 4 \frac{\tilde{d}}{\tilde{a}} \frac{\tilde{d}}{\tilde{a}} - \frac{\tilde{d}}{\tilde{a}} \frac{\tilde{d}}{\tilde{a}} + \frac{\tilde{d}}{\tilde{a}} \frac{\tilde{d}}{\tilde{a}} \right] - n^2 \left[ \frac{\tilde{d}}{\tilde{a}} \left( \frac{\tilde{d}}{\tilde{a}} + \frac{\tilde{d}}{\tilde{a}} \right) - \frac{\tilde{d}^2}{\tilde{d}^2} \right] 
= \frac{\delta^2}{n^2} \left[ \frac{\tilde{d}}{\tilde{a}} \left( \frac{\tilde{d}}{\tilde{a}} + \frac{\tilde{d}}{\tilde{a}} \right) - \frac{\tilde{d}^2}{\tilde{d}^2} \right]. \tag{15} \]

In order to separate Eqs. (10)–(15) one has to make a choice \( \delta = \epsilon \) and if one additionally choose a gauge in the form \( n = 1 \) as in Ref. [10] one gets the following set of time-dependent field equations (note that Eqs. (14) and (15) become identical so we reduce the number of equations to five \(^2\))
\[
\frac{\tilde{a}^2}{\tilde{d}^2} \left[ \frac{\tilde{d}}{\tilde{a}} \frac{\tilde{d}}{\tilde{a}} - \frac{\tilde{d}}{\tilde{a}} \frac{\tilde{d}}{\tilde{a}} \right] - n^2 \left[ \frac{\tilde{d}}{\tilde{a}} \left( \frac{\tilde{d}}{\tilde{a}} + \frac{\tilde{d}}{\tilde{a}} \right) - \frac{\tilde{d}^2}{\tilde{d}^2} \right] 
= \frac{\delta^2}{n^2} \left[ \frac{\tilde{d}}{\tilde{a}} \left( \frac{\tilde{d}}{\tilde{a}} + \frac{\tilde{d}}{\tilde{a}} \right) - \frac{\tilde{d}^2}{\tilde{d}^2} \right] \tag{16} \]
\(^2\)This, of course, reduces the generality of our discussion in the context of [17] since we identify an extra dimension with a scalar field.
\[ \alpha + \frac{\dot{\alpha}}{\alpha} = \frac{1}{2} \left( \frac{\dot{\beta}}{\beta} + \frac{\gamma}{\gamma} + \frac{\delta}{\delta} \right) \]
\[ = \frac{1}{2} \alpha \beta^2 \gamma^2 \left( \beta^2 - \gamma^2 \right)^2 - \alpha^4, \quad (17) \]
\[ \frac{\dot{\beta}}{\beta} + \frac{\dot{\delta}}{\delta} \left( \frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} \right) \]
\[ = \frac{1}{2} \alpha \beta^2 \gamma^2 \left( \beta^2 - \gamma^2 \right)^2 - \beta^4, \quad (18) \]
\[ \frac{\dot{\gamma}}{\gamma} + \frac{\dot{\delta}}{\delta} \left( \frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} \right) \]
\[ = \frac{1}{2} \alpha \beta^2 \gamma^2 \left( \beta^2 - \gamma^2 \right)^2 - \gamma^4, \quad (19) \]
\[ \frac{\dot{\delta}}{\delta} + \frac{\dot{\delta}}{\delta} \left( \frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} \right) = 0, \quad (20) \]

which, except for the right-hand side of Eq. (16), is the same set as the set of Eqs. (3.7)–(3.11) of [12] for bosonic low-energy-effective-action cosmology in string frame, provided we take dilaton field \( \phi \) as defined in [12] to be equal to \(-\ln \delta\) and also neglect axion i.e. take \( A = 0 \) in Ref. [12].

A new time coordinate is introduced to simplify the field equations by (compare Eq. (3.12) of [12])
\[ d\eta = \frac{d\tau}{\alpha \beta \gamma \delta}. \quad (21) \]
From now on we will use the notation \((\ldots)_{\eta} = d/d\eta\). To further simplify the equations we additionally define
\[ \tilde{\alpha} = \ln \alpha \tilde{\beta} = \ln \beta \tilde{\gamma} = \ln \gamma \tilde{\delta} = \ln \delta, \quad (22) \]
so that the set of Eq. (16)–(20) reads as
\[(\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} + \tilde{\delta})_{\eta} + \frac{\dot{\tilde{\delta}}}{\tilde{\delta}} \]
\[ = 2 \left( \tilde{\alpha}_{\eta} \tilde{\beta}_{\eta} + \tilde{\alpha}_{\eta} \tilde{\gamma}_{\eta} + \tilde{\beta}_{\eta} \tilde{\gamma}_{\eta} \right) \]
\[ + 2 \left( \tilde{\alpha}_{\eta} + \tilde{\beta}_{\eta} + \tilde{\gamma}_{\eta} \right) \tilde{\delta}_{\eta}, \quad (23) \]
\[ 2 \tilde{\alpha}_{\eta} = \left[ \left( \beta^2 - \gamma^2 \right)^2 - \alpha^4 \right] \tilde{\delta}^2, \quad (24) \]
\[ 2 \tilde{\beta}_{\eta} = \left[ \left( \alpha^2 - \gamma^2 \right)^2 - \beta^4 \right] \tilde{\delta}^2, \quad (25) \]
\[ 2 \tilde{\gamma}_{\eta} = \left[ \left( \beta^2 - \gamma^2 \right)^2 - \alpha^4 \right] \tilde{\delta}^2, \quad (26) \]
\[ \tilde{\delta}_{\eta} = 0. \quad (27) \]

These equations are the same as pre-big-bang cosmology Mixmaster equations in string frame (3.19)–(3.22) of Ref. [12] if we take \( \tilde{\delta} = -\phi = -M\eta + \text{const} \) or as 5-dimensional vacuum Mixmaster Eqs. (31) and (32) of Ref. [17].

Now we consider suitable initial conditions expressed in terms of the Kasner parameters and discuss the general behaviour of Bianchi type IX Horava–Witten cosmology on the approach to singularity.

The Kasner solutions are obtained as approximate solutions of the Eq. (16)–(20) when the right-hand sides (describing the curvature anisotropies) are neglected. In terms of \( \tau \)-time, they are
\[ \alpha = \alpha_0 \tau^{p_1}, \quad (28) \]
\[ \beta = \beta_0 \tau^{p_2}, \]
\[ \gamma = \gamma_0 \tau^{p_3}, \]
\[ \delta = \delta_0 \tau^{p_4}, \]
while
\[ \tilde{\delta} = -\ln \delta_0 - p_4 \ln \tau. \quad (29) \]

From (23)–(27) we have the following algebraic conditions for the Kasner indices, \( p_i \):
\[ p_1 + p_2 + p_3 + p_4 = 1, \quad (30) \]
and
\[ p_1^2 + p_2^2 + p_3^2 + \frac{1}{2} p_4^2 = 1. \quad (31) \]

This, in particular, proves that the isotropic Friedmann case as obtained by Lukas et al. [10] is given by
\[ p_1 = p_2 = p_3 = p_4 = \frac{3}{4} \pi^2 \frac{4}{11 \sqrt{3}}, \quad (32) \]
\[ p_4 = q_+ = \frac{3}{2} \pi \pm \frac{4 \sqrt{3}}{11}. \quad (33) \]

The meaning of such isotropic solutions was discussed in Ref. [10] (note that numerically \( p_+ = 0.48 \), \( p_- = 0.06 \) and \( q_- = -0.45 \), \( q_+ = 0.81 \)). In fact, there are two branches each one for negative and positive values of time coordinate \( \tau \) (negative values of time
can be achieved by taking \(-\tau\) instead of \(+\tau\) in (28), or, simply by taking the modulus. If \(\tau < 0\) one has \((-\) branch and if \(\tau > 0\) one has \((+) \) branch. For \((-)\) branch both the worldvolume of (3-dimensional space) and the orbifold contract for \(p_+\) and \(q_+\) while the worldvolume contracts and the orbifold expands (superinflationary) for \(p_+\) and \(q_-\). For \((+)\) branch the worldvolume and the orbifold expand for \(p_-\) and \(q_-\) while the worldvolume expands and the orbifold contracts for \(p_+\) and \(q_-\).

Notice that the conditions (30), (31) are different from the conditions which emerge in pre-big-bang cosmology where the role of the fifth coordinate is played by the dilaton (see Eqs. (3.50), (3.51) of [12]). They are also different from Mixmaster Kaluza–Klein five-dimensional models where the homogeneity group acts on four-dimensional hypersurfaces of constant time (see Eq. (33) of [17]). The reason for that is simply the fact the fifth coordinate in Hořava–Witten cosmology is an orbifold. The isotropization of the models under consideration means that the Kasner indices reach the values \(32\), \(34\) in Horava–Witten cosmology is an orbifold. The isotropization of 5-dimensional Kaluza–Klein models in supergravity as first considered by Chodos and Detweiler [18] would require the different values of the Kasner indices namely \(p_1 = p_2 = p_3 = -p_4 = 1/2\) which fulfill the conditions (30) and (31) without a factor \(3/2\) in front of \(p_4\) in (31).

Having given the conditions (30), (31), one can express the indices \(p_2\) and \(p_3\) by using \(p_1\) and \(p_4\), i.e.,

\[
\begin{align*}
p_2 &= \frac{1}{2}\left(1 - p_1 - p_4\right) \\
&\quad - \sqrt{-3p_1^2 + 2p_1(1 - p_4) + 1 + 2p_4(1 - 2p_4)} \\

p_3 &= \frac{1}{2}\left(1 - p_1 - p_4\right) \\
&\quad + \sqrt{-3p_1^2 + 2p_1(1 - p_4) + 1 + 2p_4(1 - 2p_4)}.
\end{align*}
\]

(34)

Since the expression under the square root in (34) should be nonnegative, one can extract the restriction on the permissible values of \(p_4\) which is

\[
q_- \leq p_4 \leq q_+.
\]

(35)

Some particular choices are of interest. If one takes \(p_4 = 0\) one recovers vacuum general relativity limit with Kasner indices \(-1/3 \leq p_1 \leq 0, 0 \leq p_2 \leq 2/3, 2/3 \leq p_3 \leq 1\). The range dividing case is for \(p_4 = 2/11\) with the following ordering of the Kasner indices

\[
\begin{align*}
\frac{3}{11} - \frac{4}{11\sqrt{3}}\sqrt{11} &\leq p_1 \leq \frac{3}{11} - \frac{2}{11\sqrt{3}}\sqrt{11}, \\
\frac{3}{11} - \frac{2}{11\sqrt{3}}\sqrt{11} &\leq p_2 \leq \frac{3}{11} + \frac{2}{11\sqrt{3}}\sqrt{11}, \\
\frac{3}{11} + \frac{2}{11\sqrt{3}}\sqrt{11} &\leq p_3 \leq \frac{3}{11} + \frac{4}{11\sqrt{3}}\sqrt{11}.
\end{align*}
\]

(36)

However, we are interested in knowing whether the curvature terms on the right-hand side of the field Eq. (24)–(26) really increase as \(\eta \to -\infty (\tau \to 0\) - approach to singularity for \((+)\) branch) since from (21) and (28) we get

\[
\eta = \eta_0 + \ln\tau,
\]

(37)

and \(\eta_0 = \text{const}\). This would require either \(\alpha^{4/3}\),

![Fig. 1. The range of Kasner indices \(p_1\) and \(p_2\) which fulfill the conditions (41), (42). The appearance of the isotropic Friedmann cases at \(p_1 = p_2 = p_-\) and \(p_1 = p_2 = p_+\) prevents chaotic oscillations in the shaded region that surrounds them. For the values of Kasner indices \(p_1\) and \(p_2\) from that region chaotic oscillations are not possible to begin. On the other hand, even if after some number of oscillations from one Kasner epoch to the other, the values of \(p_1\) and \(p_2\) will fall into that region, the chaotic oscillations of the scale factors stop which reflects nonchaotic behaviour of such Bianchi IX Hořava–Witten cosmologies.](image-url)
\(\beta \delta^2\), or \(\gamma \delta^2\) to increase if the transition to another Kasner epoch is to occur [17,16]. Since

\[
\begin{align*}
\alpha \delta^2 &\propto \tau^{(p_1+p_3)}, \\
\beta \delta^2 &\propto \tau^{(1+p_1-p_2-p_3)}, \\
\gamma \delta^2 &\propto \tau^{(1+p_3-p_2)}.
\end{align*}
\]

we need one of the following three conditions to be fulfilled

\(p_1 + p_4 = 1 + p_1 - p_2 - p_3 < 0\),

\(2p_2 + p_4 = 1 + p_2 - p_1 < 0\),

\(2p_3 + p_4 = 1 + p_3 - p_1 - p_2 < 0\). \hspace{1cm} (39)

From Kasner conditions (30), (31) we are free to choose only two parameters so that we can write

\(p_4 = 1 - p_1 - p_2 - p_3\),

\(p_1 \propto (1 - p_1 - p_2) \pm \frac{1}{2}(1 + p_1 - p_2 - p_3)^{1/2},\) \hspace{1cm} (40)

which gives also the condition for \(p_3\) to be real as follows

\(1 - 4\left(p_1^2 + p_3^2\right) - 3p_1 + 3p_2 + 1 \geq 0.\) \hspace{1cm} (41)

In (40) we take the + sign for \(p_4 < 2/11\) and the − sign for \(p_4 > 2/11\). The conditions (39) become (compare [17])

\(p_1^2 + 4p_2^2 - p_1 + 2p_2 - p_1^2 < 0,\)

\(4p_1^2 + p_2^2 - p_1 + 2p_2 - p_1^2 < 0,\)

\(4(p_1^2 + p_2^2) + 7p_1 - 7(p_1 + p_2) + 3 > 0.\) \hspace{1cm} (42)

One should remind here that for (-) branch we have to take (−τ) in (28), (29) and (38) which leads to the same conditions (42).

The plot of the conditions (41), (42) is given in Fig. 1. The chaotic oscillations (Kasner-to-Kasner transitions) can start in any region except the narrow range surrounding the isotropic points \(p_1 = p_2 = p_3 = q = q\) and \(p_1 = p_2 = p_3 = q = q\). However, such chaotic oscillations would continue indefinitely provided there were no such regions at all (this is the case of vacuum general relativity, for example). Here, once the Kasner parameters fall into the region surrounding the isotropic Friedmann \(p_+\) or \(p_-\) solutions of Lukas et al. [10] the chaotic oscillations cease so that there is no chaos in such Hořava–Witten cosmologies.

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Note added. During the publication process of this paper it emerged that the properties of the most general class of anisotropic and inhomogeneous separable Hořava–Witten cosmologies were studied in [19]. Other exact cosmologies were also found in [20].

References


Ward identities and the vanishing theorem for loop amplitudes of the closed $N = 2$ string

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Abstract

The existence of a ground ring of ghost number zero operators in the chiral BRST cohomology of the $N = 2$ string is used to derive an infinite set of Ward identities for the closed-string scattering amplitudes at arbitrary genus. These identities are sufficient to rederive the well known vanishing theorem for loop amplitudes with more than three external legs. © 2000 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

The explicit computation of loop amplitudes in string theory is notoriously difficult. Even for the technically most simple theory – the bosonic string – the level of mathematical complexity is impressing if one tries to go beyond one loop. Adding fermions and supersymmetry on the world-sheet does not improve the situation. On the contrary, the calculations become still more intricate and only a few explicit results exist. It seems that even the general formalism has not yet been fully worked out [1].

Fortunately, explicit computations can sometimes be replaced by more indirect methods, often related to symmetry arguments. It is thus not surprising that for the $N = 2$ string (i.e. the theory based on extended supersymmetry on the world-sheet; see [2] for a general review and [3] for a discussion of loop amplitudes) Berkovits and Vafa succeeded to avoid the evaluation of the path integral and obtained powerful results for loop amplitudes by embedding the theory into an $N = 4$ topological theory [4]. In fact, they found that all amplitudes with more than three external legs vanish to all orders in the loop expansion. The purpose of this letter is to give an alternative derivation of this result. Our approach has the advantage that conceptually it is very clear what is going on since the equations used to derive the vanishing of the amplitudes can nicely be interpreted as Ward identities of an infinite set of unbroken symmetries in target space. Another interesting point is that from a technical point of view our analysis rests on the picture dependence of the BRST cohomology of the $N = 2$ string at zero momentum and demonstrates what kind of information may be stored in the still somewhat obscure picture phenomenon [5]. Maybe, this lesson can also be useful in some way for the $N = 1$ string.
The letter is organised as follows: In the next section we recall some facts about the BRST cohomology of the $N=2$ string. These results will be used in section three to derive an infinite set of target space Ward identities which will then be explicitly evaluated so that the vanishing of the loop amplitudes directly follows. We conclude with some further remarks and a brief discussion of the reliability of our arguments.

2. Symmetries and ground ring of the $N=2$ string

One of the attractive features of the BRST approach to closed string theory is that it provides an efficient means to analyse symmetries in target space. More precisely, unbroken target space symmetries generally lead to the existence of ghost number one cohomology classes (in conventions where physical states have ghost number two). A detailed explanation of this fact is given in [7] (see also [8]) where in addition an elegant method to derive the corresponding Ward identities – briefly reviewed below – is described.

Due to the fact that the closed string Fock space factorises into right- and left-moving parts ghost number one cohomology classes can be further characterised: they are most conveniently constructed as a product of a holomorphic piece of ghost number zero and an antiholomorphic piece of ghost number one. The latter is usually the right-moving part of a physical vertex operator, taken at some discrete value of the momentum whereas the former very often is just the unit operator. If, however, the chiral (= left-moving) cohomology at ghost number zero contains further elements besides the unit operator more closed string operators of ghost number one can be constructed resulting in a much richer symmetry structure. An example is the bosonic string in two dimensions [9]. Moreover, interesting algebraic structures emerge. The BRST cohomology possesses a natural multiplication rule, additive in ghost number. The ghost number zero cohomology therefore forms a ring under this multiplication (the so-called ground ring). As has been emphasised in [10] the structure constants of this ground ring encode much information about the symmetry of the theory.

The $N=2$ string has been studied along these lines in [12–14]. Based on the fact that so many of its scattering amplitudes are known or conjectured to vanish [4,15,16] and comparison with the field theory that reproduces tree-level scattering [15,17] it seemed very plausible that in this theory a large symmetry group is realized. In fact, a ground ring of the $N=2$ string has recently been found in [14] and will now briefly be reviewed. The construction looks somewhat unconventional because it does not restrict to operators of a single picture only, but takes into account the full picture degeneracy of the Fock space. However, starting from this ground ring one may derive powerful Ward identities as has been shown for tree amplitudes in [14] and will be demonstrated in this letter for loop amplitudes.

At zero ghost number chiral cohomology classes occur only for vanishing momentum. For low-lying picture numbers and ghost number zero the cohomology problem is rather straightforward to solve:

- The cohomology is empty for pictures $(\pi^+, \pi^-) = ((-1,-1),(-1,0),(0,-1))$ and consists of the unit operator in the $(0,0)$ picture.

- In the pictures $(-1,1)$ and $(1,-1)$ the cohomology consists of the spectral flow operators

\[
A(z) = (1 - cb') J^{-+} e^{\psi^+} e^{-\psi^-} (z) \quad (1)
\]
and

\[
A^{-1}(z) = (1 + cb') J^{++} e^{-\psi^+} e^{\psi^-} (z),
\]

with $J^{++} = \frac{1}{2} \epsilon_{a b} \psi^a \psi^b$ and $J^{-+} = -\frac{1}{2} \epsilon_{a b} \psi^a \psi^b$ (see [14] for conventions and a

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\[\text{There exist two further operations} \quad \text{the Gerstenhaber-bracket and the} \quad \Delta \quad \text{operation which, together with the ring multiplication, give the BRST cohomology the structure of a BV-algebra [11].}\]

\[\text{This construction is non-trivial due to the picture dependence of the BRST cohomology of the} \quad N=2 \quad \text{string at zero momentum [18].}\]

Pointcaré-duality provides an isomorphism between the cohomologies for pictures $(\pi^+, \pi^-)$ and $(-\pi^- - 2, -\pi^+ - 2)$ [19]. Moreover, the cohomologies for pictures $(\pi^+ + \rho, \pi^- - \rho)$ with $\rho \in \frac{1}{2} \mathbb{Z}$ coincide due to spectral flow. It is therefore sufficient to consider the case $\pi^+ \geq -1$ only.
description of the $N = 2$ string ghost system. One may check that $A$ and $A^{-1}$ are inverse to each other with respect to ring multiplication.

- In the $(1,0)$ picture the cohomology consists of the picture changing operator

$$X^+(z) = \{Q, \xi^+(z)\}$$

and the operator

$$A \cdot X^- \quad \text{with} \quad X^-(z) = \{Q, \xi^-(z)\}.$$ 

It should be emphasised that $A \cdot X^-$ is BRST inequivalent to $X^+$. Analogously, the $(0,1)$ cohomology consists of the operators $X^-$ and $A^{-1} \cdot X^- \cdot X^+$. We see that the size of the cohomology grows as the picture increases. To obtain cohomology classes with higher integral picture numbers one may simply consider polynomials of the operators $A$, $A^{-1}$ and $X^\pm$.

$$(X^+)^k \cdot (X^-)^l \cdot A^n, \quad k, l \in \mathbb{N}, \quad n \in \mathbb{Z}.$$ 

Note that $k$ and $l$ must not be negative since, contrary to $N = 1$ strings, there do not exist local inverse picture changing operators for the $N = 2$ string [20] (the cohomology at vanishing momentum and ghost number is empty for picture numbers $(-1,0)$ and $(0, -1)$).

It has been shown in [14] that all these operators are BRST inequivalent! For a given picture $(\pi^+, \pi^-)$ we thus have constructed $\pi^+ + \pi^- + 1$ operators,

$$\sigma^{\pi^+, \pi^-}_{a, a', n} = (X^+)^{\pi^+ + n} \cdot (X^-)^{\pi^- - n} \cdot A^n,$$

$$n = -\pi^+, \ldots, \pi^-.$$ 

To obtain ghost number one cohomology classes of the closed string connected to the symmetries of the theory the operators in (2) have to be combined with right-moving cohomology classes of zero momentum and ghost number one. These operators can be found in a similar way: In [21] it has been shown that the relevant cohomology in the $(0,0)$ picture is spanned by the four elements

$$-i \mathcal{P}^a = c \delta Z^a - 2 \gamma^a \psi^a, \quad -i \mathcal{P}^\alpha = c \delta \bar{Z}^\alpha - 2 \gamma^\alpha \psi^\alpha.$$ 

Here the target space Lorentz indices $a$ and $\bar{a}$ range from 0 to 1. Multiplication with $\sigma^{\pi^+, \pi^-}_{a, a', n}$ gives similar operators in higher pictures:

$$\mathcal{P}^a \sigma^{\pi^+, \pi^-}_{a, a', n} = \sigma^{\pi^+, \pi^-}_{a, a', n} \cdot \mathcal{P}^a, \quad \mathcal{P}^\alpha \sigma^{\pi^+, \pi^-}_{a, a', n} = \sigma^{\pi^+, \pi^-}_{a, a', n} \cdot \mathcal{P}^\alpha.$$ 

We are now ready to write down the sought for closed string cohomology classes of ghost number one:

$$\Sigma^{a, a', \pi^+, \pi^-}_{m, n} = \sigma^{\pi^+, \pi^-}_{a, a', n} (z) \mathcal{P}^a_{a', a''} (z), \quad m, n = -\pi^+, \ldots, \pi^-.$$ 

To save space the analogous operators $\Sigma^{\alpha}$ will not be explicitly mentioned in the following. Using the descent equations one may now construct an infinite set of symmetry charges and work out the transformation laws of the physical state. This has been done in [14].

We conclude this section with one further remark. So far, we have only considered the relative cohomology of states that are annihilated by the zero modes of all fermionic antighosts. It would, however, be more appropriate also to take into account states that are not annihilated by $b_0 + \bar{b}_0$, which defines the so-called semi-relative cohomology (one way to see that this is the right space to consider is to write down a kinetic term in a string field formalism). Allowing for more states generally changes the cohomology. But fortunately, one can show that the operators (5) are still non-trivial in the semi-relative cohomology. One may also wonder whether new cohomology classes turn up, as happens for the bosonic string in two dimensions [7]. We do not know the general answer to this question, but explicit calculations for low-lying pictures indicate that this is not the case.

3. Ward identities

We will now use the results from the previous section to derive Ward identities for $N = 2$ string

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Multiplication of two operators, denoted by a dot in the following, means to take the regular term in their operator product expansion [10].
amplitudes at arbitrary genus. Actually, an \( N = 2 \) string scattering amplitude is further characterised by a Chern number classifying \( U(1) \) bundles over the world-sheet Riemann surface. It is, however, sufficient to focus on vanishing Chern number in the following. This will be justified in section four. For reasons of space the general formalism will not be reviewed in detail here. Instead, we refer to [7,22,23] for more extensive explanations.

The basic object involved in the computation of scattering amplitudes is the vertex operator of the single degree of freedom in the theory. As usual, it splits into holomorphic and antiholomorphic parts:

\[
V(z,\bar{z},k) = V^{\text{left}}(z,k)\bar{V}^{\text{right}}(\bar{z},k).
\]

The left-moving operator is

\[
V^{\text{left}}_{(-1,-1)}(z,k) = c\bar{c}e^{-a\bar{z}}e^{-\bar{a}z}e^{i\bar{k}z\bar{k}^*},
\]

in the \((-1,-1)\) picture and

\[
V^{\text{left}}(\sigma^+,\sigma^-)(z,k) = (X^+)^{n+1} \cdot (X^-)^{n+1} \cdot V^{\text{left}}_{(-1,-1)}(z,k)
\]

in higher pictures (the right-moving piece \( \bar{V}^{\text{right}} \) looks similar)\(^5\). Counting both metric and \( U(1) \) but not supersymmetry ghost number vertex operators in closed \( N = 2 \) string theory therefore have ghost number four (in our conventions picture changing operators have ghost number zero, see [14]). Moreover, they are not annihilated by the zero modes \( b_0 \) and \( \bar{b}_0 \) of the \( U(1) \) antighosts. On the other hand the ghost number one operators constructed in the previous section are all elements of the relative cohomology, i.e. they are all killed by the zero modes of all fermionic antighosts. It is, however, not too difficult to relate relative cohomology classes to operators of higher ghost number, essentially by multiplying with the relevant ghosts. In this way we can construct from the ghost number one operators in Eq. (5) new cohomology classes of ghost number three:

\[
\Sigma^{a\bar{a}}_{a\bar{a}^{-},m,n} \rightarrow \Omega^{a\bar{a}}_{a\bar{a}^{-},m,n} = c\bar{c} \Sigma^{a\bar{a}}_{a\bar{a}^{-},m,n} \cdot \ldots.
\]

Here the dots refer to further terms that might be necessary to achieve BRST invariance but are unimportant otherwise.

We are now ready to derive a Ward identity involving a genus \( g \) scattering amplitude of \( N \) external states with momenta \( k_1,\ldots,k_N \) (denoted \( A^{\mu}_{g}(k_1,\ldots,k_N) \) in the following). One starts with the correlator \(^6\)

\[
\left\langle \Omega^{a\bar{a}}_{a\bar{a}^{-},m,n}(z,\bar{z}) \prod_{i=1}^{N} V^{\mu}_{a\bar{a}^{-},l}(z_i,\bar{z}_i,k_i) \right\rangle.
\]

where \((\mu,B)\) and \((\bar{\mu},\bar{B})\) are the the appropriate Beltrami differentials integrated with the corresponding antighosts and the index \( g \) indicates that the correlator is meant to be evaluated with respect to the conformal field theory living on a Riemann surface of genus \( g \). The antighosts can be applied to the vertex operators and the integrations can be pulled out of the brackets. Let us denote the remaining integrand by \( \Theta \). If the operator \( \Omega \) in (6) were replaced by an ordinary physical vertex operator \( V \) one could integrate \( \Theta \) over the moduli space of a genus \( g \) surface with \( N + 1 \) punctures. From counting dimensions and ghost numbers it follows, however, that \( \Theta \) as defined by (6) can be integrated only over the boundary of moduli space. In fact, it can be considered as a differential form on moduli space of codimension one. Since \( \Theta \) can also be shown to be a closed form [7] Stokes’ theorem leads to the desired Ward identity

\[
\int_{\Delta^{g,N+1}} \Theta = \int_{\Delta^{g,N+1}} d\Theta = 0.
\]

The next step is to have a closer look at the \( N = 2 \) string moduli space \( \mathcal{M}^{g,N+1} \), i.e. the moduli space of \( N = 2 \) super Riemann surfaces with genus \( g \) and \( N + 1 \) punctures (and vanishing Chern number in our case). In addition to the usual metric and super moduli, we also have to consider the so-called \( U(1) \) moduli [3] describing a continuum of possible monodromy phases for the world-sheet fermions arising from their transport along non-trivial homology cycles. However, the \( U(1) \) moduli space is compact (it always has the topology of a torus) and therefore

\(^{5}\)Application of spectral flow only leads to vertex operators proportional to those above.

\(^{6}\)For simplicity we only consider closed string operators whose left- and right-moving picture numbers coincide.
does not contribute to the boundary of moduli space. As a result, in our Ward identity (7) only the familiar boundary components of the metric moduli space appear.

The metric moduli are of two different types. One corresponds to the shape of the underlying Riemann surface whereas the other describes punctures, i.e., the locations of the vertex operators. If we move to the boundary of moduli space the Riemann surface degenerates in some way. In the following it is convenient to distinguish four different cases: First of all, the underlying surface may pinch either along a trivial or a non-trivial homology cycle. If a genus \( g \) surface pinches along a non-trivial cycle it becomes a surface of genus \( g - 1 \) with two points coinciding. If it pinches along a trivial cycle the result is a connected pair of Riemann surfaces with genera \( g \) and \( g' \) such that \( g + g' = g \). For a \( g = 2 \) surface with four punctures these two cases are illustrated in the top row of the Fig. 1. It may also happen that a number of punctures approach each other. This is conformally equivalent to a situation where a sphere containing the relevant punctures splits off of the rest of the surface. This is illustrated in the bottom line of the figure, where we also distinguished whether two vertex operators \( V \) approach each other or one \( V \) approaches the ghost number three operator \( \Omega \).

To see how a pinch (denoted by \( P \) in the figure) can properly be included in the computation let us recall that it can equivalently be described by an infinitely long cylinder. This cylinder can be taken into account by inserting a complete set of physical states. In this formulation the twist angle of the cylinder is one of the moduli leading to an insertion of the metric antighost combination \( h(z) - \bar{h}(\bar{z}) \). So the pinch can be represented by the sum

\[
\sum_i |\hat{O}_i\rangle \langle O_i|, \tag{8}
\]

where \( i \) labels a basis of the absolute BRST cohomology and

\[
\langle O_i | O' \rangle = \delta_i^j, \quad |\hat{O}_i\rangle = (b_{0_{i}} - \bar{b}_{0_{i}}) |O_i\rangle. \tag{9}
\]

What about the fermionic and \( U(1) \) moduli? The former are correctly taken into account by obeying the right selection rules for picture numbers [24]. Moreover, a pinch contributes one complex \( U(1) \) modulus. This corresponds to the fact that the complete set of states (8) carries two units of \( U(1) \) ghost number – just enough to compensate the antighost insertion due to the \( U(1) \) modulus of the pinch.

Let us now become more explicit: We assume \( N \geq 3 \), i.e., the presence of at least three vertex operators, and genus \( g > 0 \) since tree-level amplitudes have been discussed in [14]. It will also be sufficient and technically simpler to consider only operators \( \Sigma \) (and the corresponding \( \Omega \)) of the form

\[
\Sigma_{\alpha}(z, \bar{z}) := \sum_{-n, n} \Sigma_{-n, n, n}(z, \bar{z}) = A^\alpha(z) \tilde{A}^\alpha(\bar{z}),
\]

which have picture numbers \((-n, n)\). The four cases mentioned above will now be discussed in turn.

3.1. Case 1: A non-trivial homology cycle pinches

Besides the \( N \) physical vertex operators already present the pinching leads to an insertion of two further vertex operators \( O_i \) and \( \hat{O}_i \), as explained above. So we have to evaluate the expression

\[
\sum_i \langle \langle \Omega_i^a \ V_1 \ldots \ V_N \hat{O}_i^j \ O_i \rangle \rangle_{g-1}. \tag{10}
\]

Here, the notation for the vertex operators has been simplified in a hopefully obvious way. The double bracket as usual denotes evaluation of the full amplitude including integration over moduli space.

To further evaluate the expression (10) let us note that it contains at least six operators (since we assumed \( N \geq 3 \) in the beginning). Regardless of the
value of \( g \) integration over moduli space leads for this number of operators to insertions of metric antighosts that transform cohomology classes into integrated vertex operators. Since this effect will be crucial in the following, we briefly review some details: Assume the operator \( A(z) \) represents a closed string cohomology class. From the explicit form of the BRST operator it follows that

\[
\mathcal{B}(z) = \frac{\delta}{\delta z} b(w) \mathcal{A}(z)
\]

satisfies the relation

\[
[Q, \mathcal{B}^{(1)}] = \partial \mathcal{B},
\]

\( Q \) being the left-moving part of the BRST operator. Since this argument goes through for the right-moving half, as well, a \( b \)-ghost insertion leads to the integrated operator

\[
\int d^2 z \mathcal{B}^{(1)}(z) \mathcal{B}^{(1)}(z),
\]

which is BRST invariant since the integrand transforms into a total derivative. In practice, going over from a cohomology class to an integrated vertex operator simply amounts to getting rid of the undifferentiated \( c \)- and \( \bar{c} \)-ghosts. If some cohomology class does not contain both these ghost fields (as for example the unit operator) its integrated form is zero.

We are always free to choose where to locate the \( b \)-ghost insertions \(^{10} \), i.e. which cohomology class to convert into an integrated operator. In the present case we can pick \( \Omega \). From the explicit form of \( A \) in Eq. (1) one sees that stripping off a \( c \)-ghost necessarily leads to the presence of a \( b' \)-ghost, for example \( A^{(1)} = -b' J - e^{b'} e^{-b} \). However, there is no corresponding \( c' \)-ghost in sight to compensate \( b' \) in a correlation function. So we learn from simple \( U(1) \) ghost number counting that the amplitude (10) vanishes! In other words, the kind of degeneration considered in this subsection does not contribute to the Ward identity.

\(^{10} \) Since we are dealing with vertex operators of non-standard ghost number, this is not completely obvious in the path integral formulation. In the operator formalism, however, one may explicitly check that the location of the \( b \)-ghost insertion is immaterial.

3.2. Case 2: A trivial homology cycle pinches

The contribution to the Ward identity of this component of the boundary is

\[
\sum_{i, \alpha} \langle \langle V_{u_1} \ldots V_{u_{\beta}} e^\alpha \mathcal{O} \rangle \rangle_{g_1} \langle \langle O V_{u_{\beta+1}} \ldots V_{u_{\alpha}} \rangle \rangle_{g_2}
\]

(11)

with \( g_1 + g_2 = g \) and \( g_1, g_2 > 0 \). The sum over \( \alpha \) runs over all possible ways to divide the set of \( N \) physical vertex operators into a subset \( \{V_{u_1}, \ldots, V_{u_\beta}\} \) on the genus \( g_1 \) surface and the remainder \( \{V_{u_{\beta+1}}, \ldots, V_{u_\alpha}\} \) located on the other surface.

Since \( g_1 \) is strictly positive and the correlation function involving \( \Omega \) contains at least one further operator the expression (11) can again be evaluated by transforming \( \Omega \) to its integrated form. As in the previous subsection the vanishing of (11) then follows from \( U(1) \) ghost number counting.

3.3. Case 3: A sphere not including \( \Omega \) splits off

In this case we have to evaluate the expression

\[
\sum_{i, \alpha} \langle \langle V_{u_1} \ldots V_{u_{\beta}} e^\alpha \mathcal{O} \rangle \rangle_{g} \langle \langle O V_{u_{\beta+1}} \ldots V_{u_{\alpha}} \rangle \rangle_{g=0}
\]

(12)

Since \( g > 0 \) by assumption the correlator involving \( \Omega \) vanishes by the same argument as above.

3.4. Case 4: A sphere including \( \Omega \) splits off

In this final case the contribution to the Ward identity reads

\[
\sum_{i, \alpha} \langle \langle V_{u_1} \ldots V_{u_{\beta}} e^\alpha \mathcal{O} \rangle \rangle_{g=0} \langle \langle O V_{u_{\beta+1}} \ldots V_{u_{\alpha}} \rangle \rangle_{g}
\]

(13)

The ghost number three operator \( \Omega \) now appears in a tree-level amplitude whose evaluation involves metric antighost insertions as soon as more than three operators are present. Correspondingly, terms in the \( \alpha \)-sum vanish by the standard argument whenever the \( g = 0 \) correlator involves more than one operator \( V \) besides \( \Omega \) and \( \mathcal{O} \). What remains are those degenerations where \( \Omega \) splits off with precisely one vertex.
operator $V$. These are the only contributions to the Ward identity:

$$
\sum_{u=1}^{N} \sum_{i} \langle V_u \Omega_u \hat{O}^i \rangle_{g=0} \times \langle \langle O V_1 \ldots V_{u-1} V_{u+1} \ldots V_N \rangle \rangle_g = 0.
$$

(14)

Obviously, the only non-vanishing term in the above sum over $i$ occurs when $O_i$ coincides with the vertex operator $V_u$. In each term of the $u$-sum the second correlator therefore is just the genus $g$ amplitude of $N$ physical states $A_g^N$. Reinserting the momenta $k_u$ allows us to rewrite the Ward identity as

$$
A_g^N(k_1, \ldots, k_N) \cdot \sum_{u=1}^{N} \langle V(k_u) \Omega_u^N \hat{V}(-k_u) \rangle_{g=0} = 0.
$$

(15)

These identities have already been derived in [14] for tree amplitudes. Eq. (15) tell us that they do not get modified for higher genera. The remaining correlator can be evaluated as

$$
\langle V(k) \Omega_u \hat{V}(-k) \rangle_{g=0} = \left( \frac{\mathbb{I}^0}{k} \right)^n \equiv h(k)^n k^n.
$$

(16)

The final identities for the genus $g$ amplitude thus read

$$
A_g^N(k_1, \ldots, k_N) \cdot \sum_{i=1}^{N} h(k_i)^n k_i^n = 0 \quad \text{for any } n \in \mathbb{Z}
$$

(17)

and imply the vanishing of all amplitudes with $N \geq 4$ [14]. The three point function, however, is generally non-zero. One may for example check that the tree-level amplitude

$$
A_{g=0}^N(k_1, k_2, k_3) = \left( k_1 \cdot k_2 - k_2 \cdot k_1 \right)^2 \equiv 0
$$

satisfies all identities without being zero. On dimensional grounds it seems very plausible that for higher genus the three point function is just a power of the tree-level result:

$$
A_{g=3}^N(k_1, k_2, k_3) = \alpha_g \left( \frac{k_1 \cdot k_2 - k_2 \cdot k_1}{k_1 \cdot k_2} \right)^{4g+2}.
$$

Here the pre-factor $\alpha_g$ depends on the genus but not on the momenta. Explicit computations at one loop show that $\alpha_{g=1}$ is divergent [25,26]. This concludes our discussion of the scattering amplitudes of the $N=2$ string.

4. Some remarks

So far we have ignored the possibility of non-vanishing Chern number $c$, corresponding to topologically non-trivial configurations of the $U(1)$ gauge field on the world-sheet. A careful evaluation of the path integral shows that a non-zero Chern number can be simulated by inserting a power of the spectral flow operator $A$ into the $c=0$ correlation function and simultaneously adjusting the picture numbers of the vertex operators [3]. Since the derivative of the spectral flow operator is BRST trivial each $A$ (or $A^{-1}$) can be moved towards one of the vertex operators and simply pulls out a momentum factor $h(k)$ (or its inverse, see Eq. (16) for a definition of $h$). Therefore, amplitudes with different Chern number are proportional to one another. Hence, it is sufficient to prove the vanishing of a scattering amplitude for one fixed value of $c$. Secondly we have ignored that, as a Riemann surface with $c=0$ degenerates and splits into two, the resulting surfaces may have non-vanishing Chern numbers $c$ and $-c$. So we actually should include in our Ward identity a summation over all such splittings [11]. However, we have just explained that this only leads to additional factors $h(k)^c$ and $h(k)^{-c}$ which cancel each other ($k$ is the momentum flowing through the pinch). This justifies our treatment where we completely neglected sectors with non-zero Chern number.

A further point that deserves to be mentioned is the question of non-linear contributions to the sym-

\[11\] This sum is finite since supersymmetry ghost zero modes kill correlators when $|c|$ exceeds a certain value.
metries. One of the remarkable features of the $N = 2$ string that make it such an interesting toy model is the fact that we know a simple field theory that reproduces the tree-level amplitudes to all orders in $\alpha'$. This field theory is well known to possess a highly non-linear symmetry structure. In [14] the linearised version of the unbroken symmetries on the field theory side was compared to the transformation rules of the $N = 2$ string vertex operators under the symmetries that lead to the above Ward identities. They were found to coincide. In fact, the Hilbert space in our formulation of the theory consists only of single string states. So it seems at first sight correct to restrict a comparison between symmetries in field theory and string theory to the linear level. However, it has been explained in [7] (Section 6) that non-linear symmetry structures can make their appearance in a first quantised string theory at the level of Ward identities. More precisely, a non-linear contribution to a Ward identity corresponds to a situation where the ghost number one three for $N = 2$ strings operator $\Omega$ splits off with more than one further vertex operator (see Fig. 2).

In this case only the overlap between the charge acting on a single vertex operator with a multi-string state is sent through the pinch. In other words, a symmetry is realized non-linearly precisely when the tree-level amplitude

$$\langle \langle \Omega V(k_1) \cdots V(k_{n-1}) \tilde{V}(k_n) \rangle \rangle_{g = 0}$$

is non-vanishing for $n \geq 3$. A model where this indeed happens is the bosonic string in two dimensions. Yet it has been argued in section three that in our case of the $N = 2$ string the relevant correlation functions vanish. As a consequence the Ward identity (17) is linear. This indicates a clear discrepancy to the field theory and suggests that the behaviour of the $N = 2$ string is not fully captured by its tree-level effective field theory.

Last but not least we should give our opinion on the reliability of our arguments. In fact, we must admit that the analysis of the boundary of the $N = 2$ string moduli space has been somewhat heuristic. It is mainly based on counting of dimensions and ghost numbers. Hidden subtleties might be detected by a more careful investigation. For example, it is conceivable that the $U(1)$ moduli space behaves in some discontinuous way as the Riemann surface degenerates. Whether or not this is the case can only be answered by studying the relevant index theorem. Other potential difficulties are related to the fermionic moduli that we have treated in a rather straightforward way, ignoring possible ambiguities due to the location of picture changing operators. In any case it would be helpful to have an explicit computation of the one-loop four point function. If that turns out to be non-vanishing it will be extremely interesting to see by which mechanism the derivation of the Ward identities must be modified.

References

An effective string theory of Abrikosov–Nielsen–Olesen vortices

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Abstract

We obtain an effective string theory of the Abrikosov–Nielsen–Olesen vortices of the Abelian Higgs model. The theory has an anomaly free effective string action which, when the extrinsic curvature is set equal to zero, yields the Nambu–Goto action. This generalizes previous work in which a string representation was obtained in the London limit, where the magnitude of the Higgs field is fixed. Viewed as a model for long distance QCD, it provides a concrete picture of the QCD string as a fluctuating Abrikosov–Nielsen–Olesen vortex of a dual superconductor on the border between type I and type II.

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1. Introduction

In the dual superconductor picture of confinement [1,2], a dual Meissner effect confines the electric color flux ($Z_3$ flux) to narrow flux tubes connecting quark–antiquark pairs. As a consequence, the energy of the pair increases linearly with their separation, and the quarks are confined in hadrons. Calculations with explicit models of this type [3] have been compared both with experimental data and with Monte Carlo simulations of QCD [4]. To a good approximation, the dual Abelian Higgs model (with a suitable color factor) can be used to describe the results of these calculations. There also is evidence for the dual superconductor picture from numerical simulations of QCD [5]. The Lagrangian $\mathcal{L}_{\text{eff}}$ describing long distance QCD in the dual superconductor picture then has the form:

$$
\mathcal{L}_{\text{eff}} = \frac{1}{2} \left( \partial^\mu C_\nu - \partial^\nu C_\mu + G_{\mu\nu}^S \right)^2 + \frac{1}{4} \partial^\mu \left( \varphi - i g C_\mu \right) \partial_\mu \left( \varphi - i g C_\mu \right) + \frac{\lambda}{4} \left( |\varphi|^2 - \varphi^0 \right)^2.
$$

(1)

The potentials $C_\mu$ are dual potentials, and $\varphi$ is a complex Higgs field carrying monopole charge, whose vacuum expectation value $\varphi_0$ is nonvanishing. All particles are massive: $M_\mu = \sqrt{2\lambda} \varphi_0$, $M_C = g \varphi_0$. The dual coupling constant is $g = \frac{\alpha}{2\pi}$, where $\alpha$ is the Yang–Mills coupling constant. The potentials $C_\mu$ couple to the $q\bar{q}$ pair via $G_{\mu\nu}^S$, a Dirac string whose ends are a source and a sink of electric color flux. The effect of the string is to create a flux tube (Abrikosov–Nielsen–Olesen (ANO) vortex [6]) along some line $L$ connecting the quark–antiquark pair, on

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which the dual Higgs field $\phi$ must vanish. As the pair moves, the line $L$ sweeps out a space time surface $\vec{x}^\mu$, whose boundary is the loop $\Gamma$ formed by the lines of the quark and antiquark trajectories. (See Fig. 1) The monopole field $\phi$ vanishes on the surface $\vec{x}^\mu(\sigma)$ parameterized by $\sigma$, $a = 1,2$: $\phi(\vec{x}^\mu(\sigma)) = 0$. 

Eq. (2) determines the location $\vec{x}^\mu$ of the field configuration $\phi(x^\mu)$.

The long distance $q\bar{q}$ interaction is determined by the functional integral $W$ of the field configuration $\phi(x^\mu)$:

$$W_{\text{eff}}[\Gamma] = \frac{1}{Z_{\text{vac}}} \int D\phi D\phi^* e^{-S_{\text{eff}}[\phi, G^\Delta_{\mu}\phi]}.$$  

The action $S$ includes a gauge fixing term $L_{GF}$,

$$S[C_{\mu}, \phi, G^\Delta_{\mu}] = \int d^4x [L_{\text{eff}} + L_{GF}].$$

$W_{\text{eff}}$ plays the role in the dual theory of the Wilson loop, and is normalized by the vacuum partition function $Z_{\text{vac}}$, in which $G^\Delta_{\mu}$ is not present.

Previous calculations of $W_{\text{eff}}$ were carried out in the classical approximation (corresponding to a flat vortex sheet $\vec{x}^\mu(\sigma)$), and showed [7] that the Landau–Ginzburg parameter $\lambda/g^2$ is approximately equal to $\frac{1}{2}$. This is consistent with recent lattice studies [8] of long distance QCD, and corresponds to a superconductor on the border between type I and type II. In this situation, both particles have the same mass $M = g \phi_0$, the string tension is $\mu = \frac{1}{4} \pi g^2 1$, and the flux tube radius is $a = \sqrt{2}/M$.

The classical approximation neglects the effect of fluctuations in the shape of the flux tube on the $q\bar{q}$ interaction. The goal of this paper is to express $W_{\text{eff}}[\Gamma]$ as a functional integral over all surfaces $\vec{x}^\mu(\sigma)$ to obtain a string representation of the Abelian Higgs model (1). Akhmedov, Chernodub, Polikarpov, and Zubkov [9] obtained such a representation in the London limit $\lambda \rightarrow \infty$, where $|\phi|$ is fixed. Our work can be regarded as an extension of their work to the full Abelian Higgs model.

2. Effective string action for ANO vortices

The integration in (3) goes over all field configurations which include a vortex sheet $\vec{x}^\mu(\sigma)$ bounded by the loop $\Gamma$. We will carry out the integrations over $C_{\mu}$ and $\phi$ in the following way:

1. We will first fix the location of a vortex sheet $\vec{x}^\mu(\sigma)$, and integrate only over field configurations for which $\phi(\vec{x}^\mu(\sigma)) = 0$.
2. We will then integrate over all possible vortex sheets $\vec{x}^\mu(\sigma)$, so that $W_{\text{eff}}$ takes the form

$$W_{\text{eff}}[\Gamma] = \int D\vec{x}^\mu e^{-S_{\text{eff}}[\vec{x}^\mu]}.$$  

In the rest of this paper we will show how to obtain the string representation (5) from the field representation (3), and will give the form of the effective action $S_{\text{eff}}$, and the meaning of the integral over all surfaces in (5).

We first introduce into the integral in Eq. (3) the factor one, written in the form

$$1 = J[\phi] \int D\vec{x}^\mu \delta[\text{Re}\phi(\vec{x}^\mu(\sigma))]$$

$$\times \delta[\text{Im}\phi(\vec{x}^\mu(\sigma))].$$

Eq. (6) defines the Jacobian $J[\phi]$. Given $\phi$, the integral (6) selects the surface $\vec{x}^\mu(\sigma)$ on which $\phi$ vanishes. Inserting (6) into (3) yields

$$W_{\text{eff}}[\Gamma] = \frac{1}{Z_{\text{vac}}} \int D\phi D\phi^* e^{-\frac{1}{2}J[\phi] \int D\vec{x}^\mu \delta}$$

$$\times \left[\text{Re}\phi(\vec{x}^\mu(\sigma))\right] \delta[\text{Im}\phi(\vec{x}^\mu(\sigma))].$$  

(7)
The field integration in (7) is over all field configurations \( \phi(x^\mu) \) which contain a vortex, while the integral over all surfaces forces \( \tilde{x}^\mu \) to lie on the surface \( \phi(x^\mu) = 0 \). We now reverse the order of the field integrals and the string integral in (7). This gives

\[
W_{\text{eff}}[\Gamma] = \frac{1}{Z_{\text{vac}}} \int \mathcal{D}C \mathcal{D} \phi \mathcal{D} \phi^* J[\phi] \delta \left[ \text{Re} \phi(\tilde{x}^\mu(\sigma)) \right] \delta \left[ \text{Im} \phi(\tilde{x}^\mu(\sigma)) \right] e^{-S}.
\]

(8)

The string integral in (8) is over all surfaces \( \tilde{x}^\mu(\sigma) \), while the field integral is over only those field configurations \( \phi(x^\mu) \) for which \( \phi(\tilde{x}^\mu(\sigma)) = 0 \). Eq. (8) has the form (5), with \( S_{\text{eff}} \) given by

\[
\frac{1}{Z_{\text{vac}}} \int \mathcal{D}C \mathcal{D} \phi \mathcal{D} \phi^* J[\phi] \delta \left[ \text{Re} \phi(\tilde{x}^\mu(\sigma)) \right] \delta \left[ \text{Im} \phi(\tilde{x}^\mu(\sigma)) \right] e^{-S}.
\]

(9)

To calculate \( W_{\text{eff}} \) we must evaluate:

(A) \( J[\phi] \), Eq. (6).

(B) The field integration in (9) determining \( S_{\text{eff}} \).

(C) The integration over all surfaces (8) determining \( W_{\text{eff}} \). This integration must be carried out the same way as the integral (6) for \( J[\phi] \).

3. Evaluating the Jacobian \( J[\phi] \)

The Jacobian \( J[\phi] \) in (9) is evaluated for field configurations which vanish on a specific surface \( \tilde{x}^\mu(\sigma) \). To distinguish this surface \( \tilde{x}^\mu(\sigma) \) from the integration variable in the integral (6) defining \( J[\phi] \), we rewrite (6) as

\[
J^{-1}[\phi] = \int \mathcal{D} \tilde{y}^\mu \delta(\text{Re} \phi(\tilde{y}^\mu(\tau))) \delta(\text{Im} \phi(\tilde{y}^\mu(\tau))) \phi(\tilde{y}^\mu(\tau)),
\]

where \( \phi(\tilde{y}^\mu(\sigma)) = 0 \). Eq. (10) expresses the Jacobian \( J[\phi] \) as the inverse of a “string theory,” defined by the integration over all surfaces \( \tilde{y}^\mu(\tau) \). Hence, the representation (8) of the functional integral (3) is a ratio of two string theories. String theories contain anomalies [10], which must not be present in field theories [9,11]. The anomalies of the two string theories appearing in the representation (8) must then cancel.

The \( \delta \) functions in (10) will select those surfaces \( \tilde{y}^\mu(\tau) \) which lie in the neighborhood of \( \tilde{x}^\mu(\sigma) \). Furthermore, the surface \( \tilde{y}^\mu(\tau) \) defines a parameterization of the surface \( \tilde{x}^\mu(\sigma), \sigma \rightarrow \sigma(\tau) \). To evaluate (10), we separate \( \tilde{y}^\mu(\tau) \) into components lying on the surface \( \tilde{x}^\mu(\sigma) \) and components \( y^\lambda_\perp \) lying along the normal to the surface,

\[
\tilde{y}^\mu(\tau) = \tilde{x}^\mu(\sigma(\tau)) + y^\lambda_\perp(\sigma(\tau)) n^\lambda_\perp(\sigma(\tau)),
\]

(11)

where the \( n^\lambda_\perp(\sigma) \) are a set of vectors normal to the sheet \( \tilde{x}^\mu \) at the point \( \sigma \). The integral over the normal components \( y^\lambda_\perp(\sigma) \) is determined by the normal derivatives \( \frac{\partial^2}{\partial \tau^2} |_{\tau=0} \) of the Higgs field evaluated at the surface \( \tilde{x}^\mu \). The integral over the functions \( \sigma(\tau) \) which parameterize components of \( \tilde{y}^\mu \) lying on the surface corresponds to an integration over coordinate parameterizations \( \sigma(\tau) \) of the surface \( \tilde{x}^\mu(\sigma) \). The resulting integral for \( J^{-1}[\phi] \) can be written in the factorized form

\[
J^{-1}[\phi] = J^{-1}_-[\phi] J^{-1}_\sigma[\tilde{x}^\mu],
\]

(12)

where

\[
J^{-1}_- = \int \mathcal{D} y^\lambda_\perp \delta(\text{Re} \phi(\tilde{y}^\mu(\tau))) \delta(\text{Im} \phi(\tilde{y}^\mu(\tau)))
\]

\[
= \text{Det}^{-1} \left[ \frac{i}{2} \left( \epsilon^{AB} \frac{\partial \phi}{\partial y^A_\perp} \frac{\partial \phi^*}{\partial y^B_\perp} \right) |_{\tau=0} \right].
\]

(13)

The quantity \( J^{-1}_\sigma[\tilde{x}^\mu] \) in (12) is the integral over the coordinate parameterizations \( \sigma(\tau) \), given by

\[
J^{-1}_\sigma[\tilde{x}^\mu] = \int \mathcal{D} \sigma \text{Det}[\sqrt{g}(\sigma(\tau))],
\]

(14)

where \( \sqrt{g} \) is the square root of the determinant of the induced metric \( g_{\alpha \beta} = \partial_\alpha \tilde{x}^\mu \partial_\beta \tilde{x}^\mu \) evaluated on the worldsheet \( (\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}) \). \( J^{-1}_\sigma \) has the form of a string theory in two dimensions.
Up to now, we have not specified how either the integral over $\sigma'(x)$ in (14) or the integral over the parameterizations of the surface $\tilde{x}^\mu(\sigma)$ in (8) is to be carried out. The important thing is that they be done in a consistent way. We have carried out these integrations using the techniques of Polyakov [10]. This procedure yields
\[
J_\parallel^{-1}[\tilde{x}^\mu] = \text{Det}_\parallel^{-1}[-\nabla^2_\sigma] \Delta_{FP}.
\] (15)
The quantity $-\nabla^2_\sigma$ is the two dimensional Laplacian on the surface $\tilde{x}^\mu(\sigma)$,
\[
-\nabla^2_\sigma = -\frac{1}{\sqrt{g}} \partial_a g^{ab} \sqrt{g} \partial_b,
\] (16)
and
\[
\Delta_{FP} = \exp\left(-\frac{26}{48\pi} \int d^2\sigma \sqrt{g} \left(\partial_a \ln \sqrt{g}\right)^2 - \mu \int d^2\sigma \sqrt{g}\right)
\] (17)
is a Faddeev–Popov determinant arising from fixing the nonphysical parameterization degrees of freedom in (14). Eqs. (12)–(17) determine $J[\phi]$. All the dependence of $J[\phi]$ on the field $\phi$ is contained in $J_\parallel[\phi]$.

4. The field integration determining $S_{\text{eff}}$

The Wilson loop $W_{\alpha\beta}[\Gamma]$ describes the $q\bar{q}$ interaction at distances greater than the flux tube radius $a$. The important fluctuations at such distances are string fluctuations described by the integral (8) over all surfaces $\tilde{x}^\mu(\sigma)$. The field integrations in (9) determining the effective string interaction must then be evaluated in the steepest descent approximation around the classical solution $\phi_\text{class}$, $\mu_\text{class}$. The boundary condition on this solution is $\phi_\text{class}(\tilde{x}^\mu(\sigma)) = 0$. The corresponding action $S_{\text{class}}[\tilde{x}^\mu]$ is the value of the action at the classical solution:
\[
S_{\text{class}}[\tilde{x}^\mu] = S[\tilde{x}^\mu, \phi_\text{class}, \mu_\text{class}].
\] (18)
The fields $\phi_\text{class}$, $\mu_\text{class}$ minimize the action for a fixed location of the vortex sheet $\tilde{x}^\mu$.

The steepest descent calculation of (9) around the classical solution gives
\[
e^{-S_{\text{eff}}[\tilde{x}^\mu]} \equiv \frac{1}{Z_{\text{vac}}} J_\parallel[\tilde{x}^\mu] \int \mathcal{D} \phi \mathcal{D} \mu \exp J_\parallel[\phi] \times \delta[\text{Re} \phi(\tilde{x}^\mu(\sigma))] \delta[\text{Im} \phi(\tilde{x}^\mu(\sigma))]
\]
\[
e^{-S_{\text{class}}[\tilde{x}^\mu]} \frac{1}{Z_{\text{vac}}} \text{Det}^{-1/2}[G^{-1}] J_\parallel[\tilde{x}^\mu],
\] (19)
where $G^{-1}$ is the inverse Green’s function determined by the quadratic terms in the expansion of the action (4) about $\phi_\text{class}$, $\mu_\text{class}$. The determinant of $G^{-1}$ must be evaluated numerically. The $\delta$ functions in (19), which specify the location of the vortex, cause the field integration to produce a Jacobian which cancels $J_\parallel$, so that only $J_\parallel$ appears on the right hand side of (19).

The effect of the determinant of $G^{-1}$ is to renormalize the parameters in $S_{\text{class}}$. Short distance renormalization effects in the dual theory are cut off at the flux tube radius $a$. These renormalizations are not very important, as all the modes in $G^{-1}$ have masses larger than $a^{-1}$.

5. Parameterizing the integral over all surfaces

In order to carry out the integration $\mathcal{D} \tilde{x}^\mu$ of $e^{-S_{\text{eff}}}$ over all surfaces, it is convenient to choose particular coordinates. We select some fixed sheet $\tilde{x}^\mu$, and define vectors $\vec{n}_A$, $A = 3, 4$, normal to the sheet:
\[
\vec{n}_A(\sigma) \partial_\sigma \tilde{x}^\mu(\sigma) = 0, \quad a = 1, 2, \quad A = 3, 4.
\] (20)
For points $x^\mu$ close to the sheet $\tilde{x}^\mu$ we can write
\[
x^\mu = \tilde{x}^\mu(\sigma) + \vec{n}_A(\sigma) x_A^A,
\] (21)
which defines the coordinate transformation $x^\mu \rightarrow \sigma, x_A^A$.

We now use these coordinates to parameterize the surface $\tilde{x}^\mu(\sigma)$. Doing this will allow us to break up
the integral (8) over \( \tilde{x}^\mu \) into an integral over distinct surfaces and an integral over parameterizations of the surface \( \tilde{x}^\mu \). For a given parameterization \( \tilde{x}^\mu(\sigma) \), we choose a reparameterization \( f(\sigma) \) defined so that

\[
\tilde{x}^\mu(f(\sigma)) = \tilde{x}^\mu(\sigma) + \tilde{\eta}_A^a(\sigma) \tilde{x}^A_\perp(\sigma) .
\]  

(22)

Eq. (22) requires that the point \( \tilde{x}^\mu(f(\sigma)) \) lie on the line normal to the surface \( \tilde{x}^\mu \) at the point \( \tilde{x}^\mu(\sigma) \). The term \( \tilde{\eta}_A^a(\sigma) \tilde{x}^A_\perp(\sigma) \) represents the displacement of the surface \( \tilde{x}^\mu \) from the surface \( \tilde{x}^\mu \). We can then write \( \tilde{x}^\mu(\sigma) \) as

\[
\tilde{x}^\mu(\sigma) = \tilde{x}^\mu(\tilde{\sigma}(\sigma)) + \tilde{\eta}_A^a(\tilde{\sigma}(\sigma)) \tilde{x}^A_\perp(\tilde{\sigma}(\sigma)) ,
\]

(23)

where \( \tilde{\sigma}(\sigma) \equiv f^{-1}(\sigma) \). This allows us to write the integration (8) over \( \tilde{x}^\mu(\sigma) \) as an integration over distinct surfaces (labeled by \( \tilde{x}^A_\perp \)) and an integration over parameterizations \( \tilde{\sigma}(\sigma) \). The integral over \( \tilde{\sigma}(\sigma) \) produces a factor \( \Delta_{pp} \text{Det}^{-1}[ - \nabla_s^2 ] = J_1^{-1} \), which cancels the factor \( J_1 \) in \( e^{-S_{\text{class}}} \), (19), and we obtain

\[
W_{\text{eff}} = \frac{1}{Z_{\text{vac}}} \int \mathcal{D} \tilde{x}^A_\perp e^{-S_{\text{class}}[\tilde{x}^\mu]} \text{Det}^{-1/2}[G^{-1}] .
\]

(24)

The integration over \( \tilde{x}^A_\perp \) is cut off at distances of the order of the string parameter \( a \).

Eq. (24), which gives the string representation of the Abelian Higgs model, is the basic result of this paper. The result (24) could also have been obtained by introducing a fixed surface \( \tilde{x}^\mu \) at an earlier stage and replacing the right hand side of Eq. (6) by the product of \( J_1[\phi] \) and an integral over \( \mathcal{D} \tilde{x}^A_\perp \). We have chosen a more general approach because we can also derive, from Eq. (19), a string representation which does not refer to local coordinates.

The action \( S_{\text{class}}[\tilde{x}^\mu] \) appearing in (24) does not depend on the parameterization \( \tilde{\sigma}(\sigma) \), and hence is expressed in terms only of \( \tilde{x}^\mu \) and \( \tilde{x}^A_\perp \) via (23). To evaluate \( S_{\text{class}}[\tilde{x}^\mu] \), we must solve the classical equations of motion for \( \phi_{\text{class}} \) and \( C^a_{\mu} \). These equations, when written in generalized coordinates \( \sigma, x^A_\perp \), explicitly contain the extrinsic curvature \( \mathcal{R}_{ab} \), defined by the equation

\[
\mathcal{R}_{ab}(\sigma) = - (\partial_a n_A(\sigma))(\partial_b \tilde{x}^\mu(\sigma)) .
\]

(25)

The \( n_A^\mu \) are normal vectors to the sheet \( \tilde{x}^\mu \).

6. The Nambu–Goto action

The classical action is a function of the extrinsic curvature. We now evaluate the action for vortex sheets which have a radius of curvature \( R_V \) much greater than the flux tube radius \( a \). In this limit, we can set the extrinsic curvature to zero in the action \( S_{\text{class}} \). Then (18) becomes

\[
S_{\text{class}} = S[ \tilde{x}^\mu, \phi_{\text{class}}, C^a_{\mu} ] = S_0 ,
\]

(26)

where \( \phi_{\text{class}}, C^a_{\mu} \) is the solution of the approximate classical equations of motion obtained by neglecting terms containing the extrinsic curvature. Evaluating \( S_0 \), we obtain

\[
S_0 = \frac{4}{\pi} \pi \phi_0^2 \int d^2s \sqrt{g} .
\]

(27)

Thus, the effective string action for ANO vortices having a radius of curvature \( R_V \) much greater than the flux tube radius \( a = \sqrt{\frac{M}{2}} \) is the Nambu–Goto action (27) with a string tension \( \mu = \frac{4}{\pi} \pi \phi_0^2 \) (the classical string tension). We can use the relation \( \alpha' = 1/2\pi\mu \) between the string tension \( \mu \) and the slope \( \alpha' \) of the leading Regge trajectory to determine the vacuum expectation value \( \phi_0 \) of the monopole condensate. Using the value \( \alpha' \approx 9(\text{GeV})^{-2} \) for the slope of the \( \rho \) trajectory gives \( \phi_0 \approx 210 \text{ MeV} \).

The difference,

\[
\delta S = S_{\text{class}} - S_0 ,
\]

(28)

gives the change in the action due to the extrinsic curvature. Since \( S_{\text{class}} \) is the value of the action at an exact solution of the equations of motion, and \( S_0 \) is its value at an approximate solution, we expect \( \delta S < 0 \).

The calculation of \( \delta S \) in not straightforward, and has been considered by a number of other authors [12], whose results are not in complete agreement. We have been working on this problem, but have not yet obtained any definite result for \( \delta S \) in the Abelian Higgs model, and therefore cannot give an explicit form for the corrections to the Nambu–Goto action.
7. Conclusions

The dual superconducting description of long distance QCD yields the effective string theory (24). It has an action which, in the limit where the extrinsic curvature is neglected, yields the Nambu–Goto action. Thus, general consequences of string models, used to describe Regge trajectories and the spectra of hybrid mesons, can also be regarded as consequences of a dual superconducting description.

Eq. (24) is the end result of a series of steps used to derive an effective string theory from the partition function of a renormalizable quantum field theory having vortex solutions. We are unaware of any other method to achieve this end. Previous work [1,9] considered only the singular London limit of the Abelian Higgs model, for which the slope of the Higgs field at the origin is infinite. Our result provides a theoretical framework which relates a low energy effective string theory to an underlying field theory.

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References

Classical string solutions in effective infrared theory of $SU(3)$
gluodynamics

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Abstract

We investigate string solutions to the classical equations of motion ("classical QCD strings") for a dual Ginzburg–Landau model corresponding to $SU(3)$ gluodynamics in an abelian projection. For a certain relation between couplings of the model the string solutions are defined by first order differential equations. These solutions are related to vortex configurations of the Abelian Higgs model in the Bogomol'nyi limit. An analytic expression for the string tension is derived and the string-string interactions are discussed. Our results imply that the vacuum of $SU(3)$ gluodynamics is near a border between type-I and type-II dual superconductivity.

1. Last years the problem of color confinement in $SU(N)$ Yang–Mills theories has been intensively studied in the framework of the abelian projection approach [1]. This approach is based on a partial gauge fixing which reduces the gauge symmetry from non-abelian gauge group to its (maximal) abelian subgroup. The diagonal elements of the gluon field transform under residual gauge transformations as abelian gauge fields, while the off-diagonal elements transform as abelian matter vector fields. Due to compactness of the Yang–Mills gauge group the residual abelian subgroup is also compact. This leads to appearance of abelian monopoles in the abelian gauge. According to the dual superconductor scenario [2] the color confinement may be explained at the classical level: if monopoles are condensed then a string forms between color charges. This string is an analog of the Abrikosov string [3] in a superconductor and the abelian monopoles are playing the role of the Cooper pairs.

The dual superconductor mechanism has been confirmed by numerous lattice simulations of $SU(2)$ Yang–Mills theory [4]. Moreover, the vacuum of $SU(2)$ Yang–Mills theory in the abelian projection was shown [5] to be close to the border between type-I and type-II dual superconductors (masses of the monopole and the dual gauge boson are approximately the same). There are also strong numerical indications that the vacuum of $SU(3)$ gluodynamics exhibits dual superconductor properties [6]. Analytical investigations [7] of the string configurations in a non-abelian dual superconductor model of $SU(3)$ gluodynamics shows that the dual vacuum is close to the border between type-I and type-II superconductivity. In Ref. [8] similar conclusion has been derived where the $SU(3)$ confining string was related to the

In this paper we investigate an effective abelian model of $SU(3)$ gluodynamics suggested in Ref. [10]. This model is based on dual superconductivity conjecture which is supposed to describe infrared properties of the $SU(3)$ gluodynamics vacuum. An essential difference between $SU(2)$ and $SU(3)$ Yang–Mills theories in an abelian projection is the presence of two independent string configurations in the later. The existence of two string types guarantees the colour neutrality of the quark bound states [11]. A possible string representation of this model in a regime of extreme type-II superconductivity (the monopole mass is infinite) has been discussed in Refs. [11,12]. The classical string configurations in the static baryon have been studied numerically in Refs. [13,14]. The results of Ref. [14] suggest that the vacuum of $SU(3)$ gluodynamics lies near a border between type-I and type-2 superconductivity. Below we study analytically the confining string corresponding to the magnetically charged particles. We choose the following anzatz for the field configuration of the static straight vortex parallel to the $z$-axis:

$$\chi_i = \eta_i \rho^{-\alpha} e^{-in\varphi}, \quad n_i \in \mathbb{Z}, \quad i = 1,2,3, \quad (3)$$

$$B^a = -\frac{1}{g\rho}(\varphi^a \cdot e_\varphi), \quad a = 3,8. \quad (4)$$

where we have used the standard notations for the cylindrical coordinates $(\varphi, \rho, z, t)$ in four dimensional space. Note that the vortex configuration does not depend on $z$- and $t$-coordinates.

The finite energy configuration must satisfy the condition $D_{\mu}^{(i)} \chi_i = 0$ at spatial infinity. This condition implies the quantization of the total flux $\Phi^a = \int \mathrm{d}x \mathrm{d}y H^a$ of the vortex magnetic field $H^a = F^{a}_{12}$:

$$\Phi_i = \sum_{a=3,8} \epsilon_i^a \Phi^a = \frac{2\pi n_i}{g}$$

$$\Phi^a = \frac{4\pi}{3g} \sum_{i=1}^{3} \epsilon_i^a n_i, \quad (5)$$

where we have used the relation $\sum_{i=1}^{3} \epsilon_i^a \epsilon_i^b = 3\delta^{ab}/2$. In particular this condition implies that the strings must carry both components, $\Phi^3$ and $\Phi^8$, of the fluxes.

\[\text{In this paper the summation over the Latin indices } i \text{ and } j \text{ is taken only if explicitly indicated.}\]
According to Eq. (2) the integer winding numbers \( n_i \) satisfy the following relation:
\[
\sum_{i=1}^{3} n_i = 0.
\]
(6)

3. To analyse the vortex solutions we begin with the Bogomol’ny method [15]. Consider the energy density per unit length of the string (string tension):
\[
\sigma = \int d^2x \left[ \frac{1}{2} (H^4)^2 + \sum_{a=1,2} \left( \frac{1}{2} \sum_{i=1}^{3} |D_a^{(i)} \xi_i|^2 
\right.
\left. + \lambda \left( |\xi_i|^2 - \eta_i^2 \right)^2 \right] \right].
\]
(7)
The first term in this equation can be rewritten as follows: \( \Sigma_{a=1,2} (H^4)^2 / 2 = \Sigma_{i=1}^{3} H_i^2 / 3 \), where \( H_i = \Sigma_{a=1}^{2} e_a H^a \). The second term in Eq. (7) can be represented as follows:
\[
\sum_{a=1,2} |D_a^{(i)} \xi_i|^2 = (D_a^{(i)} + iD_a^{(i)}) \xi_i^2 
\]
\[
\implies 2 \epsilon_{a\beta} \partial_a J_\beta^{(i)} 
\]
\[
\pm i \xi \xi = D_a^{(i)} D_a^{(i)} \xi_i,
\]
(8)
where the current \( \vec{J}_{a}^{(i)} = i \vec{\xi}^{(i)} D_{a}^{(i)} \vec{\xi} \) vanishes faster than \( \rho^{-1} \) as \( \rho \to \infty \) for the configuration with a finite energy. Therefore the second term in the r.h.s. of Eq. (8) gives zero contribution to the string tension (7). The last term in Eq. (8) can be simplified:
\[
i \xi \xi = -gH_i |\xi|^2.
\]
Thus the string tension (7) can be expressed in the form:
\[
\sigma = \int d^2x \left[ \frac{3}{2} (D_a^{(i)} + iD_a^{(i)}) \xi_i^2 
\right.
\left. + \frac{1}{2} H_i^2 + \frac{3}{2} g H_i |\xi|^2 + \lambda \left( |\xi|^2 - \eta_i^2 \right)^2 \right] 
\]
\[
= \int d^2x \left[ \frac{3}{2} (D_a^{(i)} + iD_a^{(i)}) \xi_i^2 
\right.
\left. + \frac{1}{2} \left( H_i^2 + \frac{3}{2} g |\xi|^2 - \eta_i^2 \right) \right] 
\]
\[
\pm \frac{3}{2} g \eta_i^2 \Phi_i + \left( \lambda - \frac{3 g^2}{16} \right) \left( |\xi|^2 - \eta_i^2 \right)^2,
\]
where the magnetic flux \( \Phi_i \) is quantized according to Eq. (5). The sign in front of the flux term is chosen so that the contribution of this term is positive.

The analogue of the Bogomol’ny limit [15] is defined by the following condition:
\[
\lambda = -\frac{3 g^2}{16},
\]
(9)
which guarantees that the potential term in Eq. (9) vanishes. Contrary to the case of Abelian Higgs model this condition does not imply an equivalence of the vector and scalar boson masses. The equivalence holds only in the physically relevant case when the vacuum expectation values of the Higgs monopole fields \( \xi_i \) are degenerate, \( \eta_i = \eta_0 \), \( i,j = 1,2,3 \) (the color symmetry is unbroken).

In this case the vortex configuration is defined by the first order field equations:
\[
(D_a^{(i)} + iD_a^{(i)}) \xi_i = 0;
\]
(10)
or, for anzatz (4),
\[
f_i(\rho) + (\nu_i(\rho) - n_i) f_i(\rho) \rho^{-1} = 0,
\]
\[
\pm \nu_i(\rho) + g^2 \eta_i^2 f_i(\rho) = 0,
\]
(11)
where \( \nu_i = \sum_{a=1,2} e_a v^a \) and the prime denotes the differentiation with respect to \( \rho \). The system of equations of motion (11) is over-defined and therefore a classical vortex solution to these equations does not exist in general case. However we will show below that for a physical case the classical solution can be found.

According to Eqs. (5) and (9) the solutions to the equations of motion (10) or (11) saturate an analog of the Bogomol’ny bound for the string tension:
\[
\sigma = \pi \sum_{i=1}^{3} n_i \eta_i^2.
\]
(12)
Thus parallel strings which carry the same flux do not interact with each other since energy of \( N \) strings with the same flux \( \Phi_n \) is the same as the energy of the single string with the flux \( N \Phi_n \). This is a general property of classical solutions in the Bogomol’ny limit. The vortices start to interact when relation (9) is not satisfied. A similar situation hap-
pens on a border between type-I and type-II superconductors.

4. The $SU(3)$ string in an abelian projection may be considered as a composite [11,12] of the three elementary strings with the winding numbers $n_i$ of the Higgs fields $\chi_i$ subjected to relation (6). The phase of the Higgs field must be singular at the center of the elementary string with non-zero winding number $n_i$. Since all the Higgs field phases are subjected to the condition (2) the center of the elementary strings must coincide. The tension of the $SU(3)$ string depends on the fluxes $\Phi^a$ carried by the string and the fluxes are being related to the winding numbers $n_i$, in accordance with the quantization condition (5).

Consider the classical field configuration for the abelian counterpart of the $SU(3)$ string in the Bogomol’ny limit. The lowest energy string configuration in the neutral quark–anti-quark system has the winding numbers 1, −1 and 0 (there are three string configurations corresponding to different colors of the quarks, or, equivalently, to different permutations of the $n_i$’s). Note that the same winding numbers are carried by segments of the $SU(3)$ classical string configuration in the baryon [11].

For the sake of generality we consider below the classical string solution with the winding numbers $n$, $−n$ and 0. The solution to the equations of motion (11) is:

$$f_i(\rho) = f_{\nu|n_i}^{ANO}(\rho, g\eta),$$

$$v^a = \frac{2}{3} \sum_{i=1}^{3} \text{sign}(n_i) e_i^a v_{\nu|n_i}^{ANO}(\rho, g\eta).$$

(13)

where $f_{\nu|n_i}^{ANO}(\rho,m)$ and $v_{\nu|n_i}^{ANO}(\rho,m)$ are the characteristic functions of the Abrikosov–Nielsen–Olesen (ANO) vortex solution [3,9] in the Bogomol’ny limit of the Abelian Higgs model (AHM). These functions are defined by the first order Bogomol’ny equations in the AHM [16]:

$$f_{\nu|n_i}^{ANO} \pm (v_{\nu|n_i}^{ANO} - n) f_{\nu|n_i}^{ANO} \rho^{-1} = 0,$$

$$\pm v_{\nu|n_i}^{ANO}(\rho) + m^2 \left( f_{\nu|n_i}^{ANO} \right)^2 - 1) \rho = 0,$$

$$n > 0.$$  

(14)

The functions $f_{\nu|n}^{ANO}$ and $v_{\nu|n}^{ANO}$ have been determined numerically in Ref. [16]. Note that $f_{\nu|n_0}^{ANO} = 1$ and $v_{\nu|n_0}^{ANO} = 0$.

In the degenerate case $\eta = \eta_i$, $i = 1,2,3$, the characteristic functions $v^a$ for the string with the winding numbers $n, −n$ and 0 have a simple form:

$$v^a(\rho) = \frac{\Phi^a}{2 \pi n} v_{\nu|n}^{ANO}(\rho, g\eta),$$

$$n > 0,$$

(15)

where the fluxes $\Phi^a$ are given by Eq. (5) and according to Eq. (12) the string tension is

$$\sigma_n = 2 \pi n \eta^2.$$  

(16)

The solutions for a general case ($n_i \neq 0, i = 1,2,3$) are unlikely to exist due the fact that system (11) is over-defined. Indeed according to the definition of the vectors $v_i$ we need the fulfillment of the following condition: $\sum_{i=1}^{3} n_i v_{\nu|n_i}^{ANO}(\rho, g\eta) = 0$ with given vorticities $n_i$ and integer parameters $\eta_i$. Since the functions $v_{\nu|n_i}^{ANO}$ are solutions of the nonlinear equations the linear dependence of these functions would be unnatural. Thus the Bogomol’ny bound (12) for the string tension is likely to be reached only for the strings composed of two elementary abelian strings (the third vorticity component must be zero).

5. The interaction between vortices is defined by their Higgs and gauge boson field profiles. The Higgs (gauge) boson mediated interaction between two parallel identical vortices is attractive (repulsive). In the Bogomol’ny limit the Higgs and gauge boson fields acts in the same range since at large distances the profile functions $v_{\nu|n}^{ANO}(\rho,m_i)$ and $f_{\nu|n}^{ANO}(\rho,m_i)$ behave like $e^{-m_i \rho}$ and $e^{-m_i \rho}$ respectively. According to Eq. (13) the mass $m_i$ equals to the mass $m_2$ in the Bogomol’ny limit. Thus the interactions compensate each other and the net force between vortices is zero \textsuperscript{3}.

When $\lambda < 3 g^2 / 16$ the vortices are attracting since the attraction due to exchange of the Higgs boson

\textsuperscript{3}Note that the $[U(1)]^2$ vortices are not interacting for all vortex-vortex separations contrary to, e.g., $Z_3$ vortices (these are also supposed [17] to be relevant to confinement in QCD.). The potential between $Z_3$ vortices may be attractive at short separations and repulsive at large distances [18].
prevails over the gauge boson attraction. In this region of the parameter space the strings with identical fluxes tend to join. For $\lambda > 3g^2/16$ the strings repel each other and therefore the strings with multiple vorticities are unstable; they tend to decay on strings with smallest vorticities.

Using similar arguments one can show that the strings with different fluxes (winding numbers) are always attractive in the type-I region. In the type-II region the gauge-mediated force between strings $A$ and $B$ is proportional to $\sum_{\alpha=1,2} F_{\alpha}^a F_{\beta}^b \propto \sum_{i=1}^n n_i^a n_i^b$. The interaction between the strings is attractive if this number is negative and repulsive otherwise.

In an ordinary type-I (type-II) superconductor the Abrikosov vortices attract (repel) each other and the Bogomol’ny limit is a border line between the two types. By analogy, relation (9) defines the border line between different types of the dual superconductivities.

The closeness of the $SU(3)$ gluodynamics vacuum to the border between type-I and type-II may be checked with the help of Eq. (16) which relates vacuum expectation values of the monopole fields $\eta$ and the string tension at the border. Using the estimate [10,13,14,19] for the vacuum expectation value of the Higgs fields, $\eta \approx 175$ MeV, we get $^3$ the value of the tension of the string with the lowest flux, $\sigma = \sigma_r = (440$ MeV) which is quite close to a phenomenological value. Thus the gluodynamics vacuum is likely to be near the border between type-I and type-II dual superconductivity.

Conclusions. The effective model of $SU(3)$ gluodynamics in an abelian projection possesses classical vortex solutions (“QCD strings”). For a certain relation between couplings of this model (16$\lambda = 3g^2$) the vortex solutions are related to the solutions of the Abelian Higgs model in the Bogomol’ny limit. This relation also defines the border between type-I and type-II dual superconductivity. The string tension in the Bogomol’ny limit is quantized according to Eq. (12). Using this formula and the phenomenological value of the vacuum expectation value of the monopole field we show that the vacuum of the $SU(3)$ gluodynamics lies near the border between type-I and type-II dual superconductivity.

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$^3$ Note that the notations of Refs. [10,13,14,19] differ from our notations.
Abelian vectors and self-dual tensors in six-dimensional supergravity

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Abstract

In this note we describe the most general coupling of abelian vector and tensor multiplets to six-dimensional (1,0) supergravity. As was recently pointed out, it is of interest to consider more general Chern-Simons couplings to abelian vectors of the type \( H^r = dB^r - 1/2 c^{r,s} A^r dA^s \), with \( c^r \) matrices that may not be simultaneously diagonalized. We show that these couplings can be related to Green-Schwarz terms of the form \( B^r c_{ab} F^a F^b \), and how the complete local Lagrangian, that embodies factorized gauge and supersymmetry anomalies to be disposed of by fermion loops is uniquely determined by Wess-Zumino consistency conditions, aside from an arbitrary quartic coupling for the gauginos.

Vector multiplets coupled to variable numbers of tensor multiplets in six-dimensional (1,0) supergravity arise naturally in perturbative type-I vacua [1], that are related by string dualities to non-perturbative heterotic and M-theory vacua. The low-energy field equations of these (1,0) models have revealed the explicit realization of a peculiar aspect of the physics of branes: singularities in the gauge couplings appear for particular values of the scalars in the tensor multiplets [2], and can be ascribed to phase transitions [3] in which a string becomes tensionless [4].

In these models, that arise as parameter-space orbifolds (orientifolds) [5] of K3 reductions of the type-IIB string, several antisymmetric tensors take part in a generalized Green-Schwarz mechanism [6,2], and the resulting residual anomaly polynomial has the form

\[
c^i c^j \eta_{rs} \text{tr}_s F^2 \text{tr}_r F^2,
\]

where the \( F^r \)’s denote collectively the Riemann curvature and the Yang-Mills field strengths, the \( c^r \)’s are a collection of constants \((x \text{ and } y \text{ run over the various semi-simple Lie factors in the gauge group and over the Lorentz group)}\) and \( \eta \) is the Minkowski metric for \( SO(1,n) \), with \( n \) the number of tensor multiplets. Taking into account only the gauge part of the anomaly polynomial, the corresponding Green-Schwarz term,

\[
B^r c^i \text{tr}_i F^2,
\]

contains two derivatives, and thus belongs to the low-energy effective action. Consequently, the Lagrangian has a “classical” gauge anomaly, that the Wess-Zumino conditions [7] relate to a “classical” supersymmetry anomaly [8]. This model is thus rather different from the other supergravities, since it is naturally determined by Wess-Zumino conditions.

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rather than by the usual requirement of local supersymmetry. As a consequence, the supersymmetry algebra contains a two-cocycle, and the resulting Lagrangian is determined up to an arbitrary term proportional to the square of a bilinear in the gauginos [9]. Moreover, the divergence of the energy-momentum tensor is non-vanishing [10], as is properly the case for a theory that has gauge anomalies but no gravitational anomalies, that could be accounted for introducing higher-derivative couplings.

The coupling of (1,0) supergravity to n tensor multiplets was originally studied in [11] to lowest order in the fermi fields, while [12] considered the coupling to a single tensor multiplet and to vector and hyper-multiplets to all orders in the fermi fields. In this case the kinetic term is proportional to \( ae^b + be^{-\phi} \) [2], where \( a \) and \( b \) are constants and \( \phi \) is the scalar in the tensor multiplet, while [12] actually deals with the particular case \( a = 0 \), in which the anomaly polynomial vanishes and no tensionless string transition occurs. The general coupling to non-abelian vectors and self-dual tensor multiplets was worked out to lowest order in the fermi fields in [2]. In this covariant formulation the requirement of supersymmetry gives non-integrable equations, and the divergence of the vector equation gives the covariant anomaly. The same model was then reconsidered again to lowest order in the fermi fields in [8] in the consistent formulation, requiring the closure of the Wess-Zumino conditions, that relate the consistent gauge anomaly to the supersymmetry anomaly. Additional couplings, as well as the inclusion of hyper-multiplets, were then considered in [13]. The complete coupling to non-abelian vector and tensor multiplets was finally obtained in [9] in the consistent formulation and in [10] in the covariant formulation.

One can actually consider a slight modification of these couplings, resulting from the inclusion in the low-energy Lagrangian of Green-Schwarz terms non-diagonal in a set of \( U(1) \) gauge groups. A direct indication of how this can be worked is provided by a recent paper of Cremmer et al. [14] where, in the search of the highest-dimensional origin of various three-dimensional scalar sigma models, it is shown that the “oxidation endpoint” of three-dimensional supergravity with scalars in \( F_4 \) sigma models is (1,0) six-dimensional supergravity coupled to two tensor multiplets and two abelian vector multiplets, with non-diagonal Chern-Simons couplings. The corresponding kinetic terms are also non-diagonal, compatibly with the abelian gauge invariances. The resulting model actually motivates the study of more general Green-Schwarz couplings to abelian vectors of the form

\[
B' c^{ab} F_a F^b ,
\]

where the indices \( a,b \) run over the different \( U(1) \) gauge groups, while the symmetric matrices \( c' \) may not be simultaneously diagonalized. In this paper we describe this setting in detail, constructing the general coupling of (1,0) six-dimensional supergravity to \( n \) tensor multiplets and abelian vector multiplets, and then pointing out the connection with the particular case considered in [14].

The \( n \) scalars in the tensor multiplets parameterize the coset space \( SO(1,n)/SO(n) \), and are described by the \( SO(1,n) \) matrix [11]

\[
V = \begin{pmatrix} v_r \\ \chi^m \end{pmatrix} .
\]

All spinors are symplectic Majorana-Weyl, the tensorinos \( \chi^m \) (\( m = 1, \ldots, n \)) being right-handed and the gravitino \( \psi_\mu \) and the gauginos \( \lambda \) being left-handed (we follow the notation of [9]). The tensor fields \( B'_{\mu
u} \) are valued in the fundamental representation of \( SO(1,n) \), and their field strengths include generalized Chern-Simons 3-forms of the vector fields [14] according to

\[
H^r_{\mu
u
\rho} = 3 \partial_{\mu} B'^r_{\nu\rho} - 3 c'^{rab} A^a_{\mu} \partial_{\nu} A^b_{\rho} ,
\]

where the \( c'^{rab} \) are constants that determine the gauge part of the residual anomaly polynomial

\[
c'^{ab} c'^{cd} F^a \wedge F^b \wedge F^c \wedge F^d .
\]

In the complete theory, the anomaly induced by this term would cancel against the contribution of fermion loops, while the irreducible part of the anomaly polynomial is directly absent in consistent models [15,2]. The gauge invariance of \( H' \) requires that

\[
\delta B' = \frac{1}{2} c'^{ab} A^a dA^b .
\]
The tensor fields satisfy (anti)self-duality conditions, conveniently summarized as [8,9]

\[ G_{\mu
u} \hat{H}_{\mu
u} = \frac{1}{6} \epsilon_{\mu\nu\rho\sigma} \hat{r}_{\mu
u}, \]

where \( G_{\mu
u} = v_{\mu}v_{\nu} + \epsilon_{\mu\nu}^{\mu\nu} \)

\[ \hat{H}_{\mu
u} = \frac{1}{8} v^m (\hat{\chi}^{\nu}_{\mu
u} \chi^m) \]

\[ + \frac{i}{8} \epsilon_{\mu
u}^{\nu\mu} (\hat{\chi}^{\mu}_{\mu\nu} \chi^\nu), \]

with \( \hat{H} \) the supercovariantization of \( H \). The model can be constructed using the method of [9], where the complete field equations for the case of arbitrary numbers of tensor and non-abelian vector multiplets were obtained requiring the closure of Wess-Zumino consistency conditions. The completion to all orders in the fermi fields of the equations of motion is obtained requiring the closure of the commutator of two supersymmetry transformations on the fermionic field equations. All the resulting equations may be conveniently derived from the Lagrangian

\[ e^{-1} \mathcal{L} = -\frac{i}{2} R + \frac{1}{16} G_{\mu
u} H^{\mu\nu} H_{\mu\nu} - \frac{1}{2} \partial^\mu v^r \partial^\mu v^r \]

\[ + \frac{i}{16} \epsilon_{\mu\nu\rho\sigma} (\hat{r}_{\mu
u}) F_{\mu\nu}, \]

\[ - \frac{i}{2} \psi_{\mu} \gamma^{\mu\nu} \partial_{\nu} D_{\mu} \left[ \frac{1}{2} (\omega + \hat{H}) \right] \psi_{\nu}, \]

\[ + \frac{i}{8} \psi_{\mu} (H + \hat{H}) \gamma^{\mu\nu} \partial_{\nu} \psi_{\nu}, \]

\[ + \frac{i}{16} \epsilon_{\mu\nu\rho\sigma} (\hat{r}_{\mu
u}) \chi_{\rho\sigma}, \]

\[ + \frac{i}{2} \chi_{\mu\nu} \partial_{\mu} \chi_{\nu}, \]

\[ + \frac{i}{24} \hat{H}^{\mu\nu} \chi_{\mu\nu}, \]

\[ + \frac{i}{4} \epsilon_{\mu\nu\rho\sigma} (\hat{r}_{\mu
u}) \chi_{\rho\sigma}, \]

\[ + \frac{i}{4} \epsilon_{\mu\nu\rho\sigma} (\hat{r}_{\mu
u}) \chi_{\rho\sigma}, \]

\[ + \alpha \epsilon_{\mu\nu\rho\sigma} (\hat{r}_{\mu
u}) \chi_{\rho\sigma}, \]

after imposing the (anti)self duality conditions. The last term, proportional to the arbitrary parameter \( \alpha \), vanishes identically in the case of a single abelian vector multiplet. Since the kinetic terms of the vector fields are non-diagonal, this generalization is only possible in the abelian case.

The variation of this Lagrangian with respect to gauge transformations gives the gauge anomaly

\[ \mathcal{A}_{\lambda} = -\frac{1}{8} \epsilon_{\mu
u\rho\sigma} c_{\lambda \epsilon} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \]
while the variation with respect to the supersymmetry transformations
\[ \delta e_{\mu} = -i(\bar{\psi}_a \gamma^\mu \psi_b), \quad \delta B_{\mu} = i\tilde{e}^c (\bar{\psi}_c \gamma_\mu \gamma^\nu \psi_d) \]
\[ + \frac{i}{2} \gamma^\nu (\bar{\psi}_a \gamma_\mu \psi_b) \epsilon - c_{\mu a b} (A_{\nu b}^a \delta A_{\nu c}^b), \]
\[ \delta \epsilon = i(\bar{\chi} \gamma^\mu \chi), \quad \delta A_{\mu} = -i \sqrt{2} (\bar{\psi}_a \gamma_\mu \lambda^a), \]
\[ \delta \bar{\psi}_a = \tilde{D}_{\bar{\psi}} e + \frac{1}{2} \gamma^\mu \gamma^\nu \gamma^\rho \bar{\psi}_a (\bar{\psi}_a \gamma^\rho \gamma^\nu \gamma^\mu \psi_b) \]
\[ - \frac{3i}{8} \gamma_a \lambda^a (\bar{\psi} \gamma^\mu \chi^a - \frac{i}{8} \gamma_a \lambda^a (\bar{\psi} \gamma^\mu \chi^a) \]
\[ + \frac{i}{16} \gamma_\mu \gamma^\nu \gamma^\rho \bar{\psi}_a (\bar{\psi} \gamma^\rho \gamma^\nu \gamma^\mu \psi_b) \]
\[ + \frac{i}{16} \gamma_\mu \gamma^\nu \gamma^\rho \bar{\psi}_a (\bar{\psi} \gamma^\rho \gamma^\nu \gamma^\mu \psi_b) \]
\[ - \frac{i}{32} \gamma_\mu \gamma^\nu \gamma^\rho \bar{\psi}_a (\bar{\psi} \gamma^\rho \gamma^\nu \gamma^\mu \psi_b). \]
\[ \delta \chi = \frac{i}{2} \chi_a (\bar{\psi}_a \gamma^\mu \psi_b) \gamma^\nu \gamma^\rho \bar{\psi}_a (\bar{\psi} \gamma^\rho \gamma^\nu \gamma^\mu \psi_b) \]
\[ + \frac{i}{2} \chi_a (\bar{\psi}_a \gamma^\mu \psi_b) \gamma^\nu \gamma^\rho \bar{\psi}_a (\bar{\psi} \gamma^\rho \gamma^\nu \gamma^\mu \psi_b) \]
\[ - \frac{1}{2} \bar{\psi}_a \gamma^\mu \psi_b (\bar{\psi} \gamma^\nu \gamma^\rho \gamma^\mu \psi_b) \]
\[ + \frac{i}{8} \bar{\psi}_a \gamma^\mu \psi_b (\bar{\psi} \gamma^\nu \gamma^\rho \gamma^\mu \psi_b) \gamma^\nu \gamma^\rho \gamma^\mu \psi_b \]
gives the supersymmetry anomaly
\[ \mathcal{A}_a = e_{\mu a b} c_{\nu c d} \left( -\frac{1}{16} e^{\mu a b} \delta A_{\nu}^a A_{\nu b}^c F_{\mu c}^d + \frac{i}{8} \delta A_{\nu}^a F_{\mu c}^d \right), \]
\[ \quad - \frac{i}{16} e^{\mu a b} \delta A_{\nu b}^a F_{\mu a}^c + \frac{i}{8} \delta A_{\nu b}^a F_{\mu c}^d \]
\[ + \frac{i}{8} \delta A_{\nu b}^a (\bar{\psi} \gamma^\mu \psi_b) F_{\nu b}^d \]
\[ + \frac{i}{4} \delta A_{\nu b}^a (\bar{\psi} \gamma^\mu \psi_b) F_{\nu b}^d \]
\[ - \frac{i}{128} e \delta A_{\nu b}^a (\bar{\psi} \gamma^\mu \psi_b) \bar{\psi} F_{\nu b}^d \]
\[ + \frac{i}{8 \sqrt{2}} e \delta A_{\nu b}^a (\bar{\psi} \gamma^\mu \psi_b) \bar{\psi} F_{\nu b}^d \]
\[ + \frac{i}{16 \sqrt{2}} e \delta A_{\nu b}^a (\bar{\psi} \gamma^\mu \psi_b) \bar{\psi} F_{\nu b}^d \]
\[ - \frac{i}{16 \sqrt{2}} e \delta A_{\nu b}^a (\bar{\psi} \gamma^\mu \psi_b) \bar{\psi} F_{\nu b}^d \]
\[ + \frac{i}{8 \sqrt{2}} e \delta A_{\nu b}^a (\bar{\psi} \gamma^\mu \psi_b) \bar{\psi} F_{\nu b}^d \]
\[ + \frac{i}{4} \delta A_{\nu b}^a (\bar{\psi} \gamma^\mu \psi_b) \bar{\psi} F_{\nu b}^d \].

Summarizing, the complete theory would contain additional non-local couplings induced by fermion loops, whose variation would cancel the anomalous contribution of the contact terms. Thus, the low-energy couplings that we are displaying are properly neither gauge-invariant nor supersymmetric. However, gauge and supersymmetry anomalies are related by Wess-Zumino consistency conditions, and
this grants the coherence of the construction. The presence of the arbitrary parameter $\alpha$ reflects the freedom of adding to the anomaly the variation of a local functional, consistently with all Wess-Zumino conditions. This anomalous behavior of the low-energy Lagrangian is related to another remarkable property of these models [9]: aside from local symmetry transformations and the equation of motion, the commutator of two supersymmetry transformations on the gauginos generates the two-cocycle

$$\delta_{(\alpha)}\lambda^a = c^{ab}_{r} \varepsilon_{ra} \left[ -\frac{1}{2} (\tilde{\epsilon}_1 \gamma_{\alpha} \lambda^e) (\tilde{\epsilon}_2 \gamma_{\alpha} \lambda^d) \gamma^\mu \gamma^b - \frac{3}{32} (\tilde{\lambda}^b \gamma^a \lambda^e) (\tilde{\epsilon}_1 \gamma_{\alpha} \lambda^d) \gamma^\mu \gamma^b + \frac{b}{32} (\tilde{\lambda}^b \gamma^a \lambda^e) (\tilde{\epsilon}_1 \gamma_{\alpha} \lambda^d) \gamma^\mu \gamma^b + \frac{1}{2} - \frac{b}{32} (\tilde{\lambda}^b \gamma^a \lambda^e) (\tilde{\epsilon}_1 \gamma_{\alpha} \lambda^d) \gamma^\mu \gamma^b \right].$$

Different from zero for any value of $\alpha$. In six dimensions the Wess-Zumino conditions close only on the field equations of the gauginos, and this two-cocycle actually makes these conditions close for any value of $\alpha$. In the case of a single vector multiplet, in which the term of the Lagrangian proportional to $\lambda^4$ disappears, the two-cocycle is still present, although it is properly independent of $\alpha$.

Although the properties of (anti)self-dual tensors imply that the $H^2$ term in the Lagrangian vanishes identically, the field equations obtained varying $\nabla$ are the ones that result from the supersymmetry algebra when the (anti)self-duality conditions are imposed. It is possible to apply the construction of Pasti, Sorokin and Tonin (PST) [16] in this case, following the results of [17], where this construction was applied to the case of (anti)self-dual tensors coupled to non-abelian vector multiplets, thus generalizing the results of [18], where only tensor multiplets were considered. Since the theory describes a single self-dual 2-form

$$\hat{\mathcal{F}}_{\mu \nu} = v \hat{H}_{\mu \nu} - \frac{i}{8} (\tilde{\lambda}^a \gamma_{\mu \nu} \lambda^b)$$

and $n$ antself-dual 2-forms

$$\hat{\mathcal{F}}^m_{\mu \nu} = x^m \hat{H}^m_{\mu \nu} + \frac{i}{8} x^m c^{r a b} (\tilde{\lambda}^a \gamma_{\mu \nu} \lambda^b),$$

the complete Lagrangian is obtained by the addition of the term

$$-\frac{1}{2} \frac{\partial \phi^a}{\partial \phi^b} \frac{\partial \phi^b}{\partial \phi^a} \left[ \hat{\mathcal{F}}_{\mu \nu} \hat{\mathcal{F}}_{\mu \nu} + \hat{\mathcal{F}}^m_{\mu \nu} \hat{\mathcal{F}}^m_{\mu \nu} \right],$$

where $\phi$ is an auxiliary scalar field [16,18] and $H^\pm = H \pm \ast H$. This lagrangian has PST gauge invariances needed to cancel the additional degrees of freedom [16]. Once the transformations of the gravitino and of the tensorinos are properly modified, the supersymmetry algebra generates also the PST gauge transformations [18]. The field equations obtained from the complete Lagrangian reduce to those obtained from the Lagrangian without the PST term, once these gauge invariances are fixed [16]. Alternative ways of obtaining a lagrangian formulation for self-dual tensors can be found in the literature. While in general these involve infinite auxiliary fields [19], Kavalov and Mkrtchyan [20] had obtained long ago a complete action for pure $d = 6 (1,0)$ supergravity in terms of a single tensor auxiliary field. Their work may be connected to the result of [18] via an ansatz relating their tensor to the PST scalar.

As supersymmetry does not constrain the values of the coefficients $c'$, we have obtained a class of models whose anomaly polynomials can contain odd powers of the individual field strengths $F^a$. It is interesting to compare these results with [14]. Although for generic values of the $c'$s the $SO(1,n)$ global symmetry is broken, the authors of [14] consider the amusing case of $n = 2$ with two abelian vector multiplets transforming in the spinorial representation of $SO(1,2)$. Identifying this group with the one that transforms the tensor fields, one obtains an $SO(1,2)$-invariant Lagrangian if $c' = \gamma^4 \gamma'$. In particular the results of [14] correspond to the Majorana representation of $SO(1,2)$:

$$\gamma^0 = \sigma_2 , \quad \gamma^1 = i \sigma_1 , \quad \gamma^2 = i \sigma_3 ,$$


and for this choice the anomaly polynomial vanishes identically.

The transition to tensionless strings corresponds to values of the scalar fields for which the gauge coupling vanishes [4]. In our Lagrangian, this would correspond to the vanishing of some eigenvalues of the matrix $v_0 c^{ab}$. In the case of [14] the moduli space is a two-dimensional hyperboloid, described by the equation $v_0^2 - v_1^2 - v_2^2 = 1$, and one can show that the eigenvalues of the matrix $v_0 c^{ab}$ are both positive for $v_0 \geq 1$ and both negative for $v_0 \leq -1$, so that the transition is not reached.

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On the \( N = 2 \) superstring BRST operator

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Abstract

We show that the BRST charge for the \( N = 2 \) superstring system can be written as

\[
Q = e^{-\beta \frac{\delta}{\delta \psi} \psi e^\gamma},
\]

where \( b \) and \( \gamma \) are super-reparametrizations ghosts. This provides a trivial proof of the nilpotence of this operator.

1. Introduction

Superstring theory can be seen as a critical \( N = 1 \) superconformal system defined on the world-sheet. It can be quantized by studying the cohomology of the nilpotent BRST operator

\[
Q = \frac{d\gamma}{2\pi i} \left[ c(T + \frac{i}{2} T^s) + \gamma(G + \frac{i}{2} G^s) \right],
\]

where the matter generators \( T \) and \( G \) satisfy the \( N = 1 \) superconformal algebra with central charge \( c = 15 \), and the ghosts generators \( T^s \) and \( G^s \) satisfy the corresponding algebra with central charge \( c = -15 \). In such a way, the ghost sector allows to fix the gauge symmetry of the theory. The ghosts generators are constructed out of a pair of fermionic fields \( b, c \) with spins \( 2 \) and \( 1 \), respectively, and a pair of bosonic fields \( \beta, \gamma \) with spins \( 3/2 \) and \( 1/2 \), respectively, that fix the world-sheet supersymmetry.

Physical states are described by vertex operators living in the cohomology of the BRST operator (1.1). In order to include spacetime spinors in the spectrum, we need to fermionize the bosonic ghosts as

\[
\beta = \partial \xi e^{-\phi}, \quad \gamma = \eta e^\phi,
\]

where \( \eta \) and \( \xi \) are free fermions of spins \( 1 \) and \( 0 \) respectively, and \( \phi \) is a chiral boson [1]. Note that this fermionization does not involve the zero mode of \( \xi \), then physical states are independent of this mode (it is due to the fact that physical states are constructed out of \( \beta \) and not out of \( \xi \), for example). The space of the physical states is called “small” Hilbert space and the whole space, involving operators constructed out of \( \xi \) zero mode, is called “large” Hilbert space. We can define the space of physical states as the set of operators that commute with \( Q \) but also must commute with \( \phi \eta \). As consistency, \( Q \) not only must be nilpotent but also must anticommute with \( \phi \eta \) [2].
A similar analysis can be carried out for the string with $N = 2$ world-sheet superconformal symmetry [3]. Physical states belong to the cohomology of the $N = 2$ BRST operator

\[ Q = \int \frac{dz}{2\pi i} \left[ \varepsilon(T + \frac{1}{2} \bar{T}) + \gamma_+ (G_+ + \frac{1}{2} G^b) + \gamma_-(G_- + \frac{1}{2} G^c) + \tilde{c}(J + \frac{1}{2} J^b) \right], \]  

(1.3)

where the matter generators $T$, $J$ and $G_{\pm}$ satisfy the $N = 2$ superconformal algebra with central charge $c = 6$ and the ghosts generators $T^b$, $J^b$ and $G^b_{\pm}$ satisfy the corresponding algebra with central charge $c = -6$. The critical system can be represented by a pair of complex chiral superfields $X((\bar{z}, z, \theta^0, \bar{\theta}^0)$ and $\bar{X}(z, \bar{z}, \bar{\theta}^0, \theta^0)$, $(i = 1, 2)$, being the signature of the background space (2.2) or Euclidean, but not Minkowskian. This theory describes self-dual systems in four dimensions (for related issues see [4]). The ghosts generators are constructed out of a pair of fermionic ghosts $b$, $c$ (with spins $2$, $-1$, respectively) that fix conformal symmetry, and another pair of fermionic ghosts $\tilde{b}$, $\tilde{c}$ (with spins $1$ and $0$, respectively) that fix the $U(1)$ gauge symmetry generated by $J$, and four bosonic ghosts $G_{\pm}$ and $\gamma_{\pm}$ (with spins $3/2$ and $-1/2$, respectively) that fix the world-sheet supersymmetries.

We need to fermionize the bosonic ghosts in order to describe spacetime spinors, the pair $\beta_+$ and $\gamma_-$ becomes

\[ \beta_+ = \partial \xi_+ e^{-\phi_+}, \quad \gamma_- = \eta_+ e^{\phi_+}, \]  

(1.4)

where $\xi_+$ and $\eta_+$ (spins $0$ and $1$, respectively) are free fermions, and $\phi_+$ is a chiral boson. For the bosonic pair $\beta_+$ and $\gamma_-$ there is an equivalent expression involving $\xi_-$, $\eta_-$ and $\phi_-$ instead $\xi_+$, $\eta_+$ and $\phi_+$. Note that, as in the $N = 1$ case, the zero modes of the ghosts $\xi_{\pm}$ are not involved in the fermionizations, then physical states are independent of such modes. We can define the space of physical states as the set of vertex operators that commute with $\beta_+$ and $\gamma_-$. This will be the analogous of the "small" Hilbert space of the $N = 1$ case, the "large" Hilbert space takes into account operators that depend on zero modes of $\xi_{\pm}$. As consistency, the BRST operator must be not only nilpotent but also anticommute with $\beta_+$.

It was shown in [5] that the $N = 1$ superstring BRST operator can be written as

\[ Q = e^{-\beta} \left( \int \frac{dz}{2\pi i} b \gamma^2 \right) e^\beta, \]  

(1.5)

which trivially proves the nilpotence of the BRST operator. This also shows that the cohomologies of $Q$ and $\beta \frac{dz}{2\pi i} b \gamma^2$ are equals. The last one is trivial in the "large" Hilbert space, then $Q$ is trivial in this space. In the "small" $Q$ is not trivial as expected.

The purpose of this paper is to extend the result (1.5) for the $N = 2$ superstring.

2. Similarity transformation for the $N = 2$ superstring

The BRST current $j_{\text{BRST}}(z)$ is given by $Q = \int \frac{dz}{2\pi i} j_{\text{BRST}}(z)$. After fermionizing the bosonic ghosts as in (1.4) and then bosonize $\xi_+ = e^{x_+}$ and $\eta_- = e^{-x_-}$, the BRST current becomes [3]

\[ j_{\text{BRST}} = cT + e^\phi_+ x T G_+ + e^\phi_- x T G_+ + \tilde{c}J + bc \partial c \]

\[ - e \partial \xi_+ \partial \eta_+ b \xi_+ \eta_+ \partial (\phi_+ - \phi_-) \]

\[ - e \partial \xi_+ \partial \eta_+ b \xi_+ \eta_+ \partial (\phi_+ - \phi_-) \]

\[ j_{\text{BRST}} = e^{-\beta} j_0 e^\beta, \]  

(2.2)

where

\[ T = - \partial^2 \phi_+ - \frac{1}{2} (\partial \eta_+)^2 + \frac{1}{2} \partial^2 \xi_+ + \frac{1}{2} (\partial \mu_+)^2 \]

\[ - \partial^2 \phi_- - \frac{1}{2} (\partial \eta_-)^2 + \frac{1}{2} \partial^2 \xi_- + \frac{1}{2} (\partial \mu_-)^2. \]

The BRST current has a total derivative term that we have no written in (2.1).

We will show that
where
\[ j_0 = -2be^{-x}\cdot e^{-x} \cdot e^{-x}. \]  

and
\[ R = e^{\Phi} \frac{dz}{2\pi i} \left[ cG_+ e^{x} + cG_- e^{-x} \right] \]
\[ - \frac{\phi}{2\pi i} \left[ (H + \phi_+ - \phi_-) e^{-x} + c\partial c\right] \]
\[ \times e^{-x} \cdot \frac{dz}{2\pi i} \quad \text{where the U(1) current is bosonized as } J = \partial H. \]

Using (2.2), \( j_{\text{BRST}} \) is trivially nilpotent since \( j_0 \) has no poles with itself.

To prove (2.2) we use the expansion
\[ e^{-x} j_0 e^x = \sum_{n=0}^{\infty} \frac{1}{n!} j_n, \quad j_n = [j_{n-1}, R]. \]

where, for \( R = \phi \frac{dz}{2\pi i} r(z) \) the commutator is computed using the rule
\[ [j_{n-1}(y), R] = \phi \frac{dz}{2\pi i} j_n(y) r(z) . \]

The term \( n = 1 \) in (2.4) is given by
\[ j_1 = - \frac{1}{2} \partial e^{-x} \cdot e^{-x} \cdot G_+ + e^{-x} \cdot G_+ \cdot bc\partial c \]
\[ - \partial (H + \phi_+ - \phi_-) + \partial (3\partial \phi_+ + 3\partial \phi_-) \]
\[ - 2\partial \phi_+ + 2\partial \phi_-) + \frac{\delta}{e^{-x} \cdot \partial (e^{-x}) \]
\[ \times \frac{dz}{2\pi i} \quad \text{the term } n = 2 \text{ is given by} \]
\[ j_2 = \frac{1}{2} \partial^2 e + 2cT - 2c\partial e - \frac{1}{2} G_+ c\partial c e^{-x} \cdot e^{-x} \]
\[ - \frac{1}{2} G_- c\partial c e^{-x} \cdot e^{-x} \cdot e^{-x} \cdot e^{-x} \cdot e^{-x} \]
\[ + 2\partial e (\partial \phi_+ - \partial \phi_- + \partial \phi_+ - \partial \phi_-) \]
\[ + 3\partial e - 2\partial \phi_+ + 2\partial \phi_- - \partial \phi_+ + \partial \phi_- \]
\[ - 3\partial \phi_+ - 3\partial \phi_- + 3\partial \phi_+ - 3\partial \phi_- \]
\[ \times \frac{dz}{2\pi i} \quad \text{the term } n = 3 \text{ is given by} \]
\[ j_3 = \frac{1}{2} G_+ c\partial c e^{-x} \cdot e^{-x} \cdot e^{-x} \cdot e^{-x} \]
\[ + 3\partial e - 2\partial \phi_+ + 2\partial \phi_- - \partial \phi_+ + \partial \phi_- \]
\[ - 3\partial \phi_+ - 3\partial \phi_- + 3\partial \phi_+ - 3\partial \phi_- \]
\[ \times \frac{dz}{2\pi i} \quad \text{the term } n = 4 \text{ is given by} \]
\[ j_4 = - 15c\partial c^2 e^{-x} \cdot e^{-x} \cdot e^{-x} \cdot e^{-x} \]

The term for \( n = 5 \) in the expansion vanishes identically since the OPE between \( j_4 \) and \( R \) has no single poles. Then, the terms of higher order in the expansion (2.4) vanish too.

It is straightforward to check that the BRST current is equal to \( j_0 + j_1 + j_2 + j_3 + j_4 \) up to total derivatives.

3. Concluding remarks

The form that we have written the BRST current (2.2), proves its nilpotence trivially. Note that the cohomology of \( \delta j_0 \) is trivial in the “large” Hilbert space, then the \( N = 2 \) cohomology is trivial in that space. However, such a property is not hold in the “small” Hilbert space since the \( N = 2 \) has non-trivial states [4].

One could be tempted to use the expansion (2.4) as BRST current for the non-critical case. However, if the central charge \( c \) is different of the critical value 6, in the expansion would appear one term proportional to \( (6 - c) \partial dz \cdot \partial c^2 e^{-x} \cdot e^{-x} \cdot e^{-x} \cdot e^{-x} \)

which does not commute with \( \delta dz \eta \).

\[ 87 \]
Therefore, the expansion (2.4) can be used as BRST charge in the critical case only.

Acknowledgements

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References

A note on the solutions of the Ginsparg-Wilson relation

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Abstract

The role of $R$ in the solutions of the Ginsparg-Wilson relation is discussed. © 2000 Published by Elsevier Science B.V. All rights reserved.

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In this note, we would like to clarify some seemingly subtle issues pertaining to the role of $R$ in the solutions of the Ginsparg-Wilson relation [1]

$$D\gamma_5 + \gamma_5 D = 2D\gamma_5 RD.$$ (1)

Here $R$ is a positive definite hermitian operator which is local in the position space and trivial in the Dirac space. Since we can sandwich (1) by a left multiplier $\sqrt{R}$ and a right multiplier $\sqrt{R}$ on both sides of (1), and define $D' = \sqrt{R}D\sqrt{R}$, then $D'$ satisfies

$$D'\gamma_5 + \gamma_5 D' = 2D'\gamma_5 D'$$ (2)

which is in the same form of Eq. (1) with $R = 1$. This seems to suggest that one can set $R = 1$ in the GW relation (1) and completely ignore the $R$ dependence in the general solution of the GW relation. However, as we will see, if $R$ is set to be the identity operator from the beginning, then some of the salient features of the general solution may be easily overlooked.

First, let us review some basics of the GW relation. The general solution to the GW relation (1) can be written as [2,3]

$$D = D_c (\mathbb{1} + RD_c)^{-1} = (\mathbb{1} + D_c R)^{-1} D_c,$$ (3)

where $D_c$ is any chirally symmetric Dirac operator, i.e.,

$$D_c \gamma_5 + \gamma_5 D_c = 0.$$ (4)

In order to have $D$ reproduce the continuum physics, $D_c$ is required to satisfy the necessary physical constraints [3]. The general solution of $D_c$ has been investigated in Ref. [4]. Conversely, for any $D$ satisfying the GW relation (1), there exists the chirally symmetric $D_c$

$$D_c = D(\mathbb{1} - RD)^{-1} = (\mathbb{1} - DR)^{-1} D.$$ (5)
We usually require that $D$ also satisfies the hermiticity condition\footnote{This implies that $\det(D)$ is real and non-negative.}

$$D^\dagger = \gamma_5 D \gamma_5.$$ \hfill (6)

Then $D_\gamma$ also satisfies the hermiticity condition $D_\gamma^\dagger = \gamma_5 D_\gamma \gamma_5$, since $R$ is hermitian and commutes with $\gamma_5$. The hermiticity condition together with the chiral symmetry of $D_\gamma$ implies that $D_\gamma$ is anti-hermitian. Thus there exists one to one correspondence between $D_\gamma$ and a unitary operator $V$ such that

$$D_\gamma = (\mathbb{1} + V)(\mathbb{1} - V)^{-1},$$

$$V = (D_\gamma - \mathbb{1})(D_\gamma + \mathbb{1})^{-1}.$$ \hfill (7)

where $V$ also satisfies the hermiticity condition $V^\dagger = \gamma_5 V \gamma_5$. Then the general solution (3) can be written as [2,3]

$$D = (\mathbb{1} + V)[(I - V) + R(I + V)]^{-1}$$

$$= [(I - V) + (I + V) R]^{-1}(I + V).$$ \hfill (8)

On the other hand, if one starts from Eq. (2), then its solution is

$$D' = \frac{1}{2}(I + V)$$

which agrees with (8) with $R = \mathbb{1}$. Then using the relation $D' = \sqrt{R} D \sqrt{R}$, one obtains

$$D = \frac{1}{\sqrt{2} R}(I + V) \frac{1}{\sqrt{2} R}.$$ \hfill (11)

However, (11) is in contradiction with (8) since the $R$ dependence in (11) can be factored out completely while that of (8) can not. So, one of them can not be true in general. If we take the limit $R \to 0$, then (8) gives that $D \to (I + V)(I - V)^{-1} = D_\gamma$, but (11) implies that $D \to \infty$. Since $D_\gamma$ is well defined (without poles) in the trivial gauge sector, it follows that (11) can not be true in general. The fallacy in (11) is due to the assumption that $D'$ is independent of $R$. From (8), one can derive the following formula

$$D = \frac{1}{\sqrt{2} R}(I + V') \frac{1}{\sqrt{2} R},$$ \hfill (12)

where

$$V' = \left[ (I + V)\sqrt{R} + (I - V) \frac{1}{\sqrt{R}} \right]^{-1}$$

$$\times \left[ (I + V)\sqrt{R} - (I - V) \frac{1}{\sqrt{R}} \right]$$

$$= \sqrt{R}(I + V) - \frac{1}{\sqrt{R}}(I - V)$$

$$\times \sqrt{R}(I + V) + \frac{1}{\sqrt{R}}(I - V)^{-1} \hfill (13)$$

which is unitary and depends on $R$. This shows that $D'$ actually depends on $R$ and equals to $\frac{1}{2}(I + V')$ rather than (10). Therefore it is erroneous to write the general solution of the GW relation in the form of (11), in which $V'$ is replaced by $V$. Consequently, (11) may mislead one to infer that $D$ does not play any significant roles in the locality of $D$, in particular when $R$ is proportional to the identity operator. However, $R$ indeed plays a very important role in determining the locality of $D$. Let us consider $R = r \mathbb{1}$ with $r > 0$. When $r \to 0$, $D \to D_\gamma$ which must be nonlocal if $D_\gamma$ is free of species doubling and has the correct behavior in the classical continuum limit [5]. As the value of $r$ moves away from zero and goes towards a finite value, $D$ may change from a non-local operator to a local operator. This has been demonstrated in Refs. [6,7]. Therefore, the general solution (3) of the GW relation can be regarded as a topologically invariant transformation (i.e., index($D$) = index($D_\gamma$)) which can transform a nonlocal $D_\gamma$ into a local $D$. Conversely, the transformation (5) can transform a local $D$ into the non-local $D_\gamma$. For a given $D_\gamma$, the set of transformations, $\{ \mathcal{T}(R) : D = D_\gamma(I + RD)^{-1} \}$, form an abelian group with parameter space $[R]$ [7]. In general, for any lattice Dirac operator $D$ (not necessarily satisfying the GW relation), we can use the topologically invariant transformation $D' = D(I + RD)^{-1}$ to manipulate its locality.

The next question is whether we can gain anything (e.g., improving the locality of $D$) by using another functional form of $R$ rather than the simplest choice $R = r \mathbb{1}$. We investigate this question by nu-
merical experiments. For simplicity, we consider the Neuberger-Dirac operator [8]
\[ D_h = \mathbb{1} + V, \]
\[ V = D_w(D_0^+D_0)^{-1/2}, \]
where \( D_w \) is the Wilson-Dirac fermion operator with negative mass \(-1\)
\[ D_e = -1 + \frac{1}{2} \left[ \gamma_\mu(\nabla_\mu + \nabla_\mu^*) - \nabla_\mu \gamma_\mu \right], \]
where \( \nabla_\mu \) and \( \nabla_\mu^* \) are the forward and backward difference operators defined in the following,
\[ \nabla_\mu \psi(x) = U_\mu(x) \psi(x + \hat{\mu}) - \psi(x), \]
\[ \nabla_\mu^* \psi(x) = \psi(x) - U_\mu^*(x - \hat{\mu}) \psi(x - \hat{\mu}). \]
The Neuberger-Dirac operator \( D_h \) satisfies the GW relation (1) with \( R = 1/2 \). Then Eq. (5) gives
\[ D_e = 2 - \frac{1}{2} - V. \]
Substituting (17) into the general solution (3), we obtain
\[ D = 2(\mathbb{1} + V)[(\mathbb{1} - V) + 2R(\mathbb{1} + V)]^{-1}. \]
For a fixed gauge background, we investigate the locality of \( D(x,y) \) versus the functional form of \( R(x,y) \). For simplicity, we consider \( D \) in a two dimensional \( \theta(1) \) background gauge field with non-zero topological charge, and we use the same notations for the background gauge field as Eqs. (7)–(11) in Ref. [9]. Although it is impossible for us to go through all different functional forms of \( R(x,y) \), we can use the exponential function
\[ R(x,y) = r \exp(-m|x-y|) \]
as a prototype to approximate other forms by varying the parameters \( r \) and \( m \). In the limit \( m \to 0 \), \( R(x,y) \) is nonlocal, while in the limit \( m \to \infty, R \to \mathbb{1} \) which is the most ultralocal. Hence, by varying the value of \( m \) from 0 to \( \alpha \gg 1 \), we can cover a wide range of \( R(x,y) \) of very different behaviors.

One of the physical quantities which are sensitive to the locality of \( D \) is the anomaly function
\[ A_\mu(x) = \text{tr} \left[ \gamma_\mu(\mathcal{R}D)(x,x) + \gamma_\mu(D\mathcal{R})(x,x) \right] \]
which can serve as an indicator of the localness of \( D \). Since the Neuberger-Dirac operator is topologi-
cally proper for smooth gauge backgrounds, the index of \( D \) in (18) is equal to the background topological charge \( Q \).
\[ \text{index}(D) = n_- - n_+ = Q. \]
This implies that the sum of the anomaly function over all sites on a finite lattice must be equal to two times of the topological charge [7]
\[ \sum_x A_\mu(x) = 2(n_- - n_+) \]
\[ = 2Q = \begin{cases} \frac{1}{16\pi^2} \sum_x \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x), & d = 4; \\
\frac{1}{2\pi} \sum_x \epsilon_{\mu\nu} F_{\mu\nu}(x), & d = 2. \end{cases} \]
(22)
This is true for any \( R \) since the index of \( D \) is invariant under the transformation (3), i.e., \( \text{index}(D) = \text{index}(D_e) \). First we consider the gauge configuration with constant field tensors. If \( D \) is local, then we can deduce that \( A_\mu(x) \) is constant for all \( x \). From (22), it follows that
\[ A_\mu(x) = \rho(x) \]
\[ = \begin{cases} \frac{1}{16\pi^2} \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x), & d = 4; \\
\frac{1}{2\pi} \epsilon_{\mu\nu} F_{\mu\nu}(x), & d = 2. \end{cases} \]
(23)
where \( \rho(x) \) is the Chern-Pontryagin density in continuum. Note that Eq. (23) also implies that \( A_\mu(x) \) is independent of \( R \) if \( D \) is local. Next we introduce local fluctuations to the constant background gauge field, with the topological charge fixed. Then we expect that (23) remains valid provided that the locality of \( D \) is not destroyed by the roughness of the gauge field. Therefore, in general, by comparing the anomaly function \( A_\mu(x) \) at each site with the Chern-Pontryagin density \( \rho(x) \), in a prescribed background gauge field, we can reveal whether \( D \) is local or not in this gauge background. This provides another scheme to examine the locality of \( D \) rather than checking how well \( |D(x,y)| \) can be fitted by an exponentially decay function. We will use both
methods in our investigations. We define the deviation of the anomaly function as

$$\delta_D \equiv \frac{1}{N_s} \sum_x |\mathcal{A}_\rho(x) - \rho(x)|,$$

where the summation runs over all sites, and $N_s$ is the total number of sites.

In Table 1, we list the deviation of the chiral anomaly function, $\delta_D$, versus $R(x,y)$ with parameters $r$ and $m$ defined in Eq. (19), on a $12 \times 12$ lattice, in a constant background gauge field with topological charge $Q = 1$. The last two rows with $m \gg 1$ corresponds to $R = r$, and thus $\delta_D$ is very large. Now we increase $m$ to 1, 2, and 5 successively, then $R$ and $D$ both become more and more local, as shown in Fig. 3, thus $\delta_D$ becomes smaller and smaller, as shown in the second, third, and fourth rows of Table 1. These results indicate that a nonlocal R does not produce a local $D$, and a local R does not make $D$ more local than that using $R = r$. For $m = 5.0$, if we decrease $r$ to 0.1, then $D$ becomes very nonlocal, as shown in Fig. 4. Consequently, its $\delta_D$ (in the fifth row) is about 10 times larger than that of $r = 0.5$ (in the fourth row).

**Table 1**

The deviation of the chiral anomaly function $\delta_D$ [Eq. (24)] versus $R(x,y)$ with parameters $r$ and $m$ defined in Eq. (19), on a $12 \times 12$ lattice, in a constant background gauge field with topological charge $Q = 1$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$m$</th>
<th>$\delta_D$</th>
<th>index($D$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>1.366</td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0</td>
<td>0.5360</td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>2.0</td>
<td>0.2086</td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>5.0</td>
<td>$7.299 \times 10^{-3}$</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>5.0</td>
<td>$6.554 \times 10^{-2}$</td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>$\geq 1$</td>
<td>$3.848 \times 10^{-4}$</td>
<td>1</td>
</tr>
<tr>
<td>1.0</td>
<td>$\geq 1$</td>
<td>$1.618 \times 10^{-4}$</td>
<td>1</td>
</tr>
</tbody>
</table>

This suggests that on a finite lattice, $r$ cannot be too small, otherwise $D$ will become nonlocal. 

**Fig. 1.** One of the Dirac components of $D(x,0)$, $|D_1(x,0)|$, is plotted as a function of $|x|$ for $R(x,y) = \delta_{xy}$. The lattice is $12 \times 12$ with periodic boundary conditions. The constant background gauge field has topological charge $Q = 1$. All data points at the same distance $|x|$ from the origin have been averaged. The solid line is an exponential fit to the data points. The same decay constant also fits very well for all other Dirac components of $D(x,y)$ and for any reference point $y$.

**Fig. 2.** One of the Dirac components of $D(x,0)$ is plotted as a function of $|x|$ with $R(x,y) = 0.5 \exp(-0.5|x-y|)$. Other descriptions are the same as Fig. 1. The non-localness of $D$ is shown clearly.
Our numerical results listed in Table 1 as well as those plotted in Figs. 1–4 strongly suggest that we do not gain anything by using other functional forms of $R(x,y)$ than the simplest choice $R(x,y) = r \delta_{x,y}$.

However, the value of $r$ plays the important role in determining the localness of $D$. We have also tested other functional forms of $R(x,y)$ as well as many different gauge configurations. The results from all these studies are consistent with the conclusion that the optimal choice for $R$ is $R(x,y) = r \delta_{x,y}$.

Now we come to the question concerning the range of proper values of $r$. We have already known that $r$ cannot be zero or very small, otherwise $D$ is nonlocal. On the other hand, $r$ cannot be too large, otherwise $D$ is highly peaked in the diagonal elements (i.e., $D_{\mu\nu}(x,y) \sim D_{\mu\nu}(x,x) \delta_{\mu\nu} \delta_{x,y}$), which is unphysical since it does not respond properly to the background gauge field (e.g., the chiral anomaly is incorrect even though the index of $D$ is equal to the background topological charge). In Table 2, we list the deviation of the chiral anomaly function, $\delta_D$, versus $R(x,y) = r \delta_{x,y}$, on a 12 × 12 lattice (the second column), in a constant background gauge field with topological charge $Q = 1$. We see that the proper values of $r$ are approximately in the range 0.5–1.2, where $D$ can reproduce the continuum chiral anomaly precisely. Next we investigate how the lattice size affects the range of proper values of $r$. The results of $\delta_D$ for lattice sizes 16 × 16 and 20 × 20 are listed in the third and the fourth columns in Table 2. They clearly show that the lower bound of $r$ can be pushed to a smaller value, ~ 0.2, when the size of the lattice is increased to 20 × 20. Therefore, it suggests that the chiral limit ($r \to 0$ and $r \to 1$ for the continuum case) can be reached when $r$ is approximately 0.55–0.6.

Table 2

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\delta_D(12 \times 12)$</th>
<th>$\delta_D(16 \times 16)$</th>
<th>$\delta_D(20 \times 20)$</th>
<th>index (D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>$6.232 \times 10^{-2}$</td>
<td>$1.792 \times 10^{-2}$</td>
<td>$4.854 \times 10^{-3}$</td>
<td>1</td>
</tr>
<tr>
<td>0.2</td>
<td>$4.818 \times 10^{-3}$</td>
<td>$6.229 \times 10^{-4}$</td>
<td>$7.750 \times 10^{-5}$</td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>$3.848 \times 10^{-4}$</td>
<td>$2.434 \times 10^{-5}$</td>
<td>$1.559 \times 10^{-6}$</td>
<td>1</td>
</tr>
<tr>
<td>0.8</td>
<td>$1.698 \times 10^{-4}$</td>
<td>$8.268 \times 10^{-6}$</td>
<td>$4.976 \times 10^{-7}$</td>
<td>1</td>
</tr>
<tr>
<td>1.0</td>
<td>$1.618 \times 10^{-4}$</td>
<td>$6.596 \times 10^{-6}$</td>
<td>$3.432 \times 10^{-7}$</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>$3.448 \times 10^{-4}$</td>
<td>$1.318 \times 10^{-5}$</td>
<td>$5.308 \times 10^{-7}$</td>
<td>1</td>
</tr>
<tr>
<td>1.5</td>
<td>$1.688 \times 10^{-3}$</td>
<td>$1.243 \times 10^{-4}$</td>
<td>$8.781 \times 10^{-6}$</td>
<td>1</td>
</tr>
<tr>
<td>2.0</td>
<td>$9.940 \times 10^{-3}$</td>
<td>$1.569 \times 10^{-3}$</td>
<td>$2.348 \times 10^{-4}$</td>
<td>1</td>
</tr>
<tr>
<td>5.0</td>
<td>0.1599</td>
<td>9.314 \times 10^{-2}</td>
<td>5.088 \times 10^{-2}</td>
<td>1</td>
</tr>
</tbody>
</table>

Fig. 3. One of the Dirac components of $D(x,0)$ is plotted as a function of $|x|$ with $R(x,y) = 0.5 \exp(-5.0|x-y|)$. Other descriptions are the same as Fig. 1.

Fig. 4. One of the Dirac components of $D(x,0)$ is plotted as a function of $|x|$ with $R(x,y) = 0.1 \exp(-5.0|x-y|)$. Other descriptions are the same as Fig. 1. The non-localness of $D$ is shown clearly.
$D \rightarrow D_r$ can be approached by decreasing the value of $r$ while increasing the size of the lattice, at finite lattice spacing. This provides a nonperturbative definition of the chiral limit for any $D$ of the general solution (3) with $D_r$ satisfying the necessary physical requirements [3].

It is evident that the range of proper values of $r$ also depends on the background gauge configuration. However, we suspect that when the background gauge configuration becomes very rough, there may not exist any values of $r$ such that the chiral anomaly function is in good agreement with the Chern-Pontryagin density. We intend to return to this question in a later publication.

In summary, we have clarified the role of $R$ in the general solution (3) of the Ginsparg-Wilson relation. It provides a topologically invariant transformation which transforms the chirally symmetric and nonlocal $D_r$ into a local $D$ which satisfies the GW relation, the exact chiral symmetry on the lattice. Having $R$ local in the position space is a necessary condition to ensure the absence of additive mass renormalization in the fermion propagator, as well as to produce a local $D$, which is vital for obtaining the correct chiral anomaly. Our numerical results strongly suggest that the optimal form of $R$ is $R(x, y) = r \delta(x, y)$. The range of proper values of $r$ depends on the background gauge configuration as well as the size of the lattice, $L = Na$. In the limit $N \rightarrow \infty$, for smooth gauge backgrounds, the lower bound of proper values of $r$ goes to zero, thus the chiral limit ($r \rightarrow 0$ and $D \rightarrow D_r$) can be approached nonperturbatively at finite lattice spacing.

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References

Magnetic interactions of D-branes and Wess-Zumino terms in Super Yang-Mills effective actions

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Abstract

There is a close relation between classical supergravity and quantum SYM descriptions of interactions between separated branes. In the case of D3 branes the equivalence of leading-order potentials is due to non-renormalization of the $F^4$ term in $\mathcal{N}=4$ SYM theory. Here we point out the existence of another special non-renormalized term in quantum SYM effective action. This term reproduces the interaction potential between electric charge of a D3-brane probe and magnetic charge of a D3-brane source, represented by the Chern-Simons part of the D-brane action. This unique Wess-Zumino term depends on all six scalar fields and originates from a phase of the euclidean fermion determinant in SYM theory. It is manifestly scale invariant (i.e. is the same for large and small separations between branes) and can not receive higher loop corrections in gauge theory. Maximally supersymmetric SYM theories in $D=4$ contain mixed WZ terms which depend on both scalar and gauge field backgrounds, and which reproduce the corresponding CS terms in the supergravity interaction potentials between separated Dp-branes for $p > 3$. Purely scalar WZ terms appear in other cases, e.g., in half less supersymmetric gauge theories in various dimensions describing magnetic interactions between Dp and D$(6-p)$ branes.

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1. Introduction

The duality between the supergravity and the world-volume descriptions of D-branes has led to important advances in understanding of dynamics of supersymmetric gauge theories. Many aspects of interactions of D-branes which are transparent in the supergravity description, translate into quite non-trivial properties of the world-volume field theory. The electric and gravitational interactions between branes have been widely studied in this context (see, e.g., [1–6] and references there). The agreement between the supergravity and the SYM descriptions of leading-order interactions between branes can be traced to the universal non-renormalization properties of the $F^4$ terms in the effective action of maximally supersymmetric gauge theories in various dimensions [7,8].

We shall discuss magnetic interactions. The prime case of interest in connection with 4-d gauge theories
is a D3 brane moving in the background of other D3 branes. The self-duality of the RR five-form field strength implies that a D3 brane carries both electric and magnetic charges. As a consequence, the probe brane will experience the Lorentz force, similar to the one an electric charge experiences in the magnetic field of a monopole. For example, the action of a D3-brane probe moving in a supergravity background produced by a D3-brane source contains [9–11] the Chern-Simons term \( \int \epsilon X^i dX^j dX^k \) which describes the interaction of an electric charge of the probe with the electric and the magnetic fields produced by the source. In the static gauge, the electric interaction produces the effective “\(-1\)” contribution to the P-even Born-Infeld part of the D3-brane action

\[
S_{\text{BI}} = S_{\text{BI}}^\text{el} + S_{\text{BI}}^\text{mag} \equiv \frac{1}{8\pi} \int d^4x \frac{X^i}{|X|} \epsilon^{ijk} \frac{\partial X^j}{\partial t} \frac{\partial X^k}{\partial t} - \frac{1}{4\pi} \int \epsilon^{ijk} \frac{\partial X^j}{\partial t} \frac{\partial X^k}{\partial t}.
\]

This term ensures the vanishing of the interaction potential between static parallel branes. The magnetic interaction term (which is real for Minkowski signature) \( S_{\text{CS}} = NS_{\text{WZ}} \approx iN/\sqrt{8\pi} \epsilon^i \sim \epsilon^i dX^i \) is non-vanishing only when all 6 scalars have non-trivial gradients.

Since many features of the magnetic D3 brane interaction are similar to those of the Lorentz interaction between an electric charge and a magnetic monopole, let us briefly review some facts about the latter case. The Lorentz force acting on the charge in the field of magnetic monopole is

\[
\mathbf{F} = q_e \epsilon^{ijk} \frac{q_m}{4\pi |X|^2} \mathbf{X}_j \mathbf{X}_k.
\]

The Dirac string singularity of the gauge potential for the monopole field does not allow the Lorentz force to be a variation of a well-defined local action functional. The variational principle for a charged particle interacting with a monopole can be formulated only by adding the non-local Wess-Zumino term to the action

\[
S_{\text{WZ}} = - \frac{q_e q_m}{8\pi} \int d^4x \epsilon^{ijk} \frac{1}{|X|^3} \mathbf{X}_j \frac{\partial}{\partial t} \mathbf{X}_k - \frac{1}{4\pi} \int \epsilon^{ijklm} \frac{1}{|X|^2} \frac{\partial X^l}{\partial t} \frac{\partial X^m}{\partial t} \frac{1}{|X|^2} = \mathcal{F}^i.
\]

Here the integration is over a domain whose boundary is the time axis. Then the variation of the WZ action reproduces the Lorentz force:

\[
\delta S_{\text{WZ}} / \delta X_i = \mathcal{F}^i.
\]

According to the standard argument, there is an ambiguity in the definition of the WZ action, because \( X_i \) is defined at the boundary of the integration domain and can be continued into the interior in an arbitrary way. Since the integrand is locally a total derivative, this ambiguity is discrete, in the sense that the values of the action for different continuations differ by an integer multiple of \( q_e q_m \). Despite the multi-valuedness of the action, the path integral remains single-valued, provided \( q_e q_m \) is an integer multiple of \( 2\pi \). The consistency of the quantum mechanics of an electrically charged particle thus leads to the Dirac quantization condition for the magnetic charge: \( q_m = 2\pi n / q_e \).

The Lorentz force experienced by a D3 brane in the background of another D3 brane (with both branes having unit charges) is

\[
\mathcal{F}^i = \frac{1}{12\pi^2} \epsilon^{\mu \nu \lambda \rho \sigma} \epsilon^{IJKLMN} \times 
\]

\[
\times \frac{1}{|X|^6} \mathbf{X}_I \frac{\partial}{\partial \mu} \mathbf{X}_J \frac{\partial}{\partial \nu} \mathbf{X}_K \frac{\partial}{\partial \lambda} \mathbf{X}_M \frac{\partial}{\partial \rho} \mathbf{X}_N,
\]

\[
\mu, \nu = 0, 1, 2, 3, \quad I, J, \ldots = 1, 2, 3, 4, 5, 6.
\]

Here \( X_I(x) \) parametrize the position of the probe brane in the 6-d space transverse to the source brane as a function of the 4 longitudinal coordinates \( x^\mu \). The Lorentz force can be represented as a variation of the five-dimensional integral (\( m = 0, 1, \ldots , 5 \)):

\[
\mathcal{F}^i = \frac{\delta S_{\text{WZ}}}{\delta X_i}.
\]

\( \text{The WZ term can be written also as a 4-d integral of a local functional by using spherical } S^5 \text{ coordinates instead of Cartesian } X^i \text{ (see [13]).} \)

\[\frac{1}{|X|^2} \frac{\partial X^i}{\partial t} \frac{\partial X^j}{\partial t} \frac{1}{|X|^2} = X^i X^j.\]
$S_{\text{WZ}} = - \frac{1}{60\pi} \int d^3x \, e^{\alpha \Phi} e^{1JKLMN} \times \frac{1}{|X|^6} \, X_I \partial_a X_J \partial_b X_K \partial_c X_L \partial_d X_M \partial_e X_N$

$= - \frac{1}{60\pi^2} \int e^{1JKLMN} n_I dn_J \wedge dn_K \wedge dn_L$

$\wedge dn_M \wedge dn_N$, \hspace{1cm} (6)

where $n_I = X_I / |X|$ parametrize $S^5 = SO(6)/SO(5)$. This action is defined up to $2\pi$, which reflects the fact that a D3 brane carries one unit of the quantized magnetic charge.

This “topological” term is scale-invariant and does not depend on gauge coupling. Its coefficient cannot be renormalized since any non-trivial dependence on the dilaton would break gauge invariance. Also, as for many other WZ terms, the renormalization of its coefficient would contradict the topological magnetic charge quantization condition. Like interaction, it cannot be renormalized since any non-trivial dependence on gauge coupling. Its coefficient does not depend on the dilaton would break gauge invariance. This “topological” term is scale-invariant and does not depend on gauge coupling.

The scale $(X \to aX)$ invariance and the topological nature of the above WZ term suggests that it should follow from the string-theory description of D-brane interaction at both large (supergravity) and small (gauge theory) distance regions. The non-renormalization of the coefficient of this term implies that it should be present in the effective action of quantum $\mathcal{N} = 4$ SU(N) SYM theory on the Coulomb branch of the moduli space in both weak and strong coupling regions.

Therefore, it should be expected, both from the weakly coupled string theory “long-distance−short-distance” duality [1] and the AdS/CFT duality [16], that, like the P-even “$F^4/X^4 +$ superpartners” term in $S_{\text{BI}}$, this P-odd term should be exactly reproduced by the 1-loop computation on the $\mathcal{N} = 4$ SYM theory side.

Our aim below is to confirm this expectation by the explicit computation of the imaginary part of the fermion determinant in the SYM theory. As far as we are aware, the derivation of this $D = 4$ WZ term from $\mathcal{N} = 4$ SYM theory was missing in the literature (the calculations presented in [20,21], though similar, led to a different class of CS terms, see below). That this term should have, by analogy with the case discussed in [17], a Berry phase interpretation was suggested to one of us (A.T.) by M. Douglas (for some related work in the context of matrix models see also [18]).

Our direct perturbative derivation of the WZ term (6) in the effective action of $\mathcal{N} = 4$ SYM theory described below in Section 2 will not be referring to the Berry phase. We will show that the WZ term originate from the same hexagon diagram which is responsible for the chiral anomaly in ten-dimensional SYM theory [19].

In Section 3 we will describe a similar computation of counterparts of the ten-dimensional anomaly graph in other supersymmetric gauge theories describing systems of separated Dp branes with $p > 3$. In contrast to the $D = 4$ SYM case (and some of its analogs discussed below) where the WZ term is purely scalar and does not have a local representation, the hexagon graphs in $D > 4$ lead to a different class of WZ terms which involve both scalars and vectors and admit a local CS-type representation. This class of WZ terms was previously derived from SYM theories in [20,21].

Section 4 will contain some concluding remarks.

2. Wess-Zumino term in the $D = 4$ $\mathcal{N} = 4$ SYM theory

The WZ term is odd in time derivatives, so it has a factor of $i$ in the Euclidean action (below we choose the Euclidean signature of the metric). The effective action induced by the bosonic SYM degrees of freedom is real, so the only potential source of the imaginary WZ term is the (gauge-invariant, $O(6)$ invariant and conformal-invariant) phase of the
The fermion determinant. The appearance of a WZ term in the phase of a fermion determinant is not unusual, and examples of chiral WZW terms induced by fermionic loop are known [22,23]. The present case of scalar fields interacting with fermions in $\mathcal{N}=4$ SYM theory was not explicitly discussed before.

The part of the 1-loop effective action in $\mathcal{N}=4$ SYM theory which is induced by the 4 Weyl fermions has the following form after continuation to Euclidean space:

$$S_{\text{form}} = -\frac{i}{2} \operatorname{Tr} \ln\left( \Gamma^0 \Gamma^\mu \partial_\mu + i \Gamma^0 \Gamma^I \langle \Phi_I, \cdot \rangle \right) \times \frac{1 + \Gamma^{11}}{2}. \quad (7)$$

Here $\Gamma^M$ are ten-dimensional Dirac matrices:

$$\{ \Gamma^M, \Gamma^N \} = 2 \delta^{MN}, \quad (\Gamma^M)^T = -\Gamma^M.$$ \quad (8)

We assume that the 10-d indices are split in the $4 + 6$ way, $M = (\mu, I)$, and that the $D = 4$ vector field has trivial background. Generalization to the case of non-trivial $A_\mu$ background in $D > 4$ is straightforward (see [20,21] and Section 3 below), but in the case of $\mathcal{N}=4$ SYM theory the WZ term happens to depend only on the six scalar fields.

The system of $N$ “slowly moving” separated D3 branes is represented by slowly varying diagonal scalar fields:

$$\Phi_I = \begin{pmatrix} X_I^1 \\ \vdots \\ X_I^N \end{pmatrix}.$$ \quad (9)

The commutator term in the Dirac operator in this background vanishes for diagonal components of fermions, and for non-diagonal it becomes $[\Phi_I, \Psi]^{ab} = (X_I^a - X_I^b)\Psi^{ab}$. The effective action thus decomposes into a sum of pairwise interactions:

$$S_{\text{form}} = \sum_{a < b} S(X^a - X^b),$$

where

$$S(X) = -\operatorname{Tr} \ln\left( \Gamma^0 \Gamma^\mu \partial_\mu + i \Gamma^0 \Gamma^I X_I \right) \frac{1 + \Gamma^{11}}{2}. \quad (10)$$

Taking the variation, we find (Sp is the trace in spinor indices):

$$\frac{\delta S}{\delta X_i(x)} = \operatorname{Sp} \left\langle x \left| \frac{1}{i \Gamma^\mu \partial_\mu - \Gamma^I X_I} \right| \Gamma^I \frac{1 + \Gamma^{11}}{2} \right\rangle. \quad (11)$$

We are interested in the imaginary part of the effective action. Since the operator $i \Gamma^\mu \partial_\mu - \Gamma^I X_I$ is Hermitian, and $\Gamma^I (1 + \Gamma^{11}) = (1 - \Gamma^{11}) \Gamma^I$, taking the difference of the above expression and its complex conjugate it is easy to see that

$$\frac{\delta \operatorname{Im} S}{\delta X_i(x)} = \frac{1}{2i} \operatorname{Sp} \left\langle x \left| \frac{1}{i \Gamma^\mu \partial_\mu - \Gamma^I X_I} \right| \Gamma^I \Gamma^{11} \right\rangle.$$ \quad (12)

For slowly varying fields, this expression can be expanded in derivatives of $X_I$. The term with $n$ derivatives will be proportional to the trace of $2n + 3$ Dirac matrices. Since this trace contains $\Gamma^{11}$, and

$$\operatorname{Sp} (\Gamma^M \cdots \Gamma^{M_k} \Gamma^{11}) = 0 \quad \text{for } k < 10,$$

the expansion will start with the four derivative term coming from the diagram (dashed lines are external scalar fields):

Analytically,

$$\frac{\delta \operatorname{Im} S}{\delta X_j(x)} = -\frac{1}{2i} \int \frac{d^4 k}{(2\pi)^4} \frac{X_j}{(k^2 + X^2)^{\frac{3}{2}}} \times \partial_{\hat{\alpha}} X_{\hat{\alpha}} \partial_{\hat{\beta}} X_{\hat{\beta}} \partial_{\hat{\mu}} X_{\hat{\mu}} \partial_{\hat{\nu}} X_{\hat{\nu}} \times \operatorname{Sp} (\Gamma^I \Gamma^T \Gamma^T \Gamma^T \Gamma^T \Gamma^T \Gamma^T \Gamma^T \Gamma^T \Gamma^{11}) + O(\partial^5). \quad (14)$$
Using the identity
\[ \text{Sp} ( \Gamma^{M_1} \cdots \Gamma^{M_n} \Gamma^{11} ) = 32 i e^{M_1} \cdots M_n , \]
and doing the momentum integral, we find that the variation of the imaginary part of the effective action reproduces exactly the Lorentz force between a pair of D3 branes (4):
\[
\frac{\delta \text{Im} S}{\delta X_I} = \frac{1}{12 \pi^2} \epsilon^{\mu \nu \lambda \rho \sigma \tau} e^{IJKLMN} \frac{1}{|X|^6} \times X_J \partial_\mu X_K \partial_\nu X_L \partial_\lambda X_M \partial_\rho X_N + O(\delta^5).
\]
For the case of a single D3 brane interacting with a cluster of \( N \) coinciding D3 branes we get \( S_{\text{term}} = \overline{NS}X(X) \), and thus rederive the magnetic Chern-Simons term in the D3 brane probe action directly from the gauge theory.

3. WZ terms in \( D > 4 \) SYM theories

The \( D = 4 \) case discussed in the previous section is special in that the WZ term there depends only on the scalar fields. An analogous but actually different class of ‘magnetic’ WZ terms appears in the effective actions of maximally supersymmetric SYM theories in higher dimensions \( 4 < D = p + 1 < 9 \) \([20,21]\). For \( D > 4 \) one needs to switch also a non-trivial gauge field background in order to get a non-zero result for the imaginary part of the fermionic determinant. This has a natural interpretation on the supergravity side: while D3 branes carry both electric and magnetic charges and thus their interaction potential contains ‘magnetic’ contribution, separated magnetic \( (p > 3) \) Dp-branes interact only ‘electrically’, unless one switches on a gauge field background which induces effective electric charges on the Dp brane probe.

For example, the action of a D5 brane probe moving in the D5 brane background contains the CS term \( (I = 1,2,3,4) \)
\[
S_{\text{CS}} \propto i \int_6 C_2 \wedge F \wedge F \propto i N
\]
\[
\times \int_7 \epsilon_{IJKL} \frac{1}{|X|^4} X^I dX^J \wedge dX^K \wedge dX^L \wedge F \wedge F ,
\]
where \( dC_2 \) is the (magnetic) gauge field strength of the D5 brane source. The corresponding WZ term indeed arises in the effective action of \( D = 6 \) SYM theory describing multiple D5 branes.

Let us consider the general case of the maximal SYM theory obtained by dimensional reduction of \( D = 10 \) SYM theory to \( D = p + 1 < 10 \), and couple the fermions to both the diagonal scalar background (9) and the abelian gauge field background \( (\mu = 0,1,2,\ldots,p) \)
\[
A_\mu = \begin{pmatrix} A_\mu^1 \\ \vdots \\ A_\mu^N \end{pmatrix}.
\]
As in the \( p = 3 \) case (10), the fermionic contribution to the 1-loop effective action factorizes
\[
S_{\text{ferm}} = \sum_{a < b} S(X^a - X^b, A^a - A^b) ,
\]
where now
\[
S(X,A) = -\text{Tr} \left[ \ln \left( \Gamma^0 \Gamma^\mu \partial_\mu + i \Gamma^0 \Gamma^\mu A_\mu + i \Gamma^0 \Gamma^I X_I \right) \right] \times \frac{1 + \Gamma^{11}}{2}.
\]
The first term in the derivative expansion of the imaginary part of this action comes from the hexagon diagram and has the form similar to Eq. (14) with \((p + 1)\)-dimensional momentum integral instead of 4-dimensional one and with some of the scalar fields replaced by the gauge potentials.

The contribution of the hexagon diagram in various dimensions \( 5 \leq D \leq 9 \) was shown to give rise to local Chern-Simons terms in the effective action \([20]\). Below we rederive these Chern-Simons terms and clarify their relation to the WZ actions.
If \( p \leq 7 \), we can take a variation with respect to the scalar field to get:

\[
\frac{\delta \text{Im } S}{\delta X_I} = - (8 - p) d_{\rho} \epsilon^{\mu_1 \cdots \mu_{p-1}} e^{\mu_1 \cdots \mu_p \cdot 1} K_{I \cdots K_{7-p}}
\]

\[
\times \frac{1}{|X|} X_I \partial_{\mu_1} X_{\mu_1} \cdots \partial_{\mu_{p-1}} X_{\mu_{p-1}}
\]

\[
\times F_{\mu_1 \mu_2 \cdots \mu_{p+1}} \cdots F_{\mu_1 \mu_2 \cdots \mu_{p+1} + 1}
\]

\[
= \frac{p(p-1)}{2} (1 - 1) .
\]

\[
d_p = \frac{1}{4(\rho - 3)!(4\pi)^{\rho/2}} \frac{1}{F} \left( \frac{10 - p}{2} \right)
\]

As a result, the effective action contains the term:

\[
\text{Im } S = d_p \int d^{p+2} x \epsilon^{\mu_0 \cdots \mu_{p+1}} e^{\mu_0 \cdots \mu_p \cdot 1} K_{I \cdots K_{7-p}}
\]

\[
\times \frac{1}{|X|} X_I \partial_{\mu_0} X_{\mu_0} \cdots \partial_{\mu_{p-1}} X_{\mu_{p-1}}
\]

\[
\times F_{\mu_1 \mu_2 \cdots \mu_{p+1}} \cdots F_{\mu_1 \mu_2 \cdots \mu_{p+1} + 1}
\]

or, equivalently \( n_{\pi} = X_I/\pi(X) \)

\[
\text{Im } S = d_p \int d^{p+3} x \epsilon^{\mu_0 \cdots \mu_3} \partial_{\mu_0} X_{\mu_0} \cdots \partial_{\mu_3} X_{\mu_3}
\]

\[
\left( \partial^\mu \partial_{\mu} \right)\left( \alpha_{\mu} \partial_{\mu} \right) \partial_{\mu} \cdots \partial_{\mu} \cdots 
\]

\[
\partial_{\mu} \cdots \partial_{\mu}
\]

This expression reduces to our previous \( D = 4 \) SYM result (6) in the case of \( p = 3 \). For \( p = 5 \) this WZ term reproduces the CS interaction (17) in the supergravity description.

Note that the nonlocal nature of the action (23) is fake. Since \( F = dA \), we can integrate by parts and that leads to the local Chern-Simons form given in [20]. For example, the local CS form of (17) is \( j_\rho dC_3(X) \wedge F \wedge A \). This ‘integration by parts’ is not possible in the case of purely scalar WZ terms.

Like the CS terms in the Dp brane actions, the WZ terms with different \( p \) in (23) are related by dimensional reduction (′smearing’ and T-duality, \( A_i \rightarrow X_I \)).

Instead of computing the variation of the effective action over \( X_I \), another way to obtain these ‘mixed’ WZ terms is to calculate the induced current [20]

\[
\mathcal{J}^\mu = \frac{\delta \text{Im } S}{\delta A_\mu} .
\]

For \( p = 9 \) the divergence of this current produces the chiral anomaly [19], which makes the ten-dimensional non-abelian SYM theory inconsistent. For \( p = 8 \), the induced current is

\[
\mathcal{J}^\mu \propto \frac{1}{12288 \pi^4} \text{sgn} (X) \epsilon^{\mu_0 \cdots \mu_8} X_{\mu_0} \cdots X_{\mu_8} .
\]

\[
\text{sgn} (X) = \frac{X_y}{|X_0|} .
\]

In [21] it was suggested that the theory of multiple M5 branes should contain a similar ‘mixed’ \( j_\rho B_2 \wedge H_4 \), \( H_4 (X) = \epsilon_{IJKLM} X^{IJ} \wedge dX^k \wedge dX^l \wedge 
\]

\[
dX^M \), CS term related by compactification on \( S^1 \) to the \( j_\rho A \wedge H_4 \) CS term in \( D = 5 \) SYM (multiple D4 brane) theory, and the one-loop microscopic derivation (equivalent to the one in [20]) of the latter term was given.

Analogous WZ terms are found in other D-brane interaction systems described by gauge theories with less than maximal supersymmetry. For example, pure-scalar WZ terms appear in the case of Dp–D(p-6) (electric-magnetic) brane interaction. It is possible to see that in the case of the D5–D1 system described by a particular \( \mathcal{J}^\mu = 4 \), \( D = 2 \) supersym-
metric gauge theory [28], the relevant fermion determinant contains the WZ term \( \int \epsilon_{ijk} \frac{1}{|\lambda|^4} X_i dX_j \wedge dX_k \wedge dX_l \) which reproduces the CS term \( \int \lambda^2 C_2 \) in the action of a D-string probe moving in the magnetic background produced by a D5-brane source. The same term can be obtained by starting with the D5–D5 system with the WZ term (17), and compactifying 4 parallel directions on a torus and assuming that the gauge field background has a non-zero magnetic flux \( \int F \wedge F \) representing the D1-brane charge. This is a particular example of (T-duality) relations between different magnetic WZ terms in (23).

The abelian WZ terms discussed above have natural non-abelian generalizations. In particular, they should reproduce the non-abelian CS terms in multiple D-brane action given in [29].

Let us note also that SYM theories defined on curved D-dimensional spaces should contain curvature-dependent WZ terms similar to (23) with \( F \) replaced by \( R \) (and other mixed terms). They should reproduce the corresponding \( R \)-dependent CS terms [30,11,31] in the Dp-brane actions (for example, \( \int R \wedge R \wedge C_2 \) in D5 brane case).

4. Discussion

We have shown that the 1-loop effective action of \( D = 4 \), \( \mathcal{N} = 4 \) SYM theory contains the unique WZ term (6) coming from the phase of the Euclidean fermion determinant. The presence of this term (and of its \( D = 4 \) analogs (22)) is related to the existence of chiral anomaly in \( D = 10 \) SYM theory. This term depends only on 6 scalar fields, is manifestly \( SO(6) \) and conformal invariant, and its coefficient should not be renormalized by higher loop corrections.

One interesting open question is how to construct a supersymmetric generalization of this term. The answer does not seem obvious since the use of either \( \mathcal{N} = 1 \) or \( \mathcal{N} = 2 \) superfield formulation of \( \mathcal{N} = 4 \) SYM theory breaks the \( SO(6) \) symmetry. The WZ term (6) is actually the integral part of the space-time supersymmetric and \( \kappa \)-symmetric action [14] for a D3 brane propagating in \( AdS_5 \times S^5 \) vacuum of type IIB supergravity. As in the similar superstring action case [32], this term must be added to the Born-Infeld part (1) of the action to ensure its \( \kappa \)-symmetry.

Fixing the static gauge and a \( \kappa \)-symmetry gauge in the action of [14] in a suitable way one should be able to read off the 4-d supersymmetric form of this WZ term. After the gauge fixing, half of the original 32 superconformal symmetry generators become non-linearly realized, and they should be relating the WZ term to the terms in the BI part of the action.

It should be possible to rederive this WZ term directly from string theory, by taking an appropriate \( \alpha' \to 0 \) limit in the 1-loop expression for the interaction potential between two separated D3-branes. The topological nature of this term suggests that it should be originating from certain fermionic zero mode contribution.

Similar purely scalar magnetic WZ term \( \int \lambda^3 C_3 \) appears in the classical action of M2 brane moving in the background of an M5 brane source. Though lack of detailed understanding of the theory of multiple M-branes prohibits us from deriving this term directly from a microscopic theory (as was possible in the D-brane case), it is natural to expect (by analogy with a related proposal about a CS term in the M5 brane theory action [21]) that this WZ term is again universal, i.e. is not renormalized.

The magnetic WZ terms are present also in some orbifold theories [33]. For example, they appear in the field theory on a stack of an equal number of electric and magnetic D3-branes in type 0 string theory [34], which is a \( \mathbb{Z}_2 \) orbifold of \( \mathcal{N} = 4 \) SYM [35]. This theory has two sets of scalar fields, \( X^{el}_{(i)} \) and \( X^{mag}_{(a)} \), which correspond to the transverse coordinates of the electric and the magnetic branes. The Yukawa couplings in the diagonal background of these scalar fields are \( \prod^{el}_{i} (X^{el}_{(i)} - X^{mag}_{(a)}) \prod^{el}_{i} \prod^{mag}_{a} \) [36]. This Yukawa interaction induces a WZ term depending on the difference \( X^{el}_{(i)} - X^{mag}_{(a)} \). The same prediction (CS term in the interaction potential) should follow from the gravity description, since there should be a Lorentz force between the separated electric and the magnetic D-branes.
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References

Fine-tuning constraints on supergravity models

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Abstract

We discuss fine-tuning constraints on supergravity models. The tightest constraints come from the experimental mass limits on two key particles: the lightest CP even Higgs boson and the gluino. We also include the lightest chargino which is relevant when universal gaugino masses are assumed. For each of these particles we show how fine-tuning increases with the experimental mass limit, for four types of supergravity model: minimal supergravity, no-scale supergravity relaxing the universal gaugino mass assumption, D-brane models and anomaly mediated supersymmetry breaking models. Among these models, the D-brane model is less fine tuned. The experimental prospects for an early discovery of Higgs and supersymmetry at LEP and the Tevatron are discussed in this framework. © 2000 Elsevier Science B.V. All rights reserved.

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input parameters [1], for example for $\tan \beta = 2.5$ we find

$$\frac{M_Z^2}{2} = -0.87 \mu^2(0) + 3.6 M_3^2(0) - 0.12 M_1^2(0)$$

$$+ 0.007 M_1^2(0) - 0.71 m_{\tilde{\chi}_1}^2(0) + 0.19 m_{\tilde{\psi}}^2(0)$$

$$+ 0.48 \left( m_{\tilde{\chi}_1}^2(0) + m_{\tilde{\psi}}^2(0) \right)$$

$$- 0.34 A_1(0) M_1(0) - 0.07 A_2(0) M_2(0)$$

$$- 0.01 A_1(0) M_1(0) + 0.09 A_1^2(0)$$

$$+ 0.25 M_2(0) M_3(0) + 0.03 M_1(0) M_0(0)$$

$$+ 0.007 M_1(0) M_2(0), \quad (1)$$

where we have implicitly assumed all the soft breaking parameters to be real, neglecting the phases ². One implication of the fact that fine-tuning is dominated by $M_3$ is the fact that the soft scalar masses can be larger than $M_3$ without increasing fine-tuning, a fact which has recently been emphasised in the framework of minimal supergravity in Ref. [2].

In this paper we shall extend the discussion in Ref. [1] in two ways. Firstly we shall study fine-tuning in various supergravity models: minimal supergravity, no-scale supergravity (relaxing the universal gaugino mass assumption), D-brane models and anomaly mediated supersymmetry breaking models (AMSB). The common feature of this class of models is that they involve a large mass scale of order the unification scale say $M_U \sim 2 \times 10^{16}$ GeV, and supersymmetry breaking is mediated via some sort of hidden sector supergravity mechanism. Thus our analysis does not extend to either gauge-mediated supersymmetry breaking models, or models where the string scale is lowered beneath the unification scale, although it may be lowered to the unification scale. The reason why we choose these models is that they contain the largest mass hierarchy, and hence face the most severe fine-tuning constraints in general. These models also preserve the gauge unification success most simply and directly.

Secondly we focus on the key particles whose experimental mass limits lead most sensitively to increases in fine-tuning. Clearly fine-tuning is not sensitive to squark and slepton masses which can be increased substantially due to the insensitivity of the Z mass formula in Eq. (1) to soft scalar masses. By contrast the lightest CP even Higgs mass is a very sensitive probe of fine-tuning, as we emphasised previously [1], and it is obvious from the foregoing discussion that the gluino mass itself is also a sensitive probe. Although we showed [1] that the right-handed mass is only a sensitive probe of fine-tuning if one assumes universal gaugino masses, we shall nevertheless include it for illustrative purposes.

The implicit sensitivity of the Z mass coming from changes in $\tan \beta$ as a result of small variations in the high energy inputs, does not appear in Eq. (1). This is addressed by the master formula of Dimopoulos and Giudice [3] which yields a fine-tuning parameter which corresponds to the fractional change in the Z mass squared per unit fractional change in the input parameter,

$$\Delta_a = \frac{a}{M_Z^2} \left( \frac{\partial M_Z^2}{\partial a} \right), \quad (2)$$

for each input parameter $a$ in the model of interest. The fine-tuning is then simply the maximum value of $\Delta_a$ over all the input parameters. Although there are many more sophisticated measures of fine-tuning available [3–6], this basic measure of fine-tuning is adequate for our purposes of comparing relative fine-tunings amongst different models.

The models we consider, and the corresponding input parameters given at the unification scale, are listed below:

1. Minimal supergravity [2].

$$a_{\text{msugra}} = \left\{ m_0, M_{1/2}, A(0), B(0), \mu(0) \right\}, \quad (3)$$

where as usual $m_0$, $M_{1/2}$ and $A(0)$ are the universal scalar mass, gaugino mass and trilinear coupling respectively, $B(0)$ is the soft breaking bilinear coupling in the Higgs potential and $\mu(0)$ is the Higgsino mass parameter.
2. No-scale supergravity [7] with non-universal gaugino masses 
\[
a_{\text{no-scale}} \in \{ M_1(0), M_2(0), M_3(0), B(0), \mu(0) \}.
\] (4)

3. D-brane model [8],
\[
a_{D-\text{brane}} \in \{ m_{3/2}, \theta, \Theta_1, \Theta_2, \Theta_3, B(0), \mu(0) \},
\] (5)

where \( \theta \) and \( \Theta_i \) are the goldstino angles, with 
\( \Theta_1^2 + \Theta_2^2 + \Theta_3^2 = 1 \), and \( m_{3/2} \) is the gravitino mass. The gaugino masses are given by
\[
M_i(0) = M_i(0) = \sqrt{3} m_{3/2} \cos \theta_i e^{-i \alpha_i},
\] (6)

and there are two types of soft scalar masses
\[
m_{3152}^2 = m_{3/2}^2 \left[ 1 - \frac{1}{2} \left( \sin^2 \theta + \cos^2 \theta \Theta_i^2 \right) \right],
\] (7)

\[
m_{31}^2 = m_{3/2}^2 \left[ 1 - 3 \sin^2 \theta \right].
\]

4. Anomaly mediated supersymmetry breaking [9].
\[
a_{\text{AMSb}} \in \{ m_{3/2}, m_0, B(0), \mu(0) \}.
\] (8)

Our numerical results are based on two-loop renormalisation group running of gauge [4], third generation Yukawa couplings and soft mass parameters [10]. The initial values of the Yukawa couplings are determined by the values of the third generation fermion masses [5]. The input soft mass parameters are then chosen in order to get electroweak symmetry breaking and the \( m_Z \) scale given by the minimisation conditions of the one-loop corrected Higgs potential [13,14],
\[
\frac{m_Z^2}{2} = m_{H_u}^2 - m_{H_d}^2 \tan \beta^2 - \Delta^2 - \mu^2,
\] (9)

where \( \tan \beta = \langle H_u \rangle / \langle H_d \rangle \), \( \Delta^2 \) is the one-loop contribution, and the parameters in Eq. (9) are evaluated at \( m_t \). In practice, for the numerical calculations we use as input \( \tan \beta \) and sign(\( \mu \)) (we always take \( \mu > 0 \)) and obtain \( \mu(0) \) and \( B(0) \) from the minimisation conditions.

Our main results are shown in Figs. 1–4, corresponding to SUGRA models 1–4 above. The results are shown for three values of \( \tan \beta = 2,3,10 \), corresponding to three sets of curves from top left to bottom right, respectively. In each case we plot the maximum sensitivity parameter \( \Delta_{\text{max}} \) as a function of particle mass, for the lightest CP even Higgs boson (short dashes), the lighter chargino (long dashes) and the gluino (solid). The lightest CP even Higgs boson is calculated using the one-loop RG-improved effective potential approach [15], which includes the leading two-loop corrections to the Higgs mass [6]. The gluino mass also includes the corrections due to gluon/gluino and quark/squark loops [17,12].

In Fig. 1 we give the fine-tuning results for mSUGRA. The present LEP2 mass limits of around 100 GeV on the Higgs and chargino compete for providing the tightest fine-tuning constraint, while the current Tevatron gluino mass limit of around 250 GeV provides a slightly less severe limit. For a Higgs mass of 100 GeV, \( \tan \beta = 10 \) allows fine-tuning to stay at around 10, but as the Higgs mass increases it rapidly overtakes the chargino mass in importance and as it approaches 110 GeV fine-tuning rises steeply to 100. Comparing Fig. 1(a) with \( m_0 = 100 \) GeV, to Fig. 1(b) with \( m_0 = 1000 \) GeV we see that for \( \tan \beta = 10 \) the curves are very similar, as emphasised in Ref. [2]. However we emphasise that for lower values of \( \tan \beta \) fine-tuning increases substantially as \( m_0 \) is increased from 100 GeV to 1000 GeV in mSUGRA. Also in Fig. 1(a) we show results

---

1 This is in fact a new model not previously considered in the literature, although the no-scale model with universal gaugino masses is of course well known. As in the usual no-scale model, this model has the attractive feature that flavour-changing neutral currents at low energies are very suppressed, since all the scalar masses are generated by radiative corrections, via the renormalisation group equations, which only depend on the gauge couplings which are of course flavour-independent.

2 When running the gauge couplings we have included complete threshold effects at order 1-loop [11] and used the step-function approximation in the 2-loops coefficients.

3 We have included one-loop susy threshold corrections, QCD and electroweak corrections when converting pole mass values to running mass values at the \( m_Z \) scale [12].

4 The expected accuracy in the computed Higgs mass is estimated to be \( \sim 2 \) GeV. A different approach to the calculation of the Higgs mass can be founded in Ref. [16].
for no-scale mSUGRA with $m_0 = 0$ for $\tan \beta = 2, 3, 10$ seen as short lines almost superimposed over the $m_0 = 100$ GeV lines. The reason why the no-scale lines are so short is that if $M_{1/2}$ is too small the right-handed slepton falls below its experimental limit of 88 GeV, while if $M_{1/2}$ is too large it becomes the LSP. Thus there is only a narrow allowed window for $M_{1/2}$ which in the case of $\tan \beta = 10$ is non-existent.

In Fig. 2 we give results for a generalised version of no-scale mSUGRA hitherto not considered in the literature in which $m_0 = 0$ as usual, but now we allow the gaugino masses to be non-universal. For definiteness we take $M_1(0) = M_2(0)$, but allow these gaugino masses to be different from the high energy gluino mass $M_3(0)$. The first point to make is that by relaxing gaugino mass universality, a larger parameter space is opened up and the constraints which forced $M_{1/2}$ into a small allowed range in the no-scale mSUGRA model are now replaced by large allowed regions in non-universal gaugino mass space. For example taking $M_1(0) = M_2(0) = 250$ GeV in Fig. 2(a) we see a large range of $M_3(0)$ is allowed. Also we find that fine-tuning is generally smaller in this model than mSUGRA for $\tan \beta = 10$. The reason is that although the Higgs curves in Fig. 2(a) are very similar to those in Fig. 1(a), the chargino curves are very different. In the no-scale model with non-universal gaugino masses a chargino mass limit of around 100 GeV implies a fine-tuning of between 10 and 20, almost independently of $\tan \beta$, whereas in the conventional no-scale model the corresponding fine-tuning is between 20 and 100.

The kinks in the gluino curves in Fig. 2(a) correspond to $\Delta_{\mu(0)}$ being replaced by $\Delta_{\mu(0)}$ as the largest fine-tuning parameter as $M_3(0)$ (and thus $\mu(0)$) is increased. For $M_3(0) < M_2(0)$ a partial cancellation occurs in Eq. (1) between $M_2(0)$ and $M_3(0)$
Fig. 2. Results for the no-scale with non-universal gaugino masses. The maximum sensitivity parameter $\Delta_{\text{max}}$ is plotted as a function of the lightest CP even Higgs mass (short dashes), gluino mass (solid line) and lightest chargino (long dashes). For each particle type, the three sets of curves correspond to $\tan \beta = 2, 3, 10$, from top left to bottom right, respectively. In panel (a) we fix $M_0 = 250$ GeV, while in panel (b) $M_0 = 500$ GeV.

[1], which renders the fine-tuning for $\mu(0)$ small. Because of that, in the region where the chargino is lighter and mainly higgsino fine-tuning is quite insensitive to its mass. This can be seen clearly in Fig. 2(b), where we show results for $M_2(0) = M_1(0) = 500$ GeV. Interestingly for $\tan \beta = 3, 10$ the chargino curves are almost flat, due to its Higgsino nature, while for $\tan \beta = 2$ the curve is much steeper. However, for a chargino mass of order 100 GeV, the overall fine-tuning is larger than in Fig. 2(a). This corresponds to the increase in $M_1(0)$ when increasing $M_2(0)$ as required by Eq. (1). For $\tan \beta = 2$ the curves are cut in the region of a light chargino because the lightest stop falls below its experimental lower bound $^7$ of around 90 GeV. For the three values of $\tan \beta$ considered, an upper limit on $M_1(0)$ is set by the requirement that the lightest neutralino mass does not exceed the slepton mass.

In Fig. 3 we give results for a D-brane scenario, where we take the Goldstino angles $\cos \theta = 1$ and $\Theta_\perp = 0$. We also set all the scalar masses equal to the universal value $m^2_{3152}$ at the high energy scale $^8$ $M_{\text{GUT}}$. The gaugino masses are again non-universal but now $M_2(0) = M_1(0)$ and the ratio of these masses to $M_1(0)$ is controlled by the Goldstino angles $\Theta_\perp$ and $\Theta_\parallel$. These are constrained to lie along a unit circle, and thus we have only the freedom to change their

$^7$ The same effect can be seen for the D-brane model in Fig. 3. $^8$ Other choices of the Goldstino angles or the scalar masses will affect mainly the low energy values of the scalar masses, and not so much those of the gauginos (a change in $\cos \theta$ can be compensated by a rescaling of the gravitino mass). This may change the region of the parameter space allowed by the experimental constraints, but it will leave practically unchanged the conclusions on fine-tuning.
Fig. 3. Results for the D-brane model. The maximum sensitivity parameter \( \Delta_{\text{max}} \) is plotted as a function of the lightest CP even Higgs mass (short dashes), gluino mass (solid line) and lightest chargino (long dashes). For each particle type, the three sets of curves correspond to \( \tan \beta = 2, 3, 10 \), from top left to bottom right, respectively. In panel (a) we fix \( M_f(0) = 250 \) GeV, while in panel (b) \( M_f(0) = 500 \) GeV.

ratio \( \Theta = \Theta_1/\Theta_2 \) when moving along the circumference. Therefore, we compute the fine-tuning for \( \Theta \) instead of those for \( \Theta_1 \) and \( \Theta_2 \). The results in Fig. 3(a) for \( M_f(0) = 250 \) GeV are quite similar to those in Fig. 2(a), and imply a similarly low fine-tuning. In Fig. 3(b) the choice \( M_f(0) = 250 \) GeV now leads to larger allowed regions than in Fig. 2(b) due to the presence of a non-zero scalar mass, with the charginos being now significantly heavier due to their gaugino component. Now the parameters that compete to give the largest fine-tuning are \( m_0 \) and \( \Theta \), and the kink in the gluino curves is due to \( \Delta_\mu \) being replaced by \( \Delta_{\mu(0)} \) as the maximum sensitivity parameter. The other functions \( \Delta_{\mu(0)} \) and \( \Delta_\theta \) can become comparable but not dominant. As in the generalised no-scale model, the maximum fine-tuning will be insensitive to a light chargino when this is mainly higgsino.

In Fig. 4 we give results for the AMSB model. In the minimal AMSB model where \( m_0 = 0 \) the sleptons are predicted to have negative mass squared, so we have followed the common procedure of simply adding a universal scalar mass squared \( m_0^2 \) by hand, ensuring that it is large enough to ensure acceptable slepton masses. In Fig. 4(a) we choose \( m_0 = 500 \) GeV, and in Fig. 4(b) we take \( m_0 = 1000 \) GeV. In both cases the fine-tuning is dominated by \( \Delta_{\mu(0)} \), and is much larger than the other models considered. Typically the value of \( \mu(0) \) required by electroweak symmetry breaking is \( O(1 \text{ TeV}) \) or larger in this models.

Comparing the results for all the models in Figs. 1–4 it is seen that there is slightly less fine-tuning associated with particle masses in the D-brane model than in the other models. However it is also apparent that the results for the no-scale model with non-universal gaugino masses are very similar to the D-brane scenario. The common feature of both these models is non-universal gaugino masses, and the reasons for the reduced fine-tuning are essentially those empha-
sised in Ref. [1] (namely that fine-tuning is most sensitive to $M_A(0)$ and so $M_A(0) < M_A(0)$ in general reduces fine-tuning.) However, in the D-brane model there is additionally the possibility of cancellations among different input parameters which help to lower the different fine-tuning parameters. For example, using one-loop semi-analytic solutions to the renormalisation group equations [18] and neglecting one-loop effective potential contributions, we find the approximate expressions for $\tan \beta = 3$:

\[
\Delta_{m_{3/2}} \approx \tilde{m}_{3/2}^2 \cos^2\theta \left(120.67 \Theta_1^2 - 8.15 \Theta_2^2 + 10.13 \Theta_2 \Theta_1 - 0.32 (1 - 3 \cos \theta^2)\right),
\]  

\[
\Delta_m(0) \approx \left[5.12 + \tilde{m}_{3/2}^2 \cos^2\theta \left(-135.93 \Theta_1^2 + 6.85 \Theta_2^2 - 11.40 \Theta_2 \Theta_1 + 1.14 \tilde{m}_{3/2}^2 (1 - 3 \cos \theta^2)\right)\right].
\]

\[
\Delta_\theta \approx \tilde{m}_{3/2}^2 \cos^2\theta \left(128.81 \Theta_1 \Theta_2 + 5.07 (\Theta_2^2 - \Theta_1^2)\right),
\]

\[
\Delta_\theta \approx \tilde{m}_{3/2}^2 \cos^2\theta \left[112.14 \Theta_1^2 - 8.15 \Theta_2^2 + 8.97 \Theta_2 \Theta_1 + 0.32\right].
\]

where $\tilde{m}_{3/2}$ is the gravitino mass scaled by $m_g$, and we have kept the dependence on $\cos \theta$ but taken $\Theta_1 = 0$. From the above expressions it is clear that choosing appropriate values for the goldstino angles, $\Delta_{m_{3/2}}$ might be arbitrarily small even for very large
values of the gravitino mass, and similarly for $\Delta_{\mu(0)}$ and $\Delta_{\mu}$.

In Fig. (5) we have plotted the contours of constant $\Delta$ given in Eqs. (10)–(13). Not surprisingly, all the contours show a hyperbolic behavior: they would more or less follow the curves of constant $M_0$, with fine-tuning increasing with the gluino mass. We have also included the contours of constant $M_0$ for the values considered in Fig. 3. Although $\Delta_{\mu(0)}, \Delta_{\mu(1/2)}$ and $\Delta_{\mu}$ are all simultaneously small for $\Theta_3 = 0.2$, parts of this region are experimentally excluded due to $\mu$ becoming too small and hence the lightest chargino becoming too light. Nevertheless, there are allowed regions in the plane $m_{3/2} - \Theta_3$, corresponding to a light gluino, where $\Delta_{\mu(0)}, \Delta_{m_{3/2}}$ and $\Delta_{\mu}$ are all simultaneously small, and the maximum sensitivity would be given by $\Delta_{\mu}$. We may try to play with the values of either $\cos \theta$ or $\Theta_3$ in order to find some region where all the fine-tuning is small. However, reducing (increasing) $\cos \theta$ ($\Theta_3$) the slepton masses tend to diminish, making it difficult if not impossible to fulfill the experimental constraints on the SUSY masses.

Any conclusions which are drawn from fine-tuning are always subject to caveats, disclaimers and health warnings. A precise value cannot be placed on fine-tuning, since the definition can always be changed and the question of how much fine-tuning is acceptable is subjective. For this reason we prefer not to give upper bounds on particle masses based on fine-tuning, but clearly subjective upper bounds can

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For example, for $\cos \theta < 0.5$ and $\tan \beta > 3$, the parameter space compatible with experiments shrinks to nothing.
be read off from our curves, for those inclined to do so. The main value of our work is to compare different SUGRA models with each other, and within each SUGRA model to compare different regions of parameter space, from the point of view of fine-tuning.

In all models, fine-tuning is reduced as $\tan\beta$ is increased, with $\tan\beta = 10$ preferred over $\tan\beta = 2.3$. Nevertheless, the present LEP2 limit on the Higgs and chargino mass of about 100 GeV and the gluino mass limit of about 250 GeV implies that $\Delta_{\text{max}}$ is of order 10 or higher. The fine-tuning increases most sharply with the Higgs mass. The Higgs fine-tuning curves are fairly model independent, and as the Higgs mass limit rises above 100 GeV come to quickly dominate the fine-tuning. We conclude that the prospects for the discovery of the Higgs boson at LEP2 are good. For each model there is a correlation between the Higgs, chargino and gluino mass, for a given value of fine-tuning. For example if the Higgs is discovered at a particular mass value, then the corresponding chargino and gluino mass for each $\tan\beta$ can be read off from Figs. 1–4.

The new general features of the results may then be summarised as follows:

- The gluino mass curves are less model dependent than the chargino curves, and this implies that in all models if the fine-tuning is not too large then the prospects for the discovery of the gluino at the Tevatron are good.

- The fine-tuning due to the chargino mass is model dependent. For example in the no-scale model with non-universal gaugino masses and the D-brane scenario the charginos may be relatively heavy compared to $m_{\text{SUGRA}}$.

- Some models have less fine-tuning than others. We may order the models on the basis of fine-tuning from the lowest fine-tuning to the highest fine-tuning: D-brane scenario $< \text{generalised no-scale SUGRA} < m_{\text{SUGRA}} < \text{AMSB}$.

- The D-brane model is less fine-tuned partly because the gaugino masses are non-universal, and partly because there are large regions where $\Delta_{W}, \Delta_{\mu}, \Delta_{U(0)}$, and $\Delta_{Q}$ are all close to zero (see Fig. 5). However in these regions the fine-tuning is dominated by $\Delta_{Q}$, and this leads to an inescapable fine-tuning constraint on the Higgs and gluino mass.

Finally we should comment on the parameter space dependence of our results. Although the results presented here are for specific choices of parameters, we have performed a detailed analysis of the parameter space of these models and found that the results are representative of the full parameter space, and the qualitative conclusions will not change. We shall present the complete analysis elsewhere [19].

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References


Periodic instantons in \( SU(2) \) Yang–Mills–Higgs theory

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Abstract

The properties of periodic instanton solutions of the classical \( SU(2) \) gauge theory with a Higgs doublet field are described analytically at low energies, and found numerically for all energies up to and beyond the sphaleron energy. Interesting new classes of bifurcating complex periodic instanton solutions to the Yang–Mills–Higgs equations are described. © 2000 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

Anomalous baryon and lepton number violating processes in the electroweak theory are dominated in the semiclassical weak coupling limit, \( \alpha \to 0 \), by field configurations which solve the classical Euclidean Euler–Lagrange equations. At zero temperature and energy the classical solutions that contribute to anomalous winding number transitions are the familiar BPST instantons/anti-instantons [1]. The rate of these vacuum tunneling transitions is exponentially suppressed \( \sim \exp(-2S_I) \), where \( 2S_I = 4\pi/\alpha_\mu \) is the Euclidean action of a widely separated instanton (I)/anti-instanton (\( \bar{I} \)) pair. At temperatures much higher than \( M_W \), the transitions are classical thermal activation transitions over the potential barrier between vacua, with a Boltzmann rate \( \sim \exp(-E_s/k_B T) \) controlled by the energy \( E_s \sim 4M_W/\alpha_\mu \) of a certain unstable classical stationary field configuration called the sphaleron [2].

At intermediate temperatures, or at finite energy (not necessarily arranged in thermal equilibrium), very little is known about the rate of anomalous transitions between states of different winding number. The first step in studying these transitions is to find the classical solutions which dominate the semiclassical rate, and calculate their action. This involves solving the classical nonlinear field equations in Euclidean time. At low energies one can construct solutions of the classical Euclidean Yang–Mills–
Higgs equations consisting of periodic chains of $\tilde{H}$ pairs arrayed along the imaginary time axis. The action of these periodic instanton solutions can be expressed as a power series in $\left( E/E_s \right)^{2/3}$ for small $E/E_s$ [3]. This perturbative treatment of low energy periodic instanton solutions can be recast as an expansion in powers of $(M_p \beta)^2$ where $\beta$ is the period of the solution. For larger energies or periods the solutions can be found numerically. In this Letter we describe the qualitative properties of and present numerical results for these periodic instanton solutions of the $SU(2)$ Yang–Mills–Higgs equations, i.e. the bosonic sector of the standard electroweak theory expressed as a power series in $\theta_s = 0$. An unexpectedly rich structure of bifurcating periodic instanton solutions, both real and complex, has been found, whose physical consequences for B and L violating transitions in the electroweak theory remains to be more fully investigated.

2. Periodic instantons at low energy

As a simple example of a periodic potential with periodic instanton solutions consider the pendulum potential,

$$V(q) = \omega^2 (1 - \cos q).$$

The zero energy instanton which interpolates between the vacuum states at $q = 0$ and $q = 2\pi$ is the kink configuration, $q_i(\tau)$, given by

$$\cos \left( \frac{q_i(\tau)}{2} \right) = - \tanh (\omega \tau),$$

which solves the classical Euclidean equations $\ddot{q}_i = V'(q_i)$, and has action $S_i = 8 \omega$. The anti-instanton solution is $q_\bar{i}(\tau) = q_i(-\tau)$ with the same action. Consider now the widely separated $I - \bar{I}$ pair configuration,

$$q_{\bar{I}}(\tau) = q_i(\tau) + \bar{q}_i(\tau - \bar{\tau}) - 2\pi,$$

with $\bar{\tau} \gg 1/\omega$. The action of this configuration can be computed to first order in the interaction between the pair, with the result

$$S[q_{\bar{I}}] = 2S_i - 32 \omega e^{-\omega \tau} + \Theta(e^{-2\omega \tau}).$$

The negative sign reflects the attractive interaction between the $I$ and $\bar{I}$.

We now consider a periodic arrangement of $I$ and $\bar{I}$ at equal intervals along the imaginary time axis, with period $\beta$. This means that the separation between nearest neighbor $I$ and $\bar{I}$ is $\bar{\tau} = \beta/2$. The attractive force between neighbors can now exactly balance and yield an extremum of the action, the periodic instanton solution. Since there are two nearest neighbor $\tilde{H}$ interactions per period we expect the action per period of this solution to be

$$S(\beta) = 16 \omega - 64 \omega e^{-\omega \beta/2} + \Theta(e^{-\omega \beta}),$$

in the limit of large $\omega \beta$. Since

$$E(\beta) = \frac{dS(\beta)}{d\beta} = 32 \omega^2 e^{-\omega \beta/2} + \ldots$$

large $\beta$ corresponds to low energy. In the one dimensional pendulum example the exact periodic instanton solution with this action and energy are easily found explicitly in terms of elliptic functions by simple quadrature. Because of the attractive interaction between the $I$ and $\bar{I}$ along the chain it is clear that there is a single negative mode of the second order fluctuation operator, $-\dddot{q}(\tau) + V''(q(\tau))$, around this periodic instanton solution, a fact that is also reflected by the second derivative of the action,

$$\frac{d^2 S(\beta)}{d\beta^2} = \frac{dE(\beta)}{d\beta} = -16 \omega^2 e^{-\omega \beta/2} + \ldots < 0.$$  

The monotonic decrease of period $\beta$ with increasing energy persists up to $E = E_s = 2 \omega^2$, where the curve of $S(\beta)$ versus $\beta$ of the periodic instanton becomes tangent to the constant sphaleron solution, corresponding in this simple model with the unstable static configuration $q_s = \pi$. This occurs at $\beta = \beta_+ = 2\pi/\omega$ equal to the period of oscillation in the inverted potential at $q = q_s$. At this $\beta$ the action of the periodic instanton is $E_{\beta_+} = 4\pi \omega < 16 \omega$, reflecting the fact that the action is monotonically decreasing as $\beta$ ranges from $\infty$ down to $\beta_-$, and as $E$ increases from 0 to $E_s$.

Beyond the point where the periodic instanton and sphaleron solutions merge, a complex solution bifurcates from the sphaleron and continues with real decreasing action. This may be understood by the amplitude of the zero mode at $\beta = \beta_-$ turning from real to pure imaginary as $\beta$ is decreased through the critical value. The generic behavior described here is
what we call type (I) behavior of the periodic instanton solutions, for which the monotonic negative sign in the first derivative of the action in (7) and $E, \beta_+ < 2S_\tau$ are characteristic.

A different pattern is possible when the instanton has additional zero modes, and therefore additional parameters enter the description. Such is the case in field theory models with exact or softly broken conformal invariance. We have studied this case in some detail in the $O(3)$ nonlinear sigma model in two dimensions, softly broken by a mass term [4]. This model shares many features with the bosonic sector of the $SU(2)$ electroweak theory. Although there is no isolated single $I$ or $\bar{I}$ solution in the broken theory, due to Derrick’s theorem (which tells us that zero scale size $\rho \to 0$ has minimum action), a periodic instanton solution does exist in which the scale size $\rho$ is adjusted to a certain value as a function of period $\beta$. At this value the attractive interaction between $I$ and $\bar{I}$ exactly balances the tendency of each individual $I$ or $\bar{I}$ to collapse to zero size. It is again possible to understand this at low energies by first finding the two-body interaction between well isolated $I$ and $\bar{I}$, and then arranging them periodically along the imaginary time axis, calculating the change in the action per period from that of a single $\bar{I}$ pair due to the sum of the first order interactions between them. In this calculation the scale size $\rho$ can be treated as a variational parameter with the value on the solution $\rho(\beta)$ determined by extremizing $S(\beta, \rho)$ with respect to $\rho$. Substituting into $S$ then gives $S(\beta)$ on a low energy periodic instanton solution. The resultant behavior depends upon the curvature of $S(\beta)$, which is of opposite sign relative to the type (I) models, and we will consequently refer to this case as type (II), i.e.

$$\frac{d^2S(\beta)}{d\beta^2} = \frac{dE(\beta)}{d\beta} > 0, \text{ type (II)}. \quad (8)$$

The periodic instanton solutions in this case have two negative modes rather than just one, with the second negative mode corresponding to variation of the scale size $\rho$ away from its extremal value $\rho(\beta)$. Such periodic instanton solutions do not contribute to thermal winding number processes at finite temperature. However, they can contribute to anomalous finite-energy non-thermal transitions.

Because of the existence of the conformal mode in $SU(2)$ BPST instantons, we would expect the Yang–Mills–Higgs theory to behave qualitatively similar to the $O(3)$ sigma model, and to also be of type (II). Indeed for low energies we can show that this is exactly what happens. The action for an isolated pure $SU(2)$ $\bar{I}$ pair with scale size $\rho$ separated by distance $\bar{\tau}$ is

$$S_{\text{int}} = S_I - 2S_\tau - \frac{96\pi^2 \bar{\rho}^4}{g^2 \bar{\tau}^4} + \sigma(\frac{\bar{\rho}^6}{g^2 \bar{\tau}^6}), \quad (9)$$

with $S_I = 8\pi^2 / g^2 = 2\pi / \alpha$ the single instanton action and the second term the well-known dipole-dipole attractive interaction between the $I$ and $\bar{I}$ aligned in the internal $SU(2)$ orientation. When the periodic chain of $I$ and $\bar{I}$ separated by $(n + \frac{1}{2})\beta$ is constructed, this leads to the total interaction,

$$S_{\text{int}} = -\frac{96\pi^2 \rho^4}{g^2 \bar{\tau}^4} \sum_{n=\infty}^n \frac{1}{(n + \frac{1}{2})^2} = -\frac{4\pi}{\alpha} \left(\frac{2\pi \rho^4}{\bar{\tau}^4}\right). \quad (10)$$

When the $SU(2)$ doublet Higgs field is added to the action it can be solved for at leading order in the $I$ or $\bar{I}$ background and gives a contribution,

$$S_{\text{Higgs}} = \frac{4\pi}{\alpha} \left(\frac{M_w^2 \rho^2}{2} + \sigma(\frac{M_w^2 \rho^4}{g^2 \rho^6})\right), \quad (11)$$

which was first calculated by ’t Hooft [6].

This positive contribution expresses the fact that Derrick’s theorem drives the single isolated $I$ or $\bar{I}$ scale size to zero $\rho$; however, at finite $\beta$ this is opposed by the dipole-dipole interaction $S_{\text{int}}$, and the variational action

$$S(\beta, \rho) = \frac{4\pi}{\alpha} \left[1 + \frac{M_w^2 \rho^2}{2} - \frac{2\pi \rho^4}{\beta^4} + \sigma(\frac{M_w^2 \rho^4}{\beta^4} \rho^6)\right], \quad (12)$$

has a non-trivial extremum at $\rho = \rho(\beta)$ where $\partial S / \partial \rho = 0$, or

$$\rho = \sqrt{\frac{2}{M_w}} \left[\chi^2 + \sigma(\chi^4)\right]. \quad (13)$$
denoting by $\chi$ the expansion parameter $M_W \beta / (2 \pi)$. This stationarity condition implies that the various next-to-leading contributions to the action from both the gauge and Higgs fields are all of order $\chi^6$. Hence the periodic instanton action,

$$S(\beta) = \frac{4\pi}{\alpha} \left[ 1 + \frac{1}{2} \chi^4 + \mathcal{O}(\chi^6) \right]$$

(14)

for small $M_W \beta$, and

$$E = \frac{dS(\beta)}{d\beta} = \frac{4M_W}{\alpha} \chi^3 \left[ 1 + \mathcal{O}(\chi^2) \right]$$

(15)

is an increasing function of period $\beta$ for small $M_W \beta$. The second derivative of $S(\beta)$ is also clearly positive. After a rather elaborate calculation [5], one can evaluate the coefficients of the next to leading terms in the perturbative expansion. One thus finds

$$S = \frac{4\pi}{\alpha} \left[ 1 + \frac{1}{2} \chi^4 + \frac{1}{8} \chi^6 + \mathcal{O}(\chi^8) \right],$$

(16)

$$E = \frac{4M_W}{\alpha} \chi^3 \left[ 1 + 4 \chi^2 + \mathcal{O}(\chi^n) \right].$$

(17)

We conclude from this that the $SU(2)$ Higgs theory starts out at low energy and small $\beta$ behaving like the type II case. However, there is an additional independent parameter in the $4D$ gauge theory, namely the quartic Higgs self-coupling $\lambda$, or equivalently the Higgs mass, $M_H$. Thus, we cannot preclude a more complicated behavior at higher energies, depending on the value of $M_H / M_W$. This is indeed what we have found in our numerical study.

3. Periodic instantons at finite energy in the $SU(2)$-Higgs theory

We consider the $SU(2)$ gauge theory with a doublet Higgs field in $4D$ Euclidean space with the action,

$$S = \frac{1}{g^2} \int d^4x \left\{ \frac{1}{2} \text{Tr} \left( F_{\mu\nu} F_{\mu\nu} \right) + (D_\mu \Phi)^\dagger (D_\mu \Phi) \\
+ \frac{\lambda}{2} \left( \Phi^\dagger \Phi - \frac{g^2 v^2}{2} \right)^2 \right\},$$

(18)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i \left[ A_\mu, A_\nu \right],$$

$$D_\mu \Phi = (\partial_\mu - iA_\mu) \Phi,$$

and $A_\mu = A_{\mu}^a \sigma^a / 2$. The corresponding classical Euler–Lagrange equations are

$$D_\mu F_{\mu\nu} + i(D_\nu \Phi^\dagger) \times \Phi - i \Phi^\dagger \times (D_\mu \Phi) = 0,$$

$$\left[-D^2 + \frac{2\lambda}{g^2} \left( \Phi^\dagger \Phi - \frac{g^2 v^2}{2} \right) \right] \Phi = 0,$$

(19)

where the covariant derivative acting on the $A$-field in the adjoint representation is $D_\mu A_\nu = \partial_\mu A_\nu - i \left[ A_\mu, A_\nu \right]$, and the $\times$ denotes the outer product of the two spinors. We use the standard conventions for the Higgs vacuum expectation value $v$ and the self-coupling $\lambda$ in which the $W$- and and Higgs-masses are $M_W = \frac{g}{2} v$ and $M_H = \sqrt{2\lambda} v$ respectively.

The spherical Ansatz is given by expressing the gauge field $A_\mu$ and the Higgs field $\Phi$ in terms of six real functions $a_0, a_1, a_2, g, u$ and $w$ of $r$ and $\tau$:

$$A_0(r, \tau) = \frac{1}{2} a_0(r, \tau) \sigma \cdot \hat{r},$$

$$A_i(r, \tau) = \frac{1}{2} \left[ a_i(r, \tau) \sigma \cdot \hat{r} \hat{k},
+ \frac{\alpha(r, \tau)}{r} (\sigma_i - \sigma \cdot \hat{k} \hat{r}),
+ \frac{\gamma(r, \tau)}{r} \epsilon_{ijk} \hat{r} \hat{k} \sigma_j \right],$$

$$\Phi(r, \tau) = \sqrt{2} M_W \left[ u(r, \tau) + i w(r, \tau) \sigma \cdot \hat{z} \right] \hat{\zeta},$$

(20)

where $\hat{\zeta}$ is an arbitrary unit two-component spinor, $\hat{z}$ is the unit three-vector in the radial spatial direction, and $\sigma$ is a three-vector of Pauli matrices.

Upon substituting (20) into the action (18) one finds [7]

$$S = \frac{4\pi}{g^2} \int dr \int_0^{2\pi} d\tau \left\{ \frac{1}{4} \bar{f}_{\mu\nu} f_{\mu\nu} + \left( \bar{D}_\mu \bar{X} \right) D_\mu \chi \\
+ r^2 \left( \bar{D}_\mu \bar{\Phi} \right) D_\mu \phi \\
+ \frac{1}{2} r^2 \left( \bar{X} \chi + 1 \right),
+ \frac{1}{2} \left( \bar{X} + 1 \right) \bar{\Phi} \phi + \text{Re} (i \bar{X} \phi^2) \\
+ \frac{\lambda}{g^2} r^2 \left( \bar{\Phi} \phi - 2 M_\Phi \right) \right\},$$

(21)
where the indices now run over 0 and 1 and
\[ f_{\mu\nu} = \partial_{\mu} a_{\nu} - \partial_{\nu} a_{\mu}, \]
\[ \chi = \alpha + i(\gamma - 1), \quad \bar{\chi} = \alpha - i(\gamma - 1), \]
\[ \phi = \sqrt{2} M_w (u + iw), \quad \bar{\phi} = \sqrt{2} M_w (u - iw), \]
\[ D_{\mu} \chi = (\partial_{\mu} - ia_{\mu}) \chi, \quad \bar{D}_{\mu} \bar{\chi} = (\partial_{\mu} + ia_{\mu}) \bar{\chi}, \]
\[ D_{\mu} \phi = \left( \partial_{\mu} - \frac{i}{2} a_{\mu} \right) \phi, \quad \bar{D}_{\mu} \bar{\phi} = \left( \partial_{\mu} + \frac{i}{2} a_{\mu} \right) \bar{\phi}. \]

The equations of motion for the reduced theory are
\[ -\partial_{\mu}(r^2 f_{\mu\nu}) = i \left[ \left( \bar{D}_{\mu} \chi \right) - \bar{\chi} D_{\mu} \chi \right] \]
\[ + \frac{i}{2} r^2 \left[ \left( \bar{D}_{\mu} \bar{\phi} \right) - \bar{\phi} D_{\mu} \bar{\phi} \right], \]
\[ -D_{\mu} D_{\mu} + \frac{1}{r^2} \left( \bar{\chi} \chi - 1 \right) + \frac{1}{2} \partial \bar{\phi} \chi = - \frac{i}{r^2} \phi^2, \]
\[ -D_{\mu} (r^2 D_{\mu}) + \frac{i}{2} \left( \bar{\chi} \chi + 1 \right) \]
\[ + \frac{2\lambda}{8} r^2 \left( \bar{\phi} \phi - 2 M_w^2 \right) \]
\[ + i \chi \bar{\phi}. \]

Note that the overbar on \( \phi, \chi \) and \( D_{\mu} \) denotes changing \( i \to -i \) in the definitions (22) above, which is the same as complex conjugation only if the six fields \( a_{\mu}, \alpha, \gamma, u \) and \( w \) are real. These equations can be obtained by either imposing the spherical Ansatz (20) on the four dimensional Eq. (19), or by varying the action (21) directly.

The spherical Ansatz (20) has a residual U(1) gauge invariance under the U(1) gauge transformation,
\[ a_{\mu} \to a_{\mu} + \partial_{\mu} \Omega, \]
\[ \chi \to e^{i\Omega} \chi, \]
\[ \phi \to e^{i\Omega/2} \phi. \]

The complex scalar fields \( \chi \) and \( \phi \) have U(1) charges of 1 and 1/2 respectively, \( a_{\mu} \) is the U(1) gauge field, \( f_{\mu\nu} \) is the field strength, and \( D_{\mu} \) is the covariant derivative. The residual U(1) gauge invariance must be fixed for numerical solution of the equations. In our numerical work we chose the temporal gauge \( a_0 = 0 \). The remaining time independent gauge freedom is fixed by a boundary condition at the quarter period time slice \( \tau = \beta/4 \) to be specified below. In the \( a_0 = 0 \) gauge the \( \nu = 0 \) component of the first of Eq. (23), i.e. the Gauss law constraint, must be imposed on the initial \( \tau = 0 \) surface, whereupon it will be satisfied for all \( \tau \).

The action of the various discrete symmetries, C, P, and T on the two dimensional fields follows directly from the spherical ansatz (20). In addition to these symmetries, we may consider the two dimensional reflection symmetry \( R: \phi \to -\phi \). We can employ these discrete symmetries to help us select the appropriate boundary conditions for the periodic instanton solution. The CPR even fields are \( \gamma \) and \( w \), whereas the other four fields are CPR odd. Since we are searching for a periodic solution which returns to itself with period \( \beta \), the time derivatives must reverse sign in the second half period relative to the first. This means that we should require the boundary condition that the \( \tau \) derivatives of all remaining five functions in \( a_0 \) gauge vanish at \( \tau = 0 \) and \( \tau = \beta/2 \). This corresponds to an instanton at \( \beta/4 \) and an anti-instanton at \( 3\beta/4 \) in the low energy limit. At the time slice \( \tau = \beta/4 \) the fields are sphaleron-like. In a gauge where the four dimensional fields are regular at the origin, the sphaleron is a CPR even configuration, and therefore the CPR odd fields change sign while the CPR even fields reach a maximum at \( \tau = \beta/4 \). Hence we actually require the solution only on the quarter interval \([0, \beta/4]\), if we specify the boundary conditions,
\[ \dot{a}_1 = \dot{\alpha} = \dot{\gamma} = \dot{u} = \dot{w} = 0, \quad \tau = 0; \]
\[ a_1 = \alpha = u = 0 = \gamma = w, \quad \tau = \beta \]
\[ a_1 = \alpha = u = 0 = \gamma = w, \quad \tau = \frac{\beta}{4}. \]

These boundary conditions eliminate the time translational zero mode.

The Euler–Lagrange Eq. (23) are also invariant under the two additional complex discrete transformations:
\[ \mathcal{E}_1: \quad \phi \to -\bar{\phi}, \quad \bar{\phi} \to -\phi, \quad \chi \to -\bar{\chi}, \]
\[ \bar{\chi} \to -\chi, \quad a \to -a. \]

\[ \mathcal{E}_2: \quad \phi \to \bar{\phi}^*, \quad \bar{\phi} \to \phi^*, \quad \chi \to \bar{\chi}^*, \]
\[ \bar{\chi} \to \chi^*, \quad a \to a^*. \]
Under $\mathcal{W}_1$, the action, $S \to S$, while under $\mathcal{W}_2$, $S \to S^\dagger$ (and similarly for the energy $E$).

For the four dimensional fields to be regular at the origin, and to approach the vacuum at $r = \infty$ we require the boundary conditions,
\[
\alpha = \gamma = w = 0 = a_i = u', \quad r = 0; \quad a_i = \alpha = u = 0, \quad \text{but} \quad \gamma = 2, w = 1, \quad r = \infty.
\]

where the last condition is necessary for a nonzero winding number, and agrees with the sphaleron boundary condition at $r = \infty$ on the $\tau = 0$ slice. The boundary conditions on $a_i$ at the origin and infinity are gauge choices, which completely eliminate the time independent gauge freedom in temporal gauge $a_0 = 0$.

4. Numerical results

With a well-defined elliptic boundary value problem standard numerical methods may be applied. Here we only outline our computational procedure, the full details of which will be presented in a separate publication [5]. We discretized the Euler–Lagrange equations in space and time by using link variables for the gauge degrees of freedom, thus preserving exact gauge invariance under space dependent gauge transformations (consistent with our choice of the $a_0 = 0$ gauge). The code allows for non-uniform grids in both space and time to better approximate the fields in the regions of fastest variation. We also allowed for the analytic continuation of the solutions into complex valued functions by using complex variables to represent the fields $a_i, \alpha, \gamma, u, w$ and by discretizing the equations in a manner compatible with analytic continuation. The coordinates $(r, \tau)$ are maintained real.

The equations were solved by the Newton–Raphson technique. Denoting a definite trial configuration of the fields by $f_i$, where $i$ stands for the discretized two dimensional lattice point $(r, \tau)$ as well as the various field components themselves, we may calculate the gauge invariant discretized action functional $\mathcal{S}[f]$, the discretized first variation, $\delta S/\delta f_i$, and the second order fluctuation operator $\delta^2 S/\delta f_i \delta f_i$ on the trial configuration. Since the first variation must vanish on the solution, one can find the first order correction to the configuration, $\delta f_i$, by solving the linear equations,
\[
\sum_i \frac{\delta^2 S}{\delta f_i \delta f_i} \delta f_i + \frac{\partial S}{\partial f_i} = 0.
\]

Adding $\delta f$ to $f$ yields a corrected trial configuration, and the process may now be iterated. Clearly, all gauge and translational zero modes must be removed from the second order variation by the gauge fixing and boundary conditions in order for the inverse of $\delta^2 S/\delta f_i \delta f_i$ to exist and the procedure to be well defined.

The algorithm converges quite rapidly, the error decreasing quadratically with the number of complete Newton–Raphson iterations. The most time and memory consuming step is the inversion of the second order fluctuation operator. With grids consisting of as many as $128 \times 1028$ points (or more) and 5 complex valued fields per point, a direct solution of the above system of linear equations is prohibitive. Because the equations of motion are local, the matrix to be inverted is a sparse banded matrix, and it is much more efficient to use the method of eliminating alternate time slices instead of a direct inversion. This effectively reduces the dimensionality of the linear system one must invert to the size of the space grid only, times the 5 components of the fields. Quite modest lattices $(64 \times 64)$ are sufficient for accuracy of order one percent, except in the small $\beta$, low energy region, where the larger, adaptive lattices $(128 \times 128$ or greater) were used. From this solution at a given $\beta$ and $\lambda$ other solutions were found by using the previous one as a trial configuration for the new values of the parameters, changing the values of period and $\lambda$ in small increments.

Our results for the dependence of the action on the period are shown in Fig. 1 for two different values of the quartic Higgs coupling, $\lambda = 0.7 g^2$ and $\lambda = 3.6 g^2$, respectively. They clearly exhibit a pattern of bifurcations.

Since the numerical method and our code works equally well for real or complex solutions, we were able to follow the action and energy of the latter as well. We observe an interesting pattern of bifurcating solutions depending on the Higgs self-coupling $\lambda$, as anticipated in Ref. [8].
There are two regimes, separated by a critical value $\lambda_{\text{crit}}$. As illustrated in Fig. 1a, for $\lambda < \lambda_{\text{crit}}$, the perturbative solutions (branch 1) merge with the sphaleron at some value $\beta = \beta_1$, which is similar in behavior to the $O(3)$ sigma model. For $\beta > \beta_1$, branch 1 can be analytically continued up to a second bifurcation point $\beta_2$ to yield complex solutions whose energy and action however remain real (branch 2). At $\beta_2$, branch 2 merges with yet another branch of complex solutions (branch 3), and the action and energy of these solutions are also real. The numerical results for this second complex branch suggest that its action decreases monotonically with decreasing $\beta$, and that at a sufficiently low (but positive) $\beta$ its action vanishes, with potentially interesting physical consequences (however, as indicated below, the energy along this branch becomes quite large). Branches 2 and 3 form a cusp $c$ at $\beta_2$, beyond which the solutions may be analytically continued onto a fourth branch. The action and the energy on branch 4 ($\beta > \beta_2$) are complex, and the cusp $c$ always lies above the sphaleron line (only the real part of the action has been graphed).

Fig. 1b illustrates the situation for $\lambda > \lambda_{\text{crit}}$. The pattern is similar, except in this case it is the complex solutions along branch 3, and not the real solutions along branch 1, that merge with the sphaleron, and whose extension beyond $\beta_1$ becomes branch 2. Branch 1 (the perturbative one) and branch 2 are real up to $\beta_2$, where they form a cusp $c$ beyond which the solutions are complex with complex energy and action (branch 4). In this case, the cusp always lies below the sphaleron line, except at a critical coupling $\lambda_{\text{crit}}$ where the cusp intersects the sphaleron. We found $\lambda_{\text{crit}} = 1.198 g^2$, corresponding to $M_W = 3.096 M_G$, in good agreement with $3.091 M_W$, obtained in Ref. [9] by a careful treatment
of perturbations away from the static sphaleron solution.

The bifurcation structure for the energy of the complex solutions is shown in Fig. 2 for two values of $\lambda$ on either side of the critical value and for the critical coupling $\lambda_{\text{crit}}$ itself. The associated sphaleron energies $E_{n}(\lambda)$ are illustrated by the short horizontal lines. The real solutions are indicated by the bold solid lines, the complex solutions with real energy by the dashed lines, and the complex solutions with complex energy by the dotted lines extending down from the cusps $c$ (only the real part of the energy has been graphed). The critical coupling $\lambda_{\text{crit}}$ occurs when the energy $E_{0}(\lambda)$ of the cusp crosses the sphaleron line, i.e. when $E_{0}(\lambda_{\text{crit}}) = E_{0}(\lambda_{\text{crit}})$. Finally, it should be noted that the equations are analytic (both in $\lambda$ and in $b$), and therefore the total number of solutions cannot change at the bifurcation points. Indeed, branches 1, 2 and 3 consist of two independent solutions each (with the same energy and action), whereas branch 4 consists of 4 solutions (in two conjugate pairs). By continuity in $\lambda$, these (unsuspected) new complex instanton solutions continue to exist even at lower $\lambda$. The physical consequences of these new solutions for rates of anomalous processes at finite energy in the electroweak theory are currently under investigation.

Finally, let us remark that the accuracy of our numerical calculations is high enough that they can provide a verification of the perturbative results for low energy, low period. In Fig. 3 we plot our numerical results for action and energy together with the results of the perturbative expansions to leading order (14)–(15) and next to leading order (16)–(17). The agreement with the next to leading order perturbative results is quite good.

5. Note added in proof

Yaffe and Frost have reported similar findings in Ref. [10].

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References


Supersymmetric sine algebra and degeneracy of Landau levels

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Abstract

Two different realizations of the supersymmetric sine algebra (SSA) are given. We show that the quantum superalgebra $sl_q(2/1)$ is derived from the SSA. We discuss the relevance of the latter result to the study of spin-$\frac{1}{2}$ Bloch electron in a constant magnetic field. The relation between the deformation parameter $q$ and the degeneracy of Landau levels is established. © 2000 Published by Elsevier Science B.V All rights reserved.

1. Introduction

For a long time all efforts to develop the theory of symmetry in physics were restricted to the linear case, i.e. Lie groups and Lie algebras. They are seen as symmetry algebras and groups which describe linear physical systems. However, the extensive machinery largely developed for the analysis of symmetries in physics breaks down in the non-linear domain.

One notices that the non-linear algebras, having received much attention in the last few years, are the so-called quantum groups [1–3]. They are obtained by deforming the ordinary Lie algebras and are used initially for solving the quantum Yang–Baxter equation [4,5]. The latter, seen mathematically as a representation of the braid group, has been applied in several areas of physics: conformal field theories [6,7], quantum inverse scattering [8], description of spin chains [9,10] and, more recently, intermediate statistics (anyon physics) [11] and the quantum Hall effect [12,13].

We also point out that recent progress in the theory of quantum groups has led to a relationship between the concept of deformation and the degeneracy of Landau levels in the description of a charged spinless particle with mass $m$ moving on a two-dimensional vector space in a constant magnetic field [14].

The aim of this work is two-fold. On one hand, we wish to convey to the reader our view that the quantum superalgebra $sl_q(2/1)$ is not only interesting from some mathematical points of view, but that it also plays an important role in the spin-$\frac{1}{2}$ Bloch electron in a uniform magnetic field. Indeed, the dimensions of the $sl_q(2/1)$ cyclic representations will be related to the degeneracy of Landau levels. Another essential result in this paper is a construc-
tion of the supersymmetric sine algebra in an original mathematical way. We also show how to obtain the quantum superalgebra \( sl_q(2/1) \) from the supersymmetric sine algebra.

The material in the present paper is organized as follows. In Section 2, we realize, via two different ways, the supersymmetric sine algebra (SSA). In the first step we construct it on the extended quantum plane [15]. This construction can be seen as a differential realization, since it is based on the introduction of a consistent non-commutative geometry on the quantum super-plane. We discuss in the same context the realization of the SSA by using the notion of non commutative quantum two-torus [16]. On the other hand, we realize the SSA by considering the \( k \)-fermion oscillators, which can be seen as statistical objects interpolating between bosons and fermions [17].

In Section 3, we show that it is possible to express the elements of the quantum superalgebra \( sl_q(2/1) \) as simple combinations of the generators of the supersymmetric sine algebra (SSA). This construction may be regarded as an extension of the results found in the works [12–14,18], where the authors introduced the embedding of the quantum algebra \( sl_q(2) \) in the sine algebra, also called the Fletcher–Floratos–Zachos (FFZ) algebra [19].

The physical application of these mathematical results is given in Section 4. We will discuss a spin- ½ Bloch electron in a constant uniform magnetic field. The relevance of the quantum symmetry \( sl_q(2/1) \) superalgebra, in the study of this simplest quantum system, is exhibited. We also show the relationship between the degeneracy of Landau levels and the deformation parameter \( q \).

Conclusions are presented in the final section.

2. Supersymmetric sine algebra (SSA)

2.1. Differential realization

In this section we realize the SSA starting from two different ways. As already mentioned in the first section, we construct it in a first step by considering a non-commutative differential calculus discussed in [15]. Let us start by recalling these differential calculi on the quantum plane. It is introduced by considering a differential operator \( d \) which satisfies:
1. the nilpotency condition: \( d^2 = 0 \);
2. \( d \) satisfies the graded Leibnitz rule;
3. the differential calculus is invariant under scale transformations of coordinate points.

In the work of [15], the authors found three different differential calculi. In this paper we consider only one of them, equivalent to the following algebraic relations:

\[
\begin{align*}
xdx &= pdxx, & xdy &= qdxy, \\
ydx &= q^{-1}dxy, & ydy &= rdyy, \\
dx dy &= -qdx dy,
\end{align*}
\]

with \( q, p \) and \( r \) arbitrary complex numbers. The variables \( x \) and \( y \) obey the following equality

\[ xy = qyx. \]

It is easy to verify that the relation

\[ x^n y^m = q^{nm} y^m x^n, \]

holds for any \((n,m) \in \mathbb{N}^2\).

The definition of the first derivatives given by

\[
\begin{align*}
D_x f(x,y) &= \frac{f(px,y) - f(x,y)}{(p-1)x}, \\
D_y f(x,y) &= \frac{f(x,ry) - f(x,y)}{(r-1)y},
\end{align*}
\]

is consistent with the differential calculi introduced above. They are known in the literature as Gauss derivatives along the directions \( x \) and \( y \). \( \frac{1}{q} \) and \( \frac{1}{r} \) are considered, in a formal sense, as objects decreasing the power of \( x \) and \( y \) in the analytic function \( f(x,y) \). Following the relations (Eqs. (2) and (4)), one can prove the commutation relation

\[ D_x D_y = qD_y D_x. \]

Now we will extend this differential calculus to the supersymmetric case by adding a variable \( \theta \) commuting with \( x \) and \( y \) and obeying the relations

\[
\begin{align*}
\theta^2 &= \theta_0^2 = 0, & \theta \theta = 1 - \theta \theta, \\
x\theta &= \theta x, & x\theta = \theta_0 x, \\
y\theta &= \theta y, & y\theta = \theta_0 y.
\end{align*}
\]

The variable \( \theta \) is called a fermionic variable, \( \theta_0 \) is the corresponding derivative.
After this brief review concerning the differential calculi on the quantum superplane, we are able to introduce the supersymmetric sine algebra (SSA) by defining the generators

\[ T_m = \left( q^{m_1 m_2 \over 2} D^{m_1} D^{m_2} \right) \otimes 1, \]

\[ S_m = \left( q^{m_1 m_2 \over 2} D^{m_1} D^{m_2} \right) \otimes (\partial \theta - \theta \partial \). \] (7)

The symbol \( \otimes \) means the tensorial product between the generators belonging to the same quantum superplane described by the Eqs. (1), (2), (5) and (6).

We show, by a straightforward calculation, that \( T_n \) and \( S_n \) satisfy

\[ [T_n, T_m] = 2i \sin \left( \frac{\pi}{k} n \times m \right) T_{n+m}, \]

\[ [T_n, S_m] = 2i \sin \left( \frac{\pi}{k} n \times m \right) S_{n+m}, \]

\[ [S_n, S_m] = 2 \cos \left( \frac{\pi}{k} n \times m \right) T_{n+m} \] (8)

where \( n = (n_1, n_2) \) are elements of \( \mathbb{Z}^2 \), \( n \times m = n_1 m_2 - n_2 m_1 \) and \( k \) is an integer which appears after demanding \( q \) to be a root of unity \((q = e^{2\pi i/k})\). We also note that \([x, y] = xy - yx\) and \([x, y]_\pm = xy \pm yx\) in these relations.

Consequently, we have obtained, through the equalities (Eq. (8)), and via the differential calculus on the quantum superplane introduced above, the supersymmetric sine algebra (SSA). This differential realization is possible only when \( q \) is a root of unity.

A similar realization can be obtained on the non-commutative quantum two-torus. In fact, the non-commutative two-torus is defined as an associative \( C^* \)-algebra \( A_\theta \) which is generated by two unitary operators \( U \) and \( V \) satisfying the relation:

\[ UV = qVU, \quad q = e^{i\theta} = e^{2\pi i/k}. \]

(9)

By considering the correspondence

\[ U \rightarrow D_k, \quad V \rightarrow D_{\bar{k}}, \] (10)

the realization of the (SSA) on the quantum non-commutative two-torus becomes obvious. Now, we describe another way to lead to the realization of the SSA. We will use the \( k \)-fermion formalism introduced in the work of [17]. The construction is completely different from the one discussed above.

2.2. \( k \)-fermionic realization

Let us first introduce the \( k \)-fermionic algebra \( \Sigma_q \) generated by an annihilation operator \( a_q \), a creation operator \( a^+_q \) and a number operator \( N \). The elements \( a_q \), \( a^+_q \) and \( N \) satisfy the following relations:

\[ a_q a^+_q - qa^+_q a_q = 1, \]

\[ N a_q - a_q N = -a_q, \]

\[ N a^+_q - a^+_q N = a^+_q. \] (11)

Here and in the following the complex number \( q \) is chosen to be a root of unity;

\[ q = e^{2\pi i/k}, \] (12)

where \( k \) is a fixed number in \( \mathbb{N}/\{0,1\} \). In this case the operator \( N \) is taken to be hermitian. We note that, owing to relations (Eq. (11)) with \( q \) being a root of unity, the operator \( a_q \) (respectively \( a_q^+ \)) is not (except for \( k = 2 \) and \( k \rightarrow \infty \)) the adjoint of \( a^+_q \) (respectively \( a_q \)).

From the relation (11), we obtain

\[ a_q (a_q^+)^l = [l]_q (a_q^+)^{l-1} + q^l (a_q^+)^l a_q, \]

\[ (a_q^+)^l a_q = [l]_q (a_q^+)^{l-1} + q^l a_q (a_q^+)^l, \] (13)

where the symbol \([l]_q\) is defined by \([x]_q = \frac{1 - q^l}{1 - q}\) and \( l = 1, 2, \ldots, k - 1 \). In the particular case \( l = k \), Eq. (13) can be put into the form

\[ a_q (a_q^+)^k = (a_q^+)^k a_q, \quad (a_q^+)^k a_q^+ = a_q^+ (a_q^+)^k. \] (14)

In addition, one can find

\[ N (a_q^+)^l = (a_q^+)^l (N + 1), \]

\[ (a_q^+)^1 N = (N + 1) (a_q^+)^1. \] (15)

We remark that the relations (13),(14) become trivial if we assume

\[ (a_q^+)^k = (a_q)^k = 0. \] (16)

In this section, we shall deal with a representation of the algebra \( \Sigma_q \) such that Eq. (16) is satisfied. We note that the algebra \( \Sigma_{-1} \) obtained for \( k = 2 \), corresponds to ordinary fermion operators with \((a_q^+)^2 = (a_q)^2 = 0\) which is nothing but the relation that reflects the Pauli exclusion principle. The algebra
\[ \Sigma_q (q = e^{2 \pi i / k}) \) corresponds to quon operators (or k-fermion operators) \( a_q \) and \( a_q^* \) interpolating between fermion and boson operators.

We continue with the situation where the constraints (Eq. (16)) hold. In this situation, we easily obtain the k-dimensional representation of \( \Sigma_q \) [20,21] defined through

\[ a_q |n\rangle = \left( \left[ n \right]_q \right)^{1/2} |n - 1\rangle, \quad a_q |0\rangle = 0, \]

\[ a_q^* |n\rangle = \left( \left[ n + \frac{1}{2} \right]_q \right)^{1/2} |n + 1\rangle, \quad a_q^* |k - 1\rangle = 0, \]

\( N|n\rangle = n|n\rangle. \quad (17) \]

where \( n = 0, 1, \ldots, k - 1 \). This representation is built on a finite-dimensional (Fock) unitary space \( F = \{ |n\rangle; n = 0, 1, \ldots, k - 1 \} \) of dimension \( k \). The space \( F \) is of dimension 2 for the fermionic algebra \( \Sigma_{-1} \) and infinite-dimensional for the bosonic algebra \( \Sigma_1 \).

We now introduce two operators \( U \) and \( V \) defined in terms of the generators of the algebra \( \Sigma_q \),

\[ U = a_q a_q^* - a_q^* a_q, \quad V = a_q + \left( a_q^* \right)^{k-1} [k-1]_q^{1/2}. \quad (18) \]

An elementary calculation leads to

\[ UV = q^{-1} VU. \quad (19) \]

The latter can be iterated to produce

\[ U^n V^m = q^{-nm} V^m U^n, \quad (n, m) \in \mathbb{N}^2. \quad (20) \]

Let us define the operators

\[ K_n = K_{(n_1, n_2)} = \left( q^{-n_1 n_2} U^{n_1} V^{n_2} \right) \otimes 1, \]

\[ F_n = F_{(n_1, n_2)} = \left( q^{-n_1 n_2} U^{n_1} V^{n_2} \right) \otimes \left( a_{-a} a_{-a}^* - a_{a} a_{a}^* \right), \quad (21) \]

with 1 the \( k \times k \) identity matrix, \( a_{-a} \) and \( a_{a}^* \) are respectively \( a_{q^{n-1}} \) and \( a_{q^{-n-1}} \) (\( k = 2 \)).

The operators \( K_n \) and \( F_n \) satisfy the following commutation relations

\[ [K_m, K_n] = 2 \sin \left( \frac{\pi}{k} n \times m \right) K_{m+n}, \]

\[ [K_m, F_n] = 2 \sin \left( \frac{\pi}{k} n \times m \right) F_{m+n}, \]

\[ [F_m, F_n] = 2 \cos \left( \frac{\pi}{k} n \times m \right) F_{m+n}. \quad (22) \]

As a conclusion, the generators \( K_n \) and \( F_n \) can be viewed as the generators of the supersymmetric sine algebra (SSA).

We recapitulate the main results obtained in this section. We realize the SSA in two different ways. In the first one we used the differential calculi on the supersymmetric quantum plane. This construction may be regarded as a geometric one, while in the second way we have considered one \( k \)-fermion and one fermion (Eq. (23)).

To close this section, we recall that the quantum algebra \( sl_q(2) \) has been obtained from sine algebra (FFZ) [12–14,17,18]. So, it is natural to think about some quantum enveloping superalgebra which can be derived from the (SSA). This idea will be treated in the following section.

3. The embedding of \( sl_q(2/1) \) in the SSA

This section is devoted to the construction of the quantum superalgebra \( sl_q(2/1) \) from the supersymmetric sine algebra (SSA). The \( sl_q(2/1) \) generators are obtained as linear combinations of those generating the SSA.

Before starting, we recall that the deformed superalgebra \( sl_q(2/1) \) is generated by the set of generators \( \{ e_i, f_i, h_i, i = 1, 2 \} \) satisfying the following commutation relations [22]

\[ q^{h_i} q^{h_j} = q^{h_j} q^{h_i}, \]

\[ q^{h_i} e_j q^{-h_i} = q^{a_{ij} e_j}, \]

\[ q^{h_i} f_j q^{-h_i} = q^{-a_{ij} f_j}, \]

\[ e_1 f_1 - f_1 e_1 = \frac{q^{2h_1} - q^{-2h_1}}{q - q^{-1}}, \]

\[ e_2 f_2 + f_2 e_2 = \frac{q^{2h_2} - q^{-2h_2}}{q - q^{-1}}, \]

\[ e_1 f_2 + f_2 e_1 = 0, \]

\[ e_2 f_1 - f_1 e_2 = 0, \]

\[ e_1^2 e_2 - (q + q^{-1}) e_1 e_2 e_1 + e_1 e_2^2 = 0, \]

\[ f_1^2 f_2 - (q + q^{-1}) f_1 f_2 f_1 + f_2 f_1^2 = 0, \]

\[ e_2^2 = 0 = f_2^2. \quad (23) \]
where the Cartan matrix \((a_{ij})\) is given by

\[
(a_{ij}) = \begin{pmatrix}
2 & -1 \\
-1 & 0
\end{pmatrix}.
\]  

(24)

To obtain the \(sl_q(2/1)\) superalgebra from the (SSA), we consider the generators

\[
L_n = L_{(n, n_2)} = T_{(n, n_2)} \otimes 1,
\]

\[
M_n = M_{(n, n_2)} = T_{(n, n_2)} \otimes (\bar{\theta} \theta - \theta \bar{\theta}),
\]

(25)

where \(\theta\) and \(\bar{\theta}\) are the Grassmann variable and its derivative, respectively. For convenience, we define the product between \(T_m\) and \(T_n\) as follows

\[
T_m T_n = q^{-\frac{m \times n}{2}} T_{m+n}
\]

(26)

where \(q\) is a root of unity (i.e. \(q = e^{2\pi i/k}\)).

By a direct computation, one can show that the generators \(L_m\) and \(M_m\) generate the SSA. From the latter, we define the operators

\[
e_1 = \frac{T_{(1,1)} + T_{(-1,1)}}{q - q^{-1}} \otimes 1,
\]

\[
f_1 = -iT_{(1,-1)} - T_{(-1,-1)} \otimes 1,
\]

\[
h_1 = iT_{(4,0)} \otimes 1,
\]

\[
h_2 = iT_{(1,0)} - T_{(-1,0)} \otimes \begin{pmatrix}
0 & 1 & 0 \\
1 & q & 0 \\
0 & 0 & q^2
\end{pmatrix},
\]

\[
q^{-h_2} = iT_{(4,0)} \otimes \frac{1}{q^2} \begin{pmatrix}
0 & 1 & 0 \\
1 & q & 0 \\
0 & 0 & q^2
\end{pmatrix},
\]

\[
e_2 = \frac{T_{(1,1)} - T_{(-1,-1)} \otimes \bar{\theta},}{(q - q^{-1})^2}
\]

(27)

\[
f_2 = \frac{T_{(-1,1)} + T_{(1,1)} \otimes \theta}{(q - q^{-1})^2},
\]

which satisfy Eq. (23) defining the quantum superalgebra \(sl_q(2/1)\). Consequently, we have presented the general construction of the quantum superalgebra from the supersymmetric sine algebra. This is one of the main results of this paper.

We point out that the construction of the quantum superalgebra \(sl_q(2/1)\) from the SSA is not only interesting from a mathematical point of view, but it also has an interesting physical application. Indeed, we will show, in what follows, that \(sl_q(2/1)\) is related through the deformation parameter \(q\) with the spin-\(\frac{1}{2}\) Bloch electron in a constant uniform magnetic field.

4. Physical application

In this section, we show that the supersymmetric sine algebra and thus the quantum superalgebra \(sl_q(2/1)\) lead to the study of the degeneracy for a physical system describing an electron moving in a constant magnetic field. In what follows we will investigate the connection between the deformation parameter \(q\) and the degeneracy. To clarify all these statements, let us introduce the Hamiltonian corresponding to a spin-\(\frac{1}{2}\) Bloch electron moving in the \((e_x, e_y)\)-plane under the influence of a constant uniform magnetic field \(B = Be_z\) (\(e_z\) is normal unit vector to plane). This Hamiltonian is

\[
H = \frac{1}{2m}(\pi_x^2 + \pi_y^2) + \frac{i}{2} \hbar \omega \sigma_z,
\]

(28)

where \(\sigma_z\) is the Pauli spin matrix and \(\omega = \frac{eB}{mc}\). The problem under consideration is supersymmetric [23] and the eigenfunctions of the Hamiltonian are labelled, in addition to the usual bosonic quantum numbers \((n, m)_n\), by a fermionic quantum number \(n_f = 0,1\). The latter introduces a two-fold degeneracy of each excited state.

In Eq. (28), the canonical momenta are defined by:

\[
\pi_x = p_x - \frac{eA}{c} A_x, \quad \pi_y = p_y - \frac{e}{c} A_y,
\]

(29)

where \(A = (A_x, A_y)\) is the potential vector and can be given by

\[
A_x = -B \frac{dy + \partial_y \Lambda}{2}, \quad A_y = B \frac{dx + \partial_x \Lambda}{2},
\]

(30)

\(\Lambda\) is an arbitrary scalar function determining the gauge we are working in. For simplicity, we choose \(\Lambda = Bcy\).
As in the case of spinless electron, we define the magnetic translation operators as follows:

$$T_m = T_{(m_1, m_2)} = \exp \left( \frac{i}{\hbar} R_m \cdot \mathbf{K} \right),$$

(31)

where $R_m = m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2$ is an arbitrary two-dimensional vector, $m = (m_1, m_2) \in \mathbb{Z}^2$, and $\mathbf{a}_1$ and $\mathbf{a}_2$ are respectively two given vectors in directions $\mathbf{e}_x$ and $\mathbf{e}_y$. The operator $\mathbf{K}$ is given by

$$K_x = \pi_x - m \omega y, \quad K_y = \pi_y + m \omega x. \quad (32)$$

The components $K_x$ and $K_y$ of $\mathbf{K}$ satisfy the following commutation relation

$$[K_x, K_y] = i \hbar \omega m. \quad (33)$$

Using the Baker–Campbell–Hausdorff formula, one gets the relation

$$T_m T_n = q^{-\frac{m \times n}{4}} T_{m+n},$$

(34)

where $q = e^{2 \pi i / k}$ and $k = -\frac{\omega}{2 \pi \hbar \omega}$. The scalars $a_1$ and $a_2$ are defined by

$$a_1 = a_1 e_x, \quad a_2 = a_2 e_y. \quad (35)$$

Now, we will display the relationship between the above magnetic translation operators $T_m$ and the generators of the supersymmetric sine algebra. Let us introduce the operators

$$T^+_m = T_m \otimes 1, \quad T^-_m = T_m \otimes (\sigma_+ \sigma_- - \sigma_- \sigma_+), \quad (36)$$

where the matrices $\sigma_+$, $\sigma_-$, and $\sigma_3$ are given by

$$\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (37)$$

The generators $T^+_m$ and $T^-_m$ satisfy

$$[T^+_m, T^+_n] = 2 \sin \left( \frac{\pi n}{2k} \times m \right) T^+_m + n, \quad (38)$$

$$[T^-_m, T^-_n] = 2 \sin \left( \frac{\pi n}{2k} \times m \right) T^-_{m+n}, \quad (39)$$

$$[T^+_m, T^-_n] = 2 \cos \left( \frac{\pi n}{2k} \times m \right) T^+_m + n.$$

One notices that in the limit $q \to 1$ we recover the supersymmetric extension of the area-preserving diffeomorphisms algebra on the two-dimensional surface. As before (Section 3, Eq. (27)), we can realize the quantum superalgebra $sl_q(2/1)$ from the SSA. This result shows the relevance of $sl_q(2/1)$ in the system under consideration. More precisely, the degeneracy of Landau levels is related to the dimension of the cyclic representation of $sl_q(2/1)$. Indeed, following the work of [24], it has been proved that the representation of this quantum superalgebra has several exotic properties for $q$, a root of unity. The representation in this case is called a cyclic representation, and it has been demonstrated that the dimension of the representation space equals $2k$. Using this result and defining the basis vectors by the ket $|n, m, n_F, \rangle$, where $n = 0, 1, \ldots, \infty$ denote the energy Landau levels, $m$ is a new quantum number labelling the different quantum states for a fixed $n$ and $n_F$ is related to the spin state ($n_F = 0, 1$). The action of the $sl_q(2/1)$ generators on the states $|n, m, n_F, \rangle$ is given by:

$$q^n |n, m, n_F, \rangle = \lambda_1^2 q^{-2n + 2m} |n, m, n_F, \rangle,$$

$$q^n |n, m, n_F, \rangle = \lambda_2^2 q^{2n + 2m} |n, m, n_F, \rangle,$$

$$f_1 |n, m, n_F, \rangle = \varphi q^{\alpha} |n, m + 1, n_F, \rangle, \quad (m \neq k - 1),$$

$$f_1 |n, m, n_F, \rangle = \varphi q^{\alpha} |n, 0, n_F, \rangle,$$

$$e_1 |n, m, n_F, \rangle = \varphi^{-1} q^{-\alpha} |m + \mu_2, n, n_F, \rangle$$

$$\times (\mu_1 + \mu_2 - m + 1 - n_F),$$

$$e_1 |n, m, n_F, \rangle = \varphi^{-1} q^{-\alpha} |m + \mu_2, n, n_F, \rangle \quad (m \neq 0),$$

$$f_2 |n, m, n_F, \rangle = (1 - n_F) q^{n-1} |m + \mu_2, n, n_F, \rangle$$

$$\times |n, m - 1, n_F + 1, \rangle \quad (k \neq 0),$$

$$f_2 |n, m, n_F, \rangle = (1 - n_F) \lambda_2 q^{-1} |m + \mu_2, n, k - 1, n_F, + 1, \rangle$$

$$e_2 |n, m, n_F, \rangle = n_F \lambda_2^{-1} q^{-m} |n, m + 1, n_F - 1, \rangle \quad (k \neq l - 1),$$

$$e_2 |n, m, n_F, \rangle = n_F \lambda_2^{-1} q^{-n} |n, k - 1, n_F, \rangle \quad (39),$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2$ and $\varphi$ are complex constants determined by the cyclic properties of the representation.

The dimension of the cyclic representation $|n, m, n_F, \rangle$, $m = 0, 1, \ldots, k - 1, n_F = 0, 1$ is $2k$ which
coincides, surprisingly enough, with the degeneracy of the Landau levels. This constitutes an important physical application of the mathematical results obtained in this work. Finally, we note that in the classical limit $q \rightarrow 1$ ($k \rightarrow \infty$), the system exhibits a continuous degeneracy, which is a well known result.

5. Conclusion

The main result of this work is of a mathematical nature, dealing with the realization of the supersymmetric sine algebra. As in the construction of all symmetry algebras in the study of physical systems, one proceeds, in general, through a differential calculus or an oscillator realization. For linear systems, these two methods as well as, sometimes, the correspondence between, are well understood. However, to establish this correspondence in the context of quantum algebras is much more difficult. This difficulty is related to the fact that one can find many different differential calculi to realize the same algebra. The reason is that the unicity of all the introduced differential calculi remains an open question in the literature. One can think of the same problem in terms of $q$-oscillator algebras.

Therefore, in this work we have constructed the supersymmetric sine algebra following two different ways. The first one has been based on a specific non-commutative differential calculus and the second one involves the $k$-fermion oscillator. But also establishing the correspondence between the two remains for us an open question that will be treated in a future work. Another important result of this paper is the realization of the $sl_q(2/1)$ superalgebra from the supersymmetric sine algebra. This result generalizes one well known in the context of (in)finite dimensional Lie algebras. Indeed, it has been shown in several works that $sl_q(2)$ can be derived from the sine algebra.

Finally, we deal with all these tools to treat one of the simplest two-dimensional quantum mechanics systems. We have proved, in fact, that the deformation parameter $q$ of the superalgebra $sl_q(2/1)$ may be related to the degeneracy of Landau levels; the degree of degeneracy of Landau levels is $2k$ which is nothing but the dimension of the cyclic representation space of $sl_q(2/1)$.

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RR flux on Calabi–Yau and partial supersymmetry breaking

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Abstract

We show how turning on Flux for RR (and NS-NS) field strengths on non-compact Calabi–Yau 3-folds can serve as a way to partially break supersymmetry from $N = 2$ to $N = 1$ by mass deformation. The freezing of the moduli of Calabi–Yau in the presence of the flux is the familiar phenomenon of freezing of fields in supersymmetric theories upon mass deformations.

1. Introduction

Type II strings compactified on Calabi–Yau threefolds give rise to $N = 2$ theories in 4 dimensions. The geometry of Calabi–Yau threefold and its moduli space provides a deep insight into the dynamics of $N = 2$ gauge theories. It is thus natural to ask if the simple operation of breaking supersymmetry from $N = 2$ to $N = 1$ (say by addition of mass terms) has a Calabi–Yau counterpart. If so, this may provide insight into the dynamics of $N = 1$ gauge theories.

A particular approach to breaking supersymmetry in the context of Type II compactification on Calabi–Yau threefolds was taken in [1] where Ramond–Ramond fields strengths were turned on. It was shown, however, that one either preserves all $N = 2$ supersymmetries (and freeze the moduli of Calabi–Yau to make it correspond to singular limits such as the conifold) or one breaks the supersymmetry completely. Furthermore it was argued in [2] that this is a general result.

On the other hand it was found in [3] that in the context of $N = 2$ quantum field theories it is possible to add $N = 2$ FI terms, and break the supersymmetry to $N = 1$. These constructions were generalized to the local case in [4] (see also [5]). There seemed, therefore, to exist a conflict between the results coming from considerations of type II compactifications on Calabi–Yau threefolds which suggested finding $N = 1$ supersymmetric theories by turning on fluxes is not possible, whereas field theory arguments suggested that some such deformations should be possible.

We will see in this paper that indeed we can obtain partial supersymmetry breaking by considering non-compact Calabi–Yau manifolds with fluxes turned on. The way this avoids the no-go theorem in [2] is by taking a certain decompactification limit, which renders some fields non-dynamical. In other words, it would have corresponded to a theory with no supersymmetric vacua in the compact situation, and where it not for making some fields non-dynamical, we could not have obtained partial super-
symmetry breaking. However, as far as geometric engineering of $N=1$ theories are concerned the non-compactness of the Calabi–Yau is a perfectly acceptable condition, and this is already the case for geometric engineering of $N=2$ theories.

The organization of this paper is as follows: In Section 2 we show how supersymmetry can be partially broken by considering a simple generalization of models of [3] where we include two $N=2$ vector multiplets. In Section 3 we consider type II compactifications on Calabi–Yau threefolds and show how turning on RR fluxes (and in addition NS flux $H$ for type IIB) is equivalent to turning on FI terms in the $N=2$ supersymmetric theory. We also review the no-go theorem of [2] and show how it may be avoided in certain non-compact limits. In Section 4 we briefly review aspects of geometric engineering of $N=2$ theories and show why the non-compactness of the Calabi–Yau is a perfectly acceptable condition, and this is already the case for $N=1$ chiral multiplets.

2. Partial supersymmetry breaking and mass generation

In this section, we present a simple generalization of the model discussed in [3] which exhibits partial supersymmetry breaking with mass generation for $N=1$ multiplets. It involves two $N=2$ vector multiplets, $S$ and $A$, with the prepotential

$$\mathcal{F}(S, A) = \frac{i\gamma}{2} S^2 + \frac{1}{2} S A^2,$$

(2.1)

where $\gamma$ is a real constant, and with the superpotential (which in general can be taken to be a linear combination of "periods")

$$W = eS + mF_S = (e + im\gamma)S + \frac{m}{2} A^2,$$

(2.2)

where $e = e_1 + ie_2$ and $m = m_1 + im_2$ are complex constants and $F_S = \partial \mathcal{F} / \partial S$. The corresponding Lagrangian is $N=2$ supersymmetric. The constants $e$ and $m$ correspond to $N=2$ electric and magnetic Fayet–Iliopoulos terms, respectively. In the manifestly $N=2$ supersymmetric notation of Ref. [3]:

$$\text{Re} E = (e_1, e_2, 0), \quad M = (m_1, m_2, 0).$$

(2.3)

The superpotential of Eq. (2.2) gives rise to the following potential for the scalars $S = a + i\sigma$ and $A = b + ia$:

$$V = \frac{|e a - \gamma m b|^2}{\gamma (\gamma \sigma - \alpha^2)},$$

(2.4)

In the above equation, we neglected an irrelevant, additive constant term. The potential has a zero-value minimum at $a = b = 0$. For generic values of $e$ and $m$, both $N=2$ supersymmetries are broken spontaneously. There are, however, two special configurations of these parameters:

$$e \pm im\gamma = 0,$$

(2.5)

for which supersymmetry is broken partially to $N=1$. In this case, both scalars $a$ and $b$, as well as the fermionic component of the $N=1$ chiral multiplet $A$ acquire an equal mass of $|m|/\sigma$. The simplest way to prove formally that such a partial breaking does indeed occur is to follow the method of [3] and examine the supersymmetry variations of fermions. In this way, one can identify the $N=2 \rightarrow N=1$ goldstino as one of the two fermionic components (gauginos) of the $S$ multiplet. In fact, the full $N=2$ vector multiplet $S$ and the $N=1$ vector component of $A$ remain massless while the $N=1$ chiral multiplet $A$ acquires a mass.

The above model can be generalized to more complicated prepotentials, of the form

$$\mathcal{F}(S, A) = f(S) + \frac{1}{2} S A^2,$$

(2.6)

As in the previous case, the potential has a minimum at $A = 0$. However, there is also another minimization, with respect to $S$, which yields two solutions

$$e + mF_{SS} = 0 \quad \text{or} \quad e + mF_{SS} = 0,$$

(2.7)

similar to (2.5). It is easy to see that the above equation is exactly the condition for partial supersymmetry breaking. Hence we conclude that an $N=1$ supersymmetric vacuum exists also in the general case. In particular, the mass $|m|/\sigma$ is generated again for the $N=1$ chiral multiplet $A$.

So far we have been discussing the case of global $N=2$ supersymmetry. In the context of string theory we of course have local $N=2$ supersymmetry. In such a case to obtain $N=2$ global limit we have to
take some particular limit, where gravity decouples, say by taking in the type II context weak limit of string coupling constant, and perhaps some other limits for other fields. In this context we can break $N = 2$ to $N = 1$ in an even simpler way. Set $\gamma = 0$, so that the prepotential is just
\[
F = \frac{1}{2} \Sigma A^2.
\]
This would have given a singular kinetic term for $S$ in the global case, but it is perfectly fine in the local case. We can think of $A$ for example as the “heterotic string coupling constant”. We now turn on FI term $\alpha F_5 + \beta F_4$. We take the limit where the vev of $S$ becomes large (i.e. weak coupling heterotic string limit). In this limit $S$ becomes non-dynamical. And the superpotential term $W = \frac{1}{2} m (A')^2$ (where $A'$ is related to $A$ by a shift) simply gives mass to the scalar $A'$, breaking $N = 2$ to $N = 1$. It is this realization of partial supersymmetry breaking that we will find applicable in the Calabi–Yau context later in this paper.

3. Type IIB on Calabi–Yau 3-fold with $H$-flux

Consider compactification of type IIB on a Calabi–Yau threefold. We would like to consider turning on flux for NS and R threeform field strengths $H_{\text{NS}}$ and $H_{\text{R}}$. This is a case already considered in [2] following the work of [1] and more recently from the viewpoint of F-theory in [6,7]. The theory has $h^{0,1}$ vector multiplets and $h^{1,1} + 1$ hypermultiplets in addition to the $N = 2$ gravitational multiplet, where $h^{p,q}$ denotes Hodge numbers of Calabi–Yau. The relevant modification to the effective action due to turning on $H$-flux is in interactions with the vector multiplets. Let $\Omega$ denote the holomorphic threeform on the Calabi–Yau. We write the effective Lagrangian we obtain in 4 dimensions in an $N = 1$ supersymmetric framework. The net effect of turning on $H$-flux is to add a superpotential of the form
\[
W = \int \Omega \wedge (\tau H_{\text{NS}} + H_{\text{R}}) \tag{3.1}
\]
in the 4-dimensional effective theory, where $\tau$ denotes the complexified coupling constant of type IIB strings. Note that $H_{\text{NS}}$ and $H_{\text{R}}$ are dual to some integral 3-cycles $C_{\text{NS}}$ and $C_{\text{R}}$ and the above formula can also be written as
\[
W = \int_{C_{\text{NS}}} \Omega + \int_{C_{\text{R}}} \Omega - \int_{\mathcal{F}_{\Lambda}} \Omega.
\]
To see how (3.1) arises note that if we consider a five brane (NS or R) wrapped around a 3-cycle $C$ in the Calabi–Yau, it corresponds to a domain wall in $3+1$ dimensional theory, whose BPS bound for tension should be given by $\Delta W$ across the domain wall. On the other hand the tension of the 5-brane should be $\int_{C} \Omega$ (times $\tau$ in the case of NS 5-brane). Since the 5-brane wrapped around $C$ changes the $H$ flux across the domain wall by a 3-form dual to the $C$ cycle we see that this gives the expected change $\Delta W$. This argument was discussed in [6] in the context of F-theory on 4-folds, and type IIB on Calabi–Yau 3-folds is a special case of it.

We can also write (3.1) explicitly if we choose a basis for $H_3(M, \mathbb{Z})$, given by $(A^1, B_2)$, $A, \Sigma = 0, \ldots, h^{2,1}$, with $A^1 \cap A^2 = B_1 \cap B_2 = 0$ and $A^4 \cap B_2 = \delta_2^1$. Sometimes we refer to $A^1$ as the electric cycles and $B_2$ as the magnetic cycles. This clearly is a basis dependent definition. Let
\[
X^1 = \int_{A^1} \Omega - \int_{B_2} \Omega.
\]
Moreover denote the dual 3-cycle to the $H$-fluxes by
\[
C_{\text{NS}} = e_1 A^1 + m^1 A^1 B_1, \quad C_{\text{R}} = e_2 A^1 + m^2 A^1 B_1, \tag{3.2}
\]
and the complex vectors $e$ and $m$ are defined as:
\[
e_1 = e_1^1 \tau + e_1^2 m^1 A^1 \tau + m^2 A^1. \tag{3.3}
\]
The superpotential (3.1) can be written explicitly as
\[
W = \int_{C_{\text{NS}}} \Omega + \int_{C_{\text{R}}} \Omega = e_1 X^1 + m^1 \mathcal{F}_X. \tag{3.4}
\]
As is well known there is a prepotential $\mathcal{F}(X)$, a homogeneous function of weight 2 in $X$ in terms of which\[
\mathcal{F}_X = \partial_X \mathcal{F}.
\]
Thus the FI terms are realized by $H$ fluxes in type IIB string compactification on Calabi–Yau threefolds.

### 3.1. Type IIA version

The same analysis can be done in the type IIA language (for the case of type IIA on Calabi–Yau 4 folds see [8]). In fact mirror symmetry already tells us what the story will be in the type IIA case. The story is much simpler in the context of just turning on the $H^6$ flux. In this case the mirror corresponds to turning on $F^2,F^4$ and $F^6$ fluxes which are dual to 4,2 and 0 cycles on Calabi–Yau 3-fold. The analog of (3.1) is now

$$W = N_0 + \int_{C_2} k + \int_{C_4} k^2,$$

where $k$ denotes the Kähler class on the Calabi–Yau threefold and $N_0$ denotes the quantum of $F^6$ flux. The above formula receives world sheet instanton correction as is well known, and in fact by mirror symmetry one can recover the instanton corrected superpotential $W$ on the Calabi–Yau.

### 3.2. Scalar potential

Our next step is to obtain the scalar potential corresponding to the superpotential (3.4). We would like to maintain a manifest $N=2$ supersymmetry, however this is not possible in the locally supersymmetric case because the superpotential is a genuinely $N=1$ quantity. Instead of turning to the fully-fledged $N=2$ supersymmetric formalism [9] (like in Ref. [2]), we can try to obtain the potential by compactifying the 10-dimensional action. Alas, this is not so simple in view of the absence of a fully covariant, off-shell formulation of type IIB supergravity. The best we can do is to start from the "non-self-dual" (NSD) action [10] employing a 4-form field strength which is not self-dual. The equations of motion of IIB supergravity follow from the NSD action after imposing the self duality constraint at the level of field equations. Using the NSD action to determine the scalar potential is somewhat questionable, nevertheless it is interesting to compare the result with the superpotential (3.4). In fact, this method will provide an independent derivation of (3.4).

In order to parameterize the $H$-fluxes, we will use the $H^i(M,Z)$ basis $(\alpha_4,\beta^2)$, dual to the $(A^i,B_z)$ basis of $H^i(M,Z)$, with \( \alpha_4 \wedge \beta^2 = \delta^2_4, \alpha_4 \wedge \alpha_4 = \beta^2 \wedge \beta^2 = 0 \). The fluxes can be written as

$$H^{NS} = e_4^1 \beta^A + m_4^1 \alpha_4, \quad H^R = e_4^2 \beta^A + m_4^2 \alpha_4.$$

(3.5)

In the presence of the fluxes, the 10-dimensional kinetic terms give rise to the potential:

$$V = (2\Im \tau)^{-1} \int (\tau H^{NS} + H^R) \wedge * (\overline{\tau} H^{NS} + H^R).$$

(3.6)

The integration over the Calabi–Yau manifold can be performed by using standard properties of $(\alpha_4,\beta^2)$ basis (see e.g. [11,12]), with the result

$$V = -(2\Im \tau)^{-1} \left[ (\Im \mathcal{M})^{-1}(e + m \Re \mathcal{M}) \right],$$

(3.7)

where $\mathcal{M}$ is the period matrix [12] while $e$ and $m$ are the complex vectors defined in (3.3). The potential can be rewritten as

$$V = -(2\Im \tau)^{-1} \left[ (e + m \overline{\mathcal{M}})(\Im \mathcal{M})^{-1}(\overline{e} + \overline{m} \mathcal{M}) \right] + m \times e,$$

(3.8)

where the constant term

$$m \times e \equiv m_4^1 e_4^2 - m_4^2 e_4^1.$$

(3.9)

As we will discuss in the next subsection $m$ and $e$ should be chosen so that $m \times e$ is zero (for cancellation of 3-brane tadpoles), which we will assume is the case.

In order to relate the above potential with the superpotential (3.4), we first use the identity [12]:

$$e^{-K(\tau)}(\Im \mathcal{M})^{-1} \alpha_4 \mathcal{M}$$

$$= -2 \overline{X} \wedge \overline{X} - 2 D_i X^4 G^{ij} D_j \overline{X}^5,$$

(3.10)

where $K$ is the Kähler potential of the $N=2$ vector multiplet moduli $z_i$, $i = 1, \ldots, h^{2,1}$, and $G^{ij}$ is the inverse metric on the vector moduli space.
parameters. In order to recover the globally supersymmetric models of the type discussed in Section 2 and in Ref. [3] from type IIB theory with $H$-fluxes, we scale the holomorphic sections as $(X^0, \mathcal{F}_0) \sim 1$ and $(X^4, \mathcal{F}_4) \sim M_{Pl}^{-1}$, $A > 0$. The Kähler potential $K(z, \overline{z})$ scales then as $M_{Pl}^2$, so the vector moduli survive as dynamical fields in the $M_{Pl} \to \infty$ limit while $\tau$ and other hypermultiplets decouple and can be treated as (complex) parameters. Furthermore, we set $e_\theta = m^0 = 0$ and adjust the remaining Fayet–Iliopoulos parameters so that the superpotential (3.4) scales as $M_{Pl}^{-3}$. In this way [9], we obtain from (3.13) a finite potential corresponding to the rigid superpotential $W = e_\theta z^4 + m^4 \mathcal{F}_4$ [3]. The procedure of taking the rigid limit can be further refined to treat some vector moduli in a special way (such as the $S$ field discussed in the previous section), in order to freeze them in a way similar to $\tau$. This will be useful in the context of geometric engineering of $N = 1$ theories, to be discussed in the next section.

### 3.3 Supersymmetric vacua

Now we analyze supersymmetric solutions with superpotential given by (3.4). The condition for getting a supersymmetric solution with $R^4$ background in this context has been studied by [2] with the conclusion that either there are no supersymmetric vacua or that the $N = 2$ is preserved at the vacua. In particular no $N = 1$ supersymmetric vacua were found in this way. Let us review these results in the $N = 1$ setup that we are considering. The condition for finding supersymmetric vacua in $R^4$ background is that

$$W = dW = 0,$$

where $dW$ denotes the derivative of $W$ with respect to all chiral fields. In the context of compact Calabi–Yau, considered in [2], turning on both $H^{NS}$ and $H^R$ can preserve supersymmetry only if

$$\int H^{NS} \wedge H^R \sim m \times e = 0. \quad (3.15)$$

Otherwise these fluxes induce anti-3-brane charge in the uncompactified spacetime (proportional to $|H^{NS} \wedge H^R|$) and to cancel it we will necessarily break supersymmetry. If $H^{NS}$ and $H^R$ satisfy (3.15) then we can choose both of them be dual to some $A$-cycles.
(i.e. \( m = 0 \)). If we denote the dual three cycles by \( N_1 A_1 \) and \( N_2 A_2 \) with periods \( \int_A \Omega = x_i \) the superpotential will take the form,

\[
W = N_1 \tau X_1 + N_2 \tau X_2.
\]

The condition that \( W = dW = 0 \) in terms of physical fields \( z_i = X_i / X_0 \), is equivalent to \( dW = 0 \) in terms of the \( X_i \) variables. Since \( X_1 \) and \( X_2 \) are independent fields, we see that condition \( dW / dX_1 = dW / dX_2 = 0 \) has no solutions, and so supersymmetry is completely broken.

So we see that if we consider smooth Calabi–Yau manifolds there are no supersymmetric solutions. However, near singular Calabi–Yau manifolds the low energy effective Lagrangian description breaks down and one could have additional light fields. The particular case of conifold was studied in [1]. In that case, say we have a vanishing \( A \) cycle, and we turn on a flux dual to that cycle. This means we have a superpotential

\[
W = \alpha a,
\]

where \( a = \int_A \Omega \) and \( \alpha = n_1 + n_2 \tau \). Let us set \( n_2 = 0 \). If \( A \) is shrinking we have in addition a light wrapped D3 brane which in \( N = 1 \) language corresponds to chiral fields \( \phi \) and \( \bar{\phi} \) with charge \( \pm 1 \) respectively under the \( U(1) \) gauge field whose supersymmetric scalar partner has vev \( \langle \phi \rangle = a \). So the actual Lagrangian superpotential should be modified to

\[
W = \alpha a + a \phi \bar{\phi}.
\]

Now the condition that \( W = dW = 0 \) has a solution and is given by

\[
a = 0, \quad \phi \bar{\phi} = -\alpha.
\]

In fact it is possible to check that this actually preserves the full \( N = 2 \) supersymmetry. Even though some other singularities of Calabi–Yau manifolds have also been considered in [2] none has been shown to lead to \( N = 1 \) unbroken supersymmetry (though a full no-go theorem is not available in this context).

3.4. How to obtain \( N = 1 \) supersymmetry?

It thus seems difficult to obtain an \( N = 1 \) supergravity solution with \( H \)-flux turned on for compact Calabi–Yau manifolds. How could we possibly relax some conditions to make this possible? The hint comes from considering \( N = 1 \) supersymmetric Yang–Mills theories. In these cases one expects to have a mass gap with some number of vacua \( c_2(G) \) given by the dual Coxeter number of the group. Moreover one can assign a superpotential to each vacuum given by

\[
W_k = \omega_k \exp\left(-S / c_2(G)\right),
\]

where \( S = 1 / g^2 \) and \( \omega_k \) is an \( c_2(G) \) root of unity. The meaning of this superpotential is that the domain walls stretched going from one vacuum to the other will have a central terms for their tension given by the difference of the corresponding values of the superpotential. In the usual \( N = 1 \) Yang–Mills the coupling constant \( S \) is not a field but a parameter. But we can actually promote it to a chiral field whose vev is given by the coupling constant. If we do that, then we will also have to consider \( dW_k / dS = 0 \) for finding supersymmetric ground states, otherwise we would get a positive energy given by

\[
g^{\frac{s^T}{s}} |\partial_s W_k|^2,
\]

where \( g^{\frac{s^T}{s}} \) is the inverse to the Kähler metric for \( S \). However, \( dW_k / dS = 0 \) has no solutions, which means that we have no supersymmetric vacuum (or any vacuum in this case). But clearly we can embed the usual \( N = 1 \) gauge theory in this theory, simply by taking the kinetic term for \( S \) field to be very large and thus effectively freezing it (this corresponds to making \( g^{\frac{s^T}{s}} \) vanishing and thus giving no energy to the vacuum). Note in this case that even if \( S \) is treated as a parameter the vacuum has an energy and so the supersymmetric background makes sense if we decouple the gravity, by taking \( M_P \rightarrow \infty \). This is in fact how we will generate \( N = 1 \) QFT’s by turning on \( H \)-fluxes; namely, as we will see later, the field \( S \) will play the role of an extra field, whose dynamics we will freeze in the limit of interest and concentrate on a decoupling limit where gravity is irrelevant. This will in particular avoid the no-go theorem of [2] for obtaining \( N = 1 \) solutions.

4. Geometric engineering for \( N = 2 \) theories and their partial breaking to \( N = 1 \)

In preparation of our discussion for turning on \( RR \)-fluxes and breaking \( N = 2 \) theories to \( N = 1 \) we
first review some relevant aspects of geometric engineering for $N=2$ gauge theories, in the context of type IIA compactification on Calabi–Yau threefold and its type IIB mirror (see [13–16] for more detail). Instead of being general, consider a simple example: Let us review how the $SU(2)$ Yang–Mills is geometrically engineered: We consider in type IIA a local CY geometry where a $P^1$ is fibered over another $P^1$. The simplest possibility is $P^1 \times P^1$. In the limit the $P^1$ goes to zero we obtain an enhanced $SU(2)$ gauge symmetry from the $A_1$ singularity. To suppress the gravity effects we take $g_s \to 0$. The area of $P^1$, $S$, determines the coupling constant for the $SU(2)$ gauge theory:

$$S = \frac{1}{g_s^2}. $$

We can identify the size of $P^1$, $a$, with the vev of a scalar in the adjoint of $SU(2)$; more precisely, we have the classical relation that

$$a^2 = \langle \text{Tr} \Phi^2 \rangle. $$

One considers the regime where $S$ is large and $a$ is small. This is the same as taking a finite size $P^1$ and zero size $P^1$ in the string frame in the limit where $g_s \to 0$ (the effective mass of $D2$ branes is given by $a = a' / g_s$, and $S = S' / g_s$, where the $a'$ and $S'$ denote the volume of the $P^1$'s in the string frame). In this limit the field $S$ becomes non-dynamical and plays only the role of a parameter in the field theory. The $N=2$ prepotential in this case has the following structure

$$\mathcal{F} = \frac{1}{2}Sa^2 + a^2 \log a^2 + \sum_n c_n a^{2-4n} \exp (-nS) $$

(4.1)

for some constants $c_n$. Note that $S$ can be absorbed into a redefinition of $a$. In fact this is the limit one is taking in the geometry, namely we take $a \to 0$ and $S \to \infty$ keeping the combination $a^2 \exp S$ fixed. The way this expression is obtained is to use mirror symmetry to compute the worldsheet instanton corrections in this type IIA background by relating it to complex structure variation in a type IIB background. Physically, the net result is that the apparent instanton corrections in (4.1) can be related to one loop corrections to the prepotential summed over all $D2$ branes wrapped around the $P^1 \times P^1$ geometry. In other words,

$$\mathcal{F} = Sa^2 + \sum_{\text{BPS m}} m^2 \log m^2, $$

and the rich structure of the instanton corrections gets mapped to an intricate structure of wrapped BPS $D2$ branes in this geometry, in the limit we are taking.

4.1. Adding the flux and breaking $N = 2 \to N = 1$

Now we are ready to add the flux. We choose the flux to be an RR 2-form flux in the direction dual to the 2-cycle $P^1$. This means according to our discussion in Section 3, generating a superpotential given by

$$W = \frac{\partial \mathcal{F}}{\partial S}. $$

In the context of type IIB it means turning on a specific RR H-flux dual to the 3-cycle representing $S$. In that context $\partial F / \partial S$ is a classical computation of periods. Now, it has been shown [17] that

$$\frac{\partial \mathcal{F}}{\partial S} = \text{constant} \cdot u, $$

where $u = \langle \text{Tr} \Phi^2 \rangle$, and the classical result $u = a^2$ receives quantum correction exactly as given by $u = \text{constant} \cdot \frac{\partial \mathcal{F}}{\partial S}$. In particular, not just $a$, but also $u$ itself is among the periods of the type IIB Calabi–Yau geometry (this was a crucial fact for extracting gauge theory implication of type IIB geometry [13]). We thus have a superpotential

$$W = \text{constant} \cdot u. $$

This can be viewed in the gauge theory language as a mass deformation (giving mass to the scalar partner $\Phi$ of the vector multiplet) breaking $N = 2 \to N = 1$. To find whether there are any supersymmetric vacua one will have to analyze it, exactly as was done originally in the field theory context by Seiberg and Witten [18]. Namely one finds near the points where there is a massless dyon, a modification of the superpotential, by including the light states. In this
case we get (somewhat analogous to the conifold case)

\[ W = mu + (a_D - a_0) \phi \tilde{\phi}, \]

and one finds an \( N = 1 \) supersymmetric solution

\[ \phi \tilde{\phi} = m \frac{\delta u}{\delta a_D}, \quad a_D = a_0. \]

In the type IIB context this corresponds to freezing some complex moduli of Calabi–Yau threefold and rendering some other moduli non-dynamical.

4.2. Generalizations

It is clear that this generalizes to a large number of \( N = 2 \) gauge models engineered in [16] (at least for the cases where the beta function is not zero). Just as was done above for the \( SU(2) \) case, one turns on in the type IIA context independent RR 2-form flux in the bases whose sizes control the coupling constants of various gauge groups. By going over to its type IIB mirror, the RR flux turns into a particular \( H^x \) flux, which again serve to freeze the moduli in the type IIB side, exactly as was done for the \( SU(2) \) case above.

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The flavour asymmetry and quark-antiquark asymmetry in the \( \Sigma^+ \)-sea

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Abstract

The sea quark content of the \( \Sigma^+ \) baryon is investigated using light-cone baryon-meson fluctuation model suggested by Brodsky and Ma. It is found that the \( \Sigma^+ \) sea is flavour asymmetric \( (\bar{d} > \bar{u} > \bar{s}) \) and quark-antiquark asymmetric \( (q \neq \bar{q}) \). Our prediction for the flavour asymmetry, \( \bar{d} > \bar{u} > \bar{s} \), is significantly different from the \( SU(3) \) prediction \( (\bar{d} < \bar{u} < \bar{s}) \), while our prediction for the \( \bar{d}-\bar{s} \) asymmetry is consistent with the \( SU(3) \) prediction.

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The nucleon sea exhibits two interesting properties: flavour asymmetry \([1-4]\) and quark-antiquark asymmetry \([5,6]\). While there have been many studies of the nucleon sea from both experiment (see e.g. \([1-6]\)) and theory (see e.g. \([7-11]\) and references therein), the studies of the sea distributions of the other baryons in the baryon octet predicted by the \( SU(3) \) quark model are very few. It is of interest to know whether the sea of the other members of the baryon octet has the same properties (flavour asymmetry and quark-antiquark asymmetry) as the nucleon sea. Also, through the study of the quark sea of the other members of the baryon octet, we can improve our understanding of the structure of the baryons and the non-perturbative properties of QCD. Alberg et al. \([14]\) pointed out that the valence and sea quark distributions of the \( \Sigma^\pm \) may exhibit large deviations from the \( SU(3) \) predictions, and these parton distributions could be obtained from Drell-Yan experiments using charged hyperon beams on proton and deuteron targets. Alberg, Falter, and Henley \([15]\) studied the flavour asymmetry in the \( \Sigma^+ \) sea employing the meson cloud model and effective Lagrangian for the baryon-meson-baryon interaction, and found large deviations from \( SU(3) \). Boros and Thomas \([16]\) calculated the quark distributions of \( \Lambda \) and \( \Sigma^\pm \) employing the MIT bag model. It was found that the valence quark distributions are quite different from the \( SU(3) \) predictions and that the quark sea is flavour asymmetric. More recently, Ma, Schmidt and Yang \([17]\) showed that there are significant differences between the predictions of perturba-
tive QCD and $SU(6)$ quark-diquark model for the flavor and spin structure of the $Λ$ baryon’s quark distributions near $x = 1$.

In this letter we shall investigate the flavour asymmetry and quark-antiquark asymmetry of the $Σ^+$ sea using the light-cone baryon-meson fluctuation model (LCM) suggested by Brodsky and Ma [12]. The baryon-meson fluctuation (meson cloud) mechanism is very successful in understanding on the flavour asymmetry and quark-antiquark asymmetry of nucleon sea. The various fluctuations can be described via corresponding baryon-meson-nucleon Lagrangians [7–10]. Recently, Brodsky and Ma [12] proposed that the baryon-meson fluctuation could be described by using a light-cone two body wave function which is a function of the invariant mass squared, the LCM is relatively simple. Furthermore our study [13] showed that the LCM can produce very similar results to the effective Lagrangians 7±10 for the description of baryon-meson fluctuations in the baryon octet and meson octet. The various fluctuations can be expressed in terms of bare nucleon and virtual baryon-meson Fock states. Although this model was proposed that the baryon-meson fluctuation could be described via corresponding baryon-meson-nucleon Lagrangians 7±10. Recently, Brodsky and Ma 12 showed that the LCM can be viewed as a bare nucleon surrounded by a mesonic cloud. The nucleon wave function can be described by using a light-cone two body wave function which is a function of the invariant mass squared, the LCM is relatively simple. Furthermore our study [13] showed that the LCM can produce very similar results to the effective Lagrangian method for a suitable choice of parameter.

The basic idea of the meson cloud model (for recent reviews see Refs. [9,10]) is that the nucleon can be viewed as a bare nucleon surrounded by a mesonic cloud. The nucleon wave function can be expressed in terms of bare nucleon and virtual baryon-meson Fock states. Although this model was developed mainly in the study of nucleon sea, applying this model to the other baryons is straightforward. For the $Σ^+$, the wave function can be written as

$$|Σ^+\rangle_{\text{physical}} = |Z\rangle |Σ^+\rangle_{\text{bare}} + \sum_{BM} \int dyd^2 k_⊥ \phi_{BM}(y,k^2_⊥) \times |B(y,k_⊥);M(1−y,−k_⊥)\rangle,$$  

where $Z$ is the wave function renormalization constant, $\phi_{BM}(y,k^2_⊥)$ is the wave function of Fock state containing a baryon ($B = Λ, Σ^0, Σ^+, p$) with longitudinal momentum fraction $y$, transverse momentum $k_⊥$, and a meson ($M = π^+, π^0, K^0$) with momentum fraction $1−y$, transverse momentum $−k_⊥$. Here we consider the most energetically-favoured fluctuations in the baryon octet and meson octet. The fluctuation $Σ^+ → Ξ^0K^+$ is neglected due to the higher mass of $Ξ^0$ ($m_Ξ = 1.32$ GeV while $m_A = 1.12$ GeV, $m_K = 1.19$ GeV).

It would seem that the fluctuation $Σ^+ → Σ^+ η$ is also important in the calculations of $d−s$ and $\bar{u}−\bar{s}$. However, applying the common $η_8−η_1$ mixing scheme

$$\eta = \cos θ \frac{1}{\sqrt{6}} (u\bar{u} + d\bar{d} − 2s\bar{s})$$  

$$− \sin θ \frac{1}{\sqrt{3}} (uu + dd + ss)$$  

and assuming $SU(3)$ symmetry for the quark distributions in the $η_8$ and $η_1$, we find that compared to the fluctuation $Σ^+ → Λπ^+$, the contributions to the $d−s$ and $\bar{u}−\bar{s}$ from the fluctuation $Σ^+ → Σ^+ η$ are suppressed by a factor of $(\frac{1}{\sqrt{6}} cos θ + \frac{1}{\sqrt{3}} sin θ)^2 = (\frac{1}{\sqrt{6}} cos θ + \frac{1}{\sqrt{3}} sin θ)^2$ which is in the range of $−0.20 ~ −0.01$ for mixing angle in the theoretically accepted range $θ = −12° ~ −20°$ [18–22]. The higher mass of the $η$ ($m_η = 0.547$ GeV, $m_1 = 0.139$ GeV) also suppresses the contribution from this fluctuation. Thus we neglect this fluctuation in our calculation.

Provided that the lifetime of a virtual baryon-meson Fock state is much longer than the strong interaction time in the Drell-Yan process, the contribution from the virtual baryon-meson Fock states to the quark and anti-quark sea of $Σ^+$ can be written as convolutions

$$q(x) = \sum_{BM} \left[ \int_x^1 dy \frac{d^2 k_⊥}{y} f_{BM}(y) q^u(\frac{x}{y}) \right.$$  

$$+ \int_x^1 dy \frac{d^2 k_⊥}{y} f_{MB}(1−y) q^d(\frac{x}{y}) \right].$$  

$$\bar{q}(x) = \sum_{BM} \left[ \int_x^1 dy \frac{d^2 k_⊥}{y} f_{BM}(1−y) \bar{q}^u(\frac{x}{y}) \right.$$  

where $f_{BM}(y) = f_{MB}(1−y)$ is fluctuation function which gives the probability for the $Σ^+$ to fluctuate into a virtual $BM$ state

$$f_{BM}(y) = \int_0^∞ dk^2_⊥ \left| \phi_{BM}(y,k^2_⊥) \right|^2.$$
A common practice in the evaluation of the wave function $\phi_{BM}(y, k^2_\perp)$ is to employ time-ordered perturbative theory in the infinite momentum frame and the effective meson-baryon-nucleon interaction Lagrangian \cite{7-10}. On the other hand, Brodsky and Ma \cite{12} suggested that this wave function can also be described by using light-cone two-body wave function which is a function of the the invariant mass squared of the baryon-meson Fock state

$$\phi_{BM}(y, k^2_\perp) = A \exp \left( \frac{1}{8\alpha^2} \left( \frac{m_B^2 + k^2_\perp}{y} + \frac{m_M^2 + k^2_\perp}{1-y} \right) \right), \quad (6)$$

where $\alpha$ is a phenomenological parameter which determines the shape of the fluctuation function. Compared to the effective Lagrangian method, Eq. (6) is quite simple. Furthermore, our study on the $s$-$\bar{s}$ asymmetry in the nucleon sea \cite{13} showed that Eq. (6) can provide similar results to the effective Lagrangian method for $\alpha = 1.0$ GeV. Because the spin structure of the baryon-meson-baryon vertex is the same for all members of the baryon octet (ignoring fluctuations to decuplet baryons), we might expect that the value of $\alpha$ should be similar for all the members of the baryon octet. We will use $\alpha = 0.3$ GeV and 1.0 GeV in our calculation as there is little constraint from experimental data or theoretical studies on the $\Sigma$ sea. The normalization $A$ in Eq. (6) can be determined by the probability for the corresponding fluctuation. We adopt the result given in Ref. \cite{16} for the probabilities of the various fluctuations \cite{3}:

$$P_{A^0} = 3.2\%, \quad P_{pK^0} = 0.4\%, \quad P_{\Sigma^0p} = 0.17\%, \quad P_{\Sigma^-*p} = 0.85\%. \quad (7)$$

In the baryon-meson fluctuation model, the non-perturbative contributions to the quark and the antiquark distributions in the $\Sigma^-$ sea come from the quarks and anti-quarks of the baryons ($\Lambda, \Sigma^-, \Sigma^0$, and $p$) and mesons ($\pi^-, \pi^0$, and $K^0$) in the virtual baryon-meson Fock states. So we need the parton distributions of the involved baryons and mesons as input. For the parton distribution in the pion, we employ the parameterization given by Glück, Reya, and Stratmann (GRS98) \cite{23} and we neglect the sea content in the meson, that is,

$$\ddot{u}^0 = u^0 = \ddot{\pi}^0 = d^0 = \frac{1}{2} v^0, \quad (9)$$

$$\ddot{u}^0 = u^0 = \ddot{\pi}^0 = d^0 = \frac{1}{2} v^0, \quad (10)$$

$$v^0(x, \mu^2_{NL}) = 1.052 x^{-0.495} (1 + 0.357\sqrt{x})(1-x)^{0.365}, \quad (11)$$

at scale $\mu^2_{NL} = 0.34$ GeV$^2$. For the $\bar{d}$ distribution in the $K^0$ we relate it to the $u$ distribution in the $K^+$ which are given in the GRS98 parameterization \cite{23} also

$$\ddot{d}^0(x, \mu^2_{NL}) = u^+(x, \mu^2_{NL}) = 0.540(1-x)^{0.7} v^0(x, \mu^2_{NL}), \quad (12)$$

at scale $\mu^2_{NL} = 0.34$ GeV$^2$.

In order to investigate the quark-antiquark asymmetry via $d(x) - \bar{d}(x)$ in the $\Sigma^-$ sea, we also need to know the $d$-quark distribution in the $\Lambda, \Sigma^-$ and $p$, for which we use the parameterization for the $d$ quark distribution in the proton given by Glück, Reya, and Vogt (GRV98) \cite{24},

$$d^0(x, \mu^2_{NL}) = 0.400 x^{-0.57}(1-x)^{4.09}(1 + 18.2 x), \quad (13)$$

at scale $\mu^2_{NL} = 0.40$ GeV$^2$.

We evolve the distributions to the scale $Q^2 = 4$ GeV$^2$ using the program of Miyama and Kumano \cite{25} in which the evolution equation is solved numerically in a brute-force method. We found that at $Q^2 = 4$ GeV$^2$ all parton distributions $v^0(x, Q^2)$, $\ddot{d}^0(x, Q^2)$ and $d^0(x, Q^2)$ can be parameterized using the following form

$$q(x, Q^2) = ax^b (1-x)^c (1 + d\sqrt{x} + ex) \quad (14)$$

with the parameters given in Table 1. We estimate the uncertainty in solving the evolution equations numerically and parameterizing the parton distribu-

\footnote{Note the relationship between the fluctuation functions for various isospin states: $f_{2u^0} = f_{2+u^0}$ and the fluctuation functions given in Ref. \cite{16} for a given type of fluctuation are defined as the sum of all isospin states: $f_{2u^0} = f_{2+u^0} + f_{2-+u^0}$.}
Table 1
Parameters in Eq. (14) at $Q^2 = 4 \text{ GeV}^2$.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c^*(x,Q^2)$</td>
<td>1.712</td>
<td>-0.518</td>
<td>1.182</td>
<td>-0.836</td>
<td>0.972</td>
</tr>
<tr>
<td>$\bar{c}^*(x,Q^2)$</td>
<td>0.910</td>
<td>-0.519</td>
<td>1.418</td>
<td>-0.910</td>
<td>1.086</td>
</tr>
<tr>
<td>$d^*(x,Q^2)$</td>
<td>0.615</td>
<td>-0.575</td>
<td>5.096</td>
<td>1.102</td>
<td>6.773</td>
</tr>
</tbody>
</table>

The effect of evolution from a lower scale to a higher scale is to make the parton distribution more concentrated in the small $x$ region. Thus we may expect that the $x$ position at which an asymmetry exhibits a maximum will move to smaller $x$ as we evolve to higher values of $Q^2$. However, we do not expect the asymmetry to "evolve away" at a higher $Q^2$ scale if it exists at a lower scale such as $m_s^2$.

We investigate the flavour asymmetry in the $\Sigma^+$ sea through calculating the differences between the antiquark distributions: $x[\bar{d}(x) - \bar{u}(x)]$, $x[\bar{q}(x) - \bar{\pi}(x)]$ and $x[\bar{\pi}(x) - \bar{\pi}(x)]$ which are given by

$$x[\bar{d}(x) - \bar{u}(x)] = x\left[\bar{d}_{\Lambda^+}\right](x) + x\left[\bar{d}_{\Sigma^+}\right](x) + x\left[\bar{d}_{\pi}\right](x)$$

$$= \int_0^1 dy \frac{x}{y} \left[ f_{\Lambda^+}(1-y) \bar{d}_{\Lambda^+}(\frac{x}{y}) + f_{\Sigma^+}(1-y) \bar{d}_{\Sigma^+}(\frac{x}{y}) + f_{\pi}(1-y) \bar{d}_{\pi}(\frac{x}{y}) \right].$$

$$x[\bar{\pi}(x) - \bar{\pi}(x)] = x\left[\bar{d}_{\Sigma^+}\right](x) + x\left[\bar{d}_{\Sigma^+}\right](x) + x\left[\bar{d}_{\pi}\right](x)$$

$$= \int_0^1 dy \frac{x}{y} \left[ f_{\Sigma^+}(1-y) \bar{d}_{\Sigma^+}(\frac{x}{y}) + f_{\pi}(1-y) \bar{d}_{\pi}(\frac{x}{y}) \right].$$

$x[\bar{\pi}(x) - \bar{\pi}(x)]$ comes from only fluctuation $\Sigma^+ \rightarrow \Sigma^+ \pi^0$, while $x[\bar{\pi}(x) - \bar{\pi}(x)]$ and $x[\bar{\pi}(x) - \bar{\pi}(x)]$ come from also $\Sigma^+ \rightarrow \Lambda \pi^+$, $\Sigma^+ \rightarrow \Sigma^0 \pi^+$ as well as $\Sigma^+ \rightarrow p \bar{K}^0$. In Fig. 1 we present our results for $x[\bar{d}(x) - \bar{u}(x)]$ at the scales $\mu_{\text{NLO}}^2 = 0.34 \text{ GeV}^2$ and $Q^2 = 4 \text{ GeV}^2$ with $\alpha = 0.3 \text{ GeV}$. It can be found that the contribution from the fluctuation $\Sigma^+ \rightarrow \Lambda \pi^+$ is about twice as large as that from $\Sigma^+ \rightarrow \Sigma^0 \pi^+$, and both are much larger than the contribution from $\Sigma^+ \rightarrow p \bar{K}^0$. Under evolution the distributions move from larger $x$ to smaller $x$ and the $x$ position at which $x[\bar{d}(x) - \bar{u}(x)]$ exhibits a maximum shifts from about 0.1 to 0.06 and the maximum decreases about 20%, which coincides with our naive expectation. The numerical results for $x[\bar{d}(x) - \bar{u}(x)]$, $x[\bar{q}(x) - \bar{\pi}(x)]$ and $x[\bar{\pi}(x) - \bar{\pi}(x)]$ are shown in Fig. 1.

Fig. 1. $x[\bar{d}(x) - \bar{u}(x)]$ with $\alpha = 0.3 \text{ GeV}$. The dashed, dotted, and dashed-dotted curves are the contributions from $\Lambda \pi$, $\Sigma \pi$ and $p K$ components respectively. The solid curve is the sum of above three contributions. The thinner and thicker curves correspond to the scales $\mu_{\text{NLO}}^2 = 0.34 \text{ GeV}^2$ and $Q^2 = 4 \text{ GeV}^2$ respectively.
and \( x[\bar{q}(x) - \bar{s}(x)] \) at \( Q^2 = 4 \text{ GeV}^2 \) are given in Figs. 2 and 3 for \( \alpha = 0.3 \text{ GeV} \) and 1.0 GeV respectively. We can see that \( \bar{d}(x) > \bar{n}(x) > \bar{s}(x) \), that is the anti-quark distribution in the \( \Sigma^+ \) sea is flavour asymmetric.

As is well known, the nucleon sea is also asymmetric and for the proton sea \( \bar{d} > \bar{n} > \bar{s} \) [7–11]. The main difference between the proton (\( uud \)) and \( \Sigma^+ (uus) \) is the replacement of a valence \( d \) quark by a valence \( s \) quark. Thus one may expect from \( SU(3) \) symmetry that the \( \Sigma^+ (uus) \) sea to be \( \bar{s} > \bar{n} > \bar{d} \). This prediction is opposite to our above conclusion \( \bar{d} > \bar{n} > \bar{s} \) from the light-cone baryon-meson fluctuation model. If the \( SU(3) \) symmetry breaking in the \( \bar{n}, \bar{d} \) and \( \bar{s} \) distributions in the \( \Sigma^+ \) sea has the same source as that for the \( u, d \), and \( s \) quark masses, we may expect that \( x[\bar{d}(x) - \bar{n}(x)] < x[\bar{n}(x) - \bar{s}(x)] \) since the mass difference between the \( u \) and \( d \) quarks is far smaller than that between the \( u \) and \( s \) quarks. However, our calculations (see Figs. 2 and 3) show that \( x[\bar{d}(x) - \bar{n}(x)] > x[\bar{n}(x) - \bar{s}(x)] \). The relation \( x[\bar{d}(x) - \bar{n}(x)] > x[\bar{n}(x) - \bar{s}(x)] \) is opposite to our above argument, which implies that the dynamics responsible for the \( SU(3) \) symmetry breaking in the quark distributions of the \( \Sigma^+ \) sea, as calculated in our model, are different from that responsible for the mass differences among the \( u, d \) and \( s \) quarks.

Another interesting question concerning the \( \Sigma^+ \) sea is the quark-antiquark asymmetry. Although the perturbative sea created from gluon-splitting is symmetric \( q = \bar{q} \) (in the leading twist approximation in perturbative calculation), the non-perturbative sea, which may exist over a long time and has a strong connection with the ‘‘bare’’ \( \Sigma^+ \), may be asymmetric \( q \neq \bar{q} \). Because of the existence of valence \( u \) and \( s \) quarks in the \( \Sigma^+ \), it is difficult to measure the differences \( u - \bar{n} \) and \( s - \bar{s} \) in the \( \Sigma^+ \) sea. The most likely experiment is to measure the difference between \( d \) and \( \bar{d} \). From the baryon-meson fluctuation model the \( d(x) - \bar{d}(x) \) turns out to be:

\[
d(x) - \bar{d}(x) = d_{1u^+}(x) - d_{1\bar{u}^+}(x) + d_{2s^+}(x) - d_{2\bar{s}^+}(x) - d_{2\bar{u}^+}(x) + d_{1\bar{s}^+}(x) + f_\rho^\pi(1 - y) \left[ f_{1\bar{u}^+}(y) + f_{2\bar{s}^+}(1 - y) \right] d^\pi(\frac{x}{y}) - f_{1\bar{s}^+}(1 - y) d^{\bar{\pi}}(\frac{x}{y}).
\]

The numerical results at scales \( \mu^2 = 4 \text{ GeV}^2 \) are presented in Fig. 4. Once again one can find that evolution ‘‘pushes’’ the distributions to the small \( x \) region. It can be seen that \( d \neq \bar{d} \) in the \( \Sigma^+ \) sea. However, the prediction for the behavior of \( d(x) - \bar{d}(x) \) depends strongly on the value of \( \alpha = 1.0 \text{ GeV} \).
for $\alpha = 0.3$ GeV $d(x) < \bar{d}(x)$ in the smaller $x$ region and $d(x) > \bar{d}(x)$ in the larger $x$ region, while for the $\alpha = 1.0$ GeV $d(x) > \bar{d}(x)$ in the smaller $x$ region and $d(x) < \bar{d}(x)$ in the larger $x$ region. This result is similar to our earlier finding on the $sx$ region employing the same light-cone baryon-meson fluctuation model, which suggests that $SU(3)$ symmetry in the sea holds in this case.

We turn to the discussion about $\alpha$-dependence in our calculation. Comparing Figs. 2 and 3 one can find that for the $x[d(x) - \bar{u}(x)]$, $x[\bar{d}(x) - \bar{s}(x)]$ and $x[u(x) - s(x)]$ the shape and maximum of asymmetries are very similar for different $\alpha$, while the $x$ position at which the asymmetries exhibit maxima shifts slightly. The calculations with $\alpha = 0.3$ GeV peak at about $x = 0.06$ while the calculations with $\alpha = 1.0$ GeV peak at about $x = 0.1$. Thus the calculations for the flavour asymmetry are not very sensitive to the value of $\alpha$, and $x$ being about 0.08 is a good region to study the flavor asymmetry in the $\Sigma^+$ sea. This observation is consistent with the prediction given in Ref. [14] – the region $0.1 \leq x \leq 0.2$ should be a good one to measure the flavour asymmetry in the $\Sigma$ sea. The calculation for the $d(x) - \bar{d}(x)$ (see Fig. 4) is much more sensitive to the value of $\alpha$ than that for the flavour asymmetry – the calculations with $\alpha = 0.3$ GeV and 1.0 GeV even give opposite predictions for the $x$-dependence of $d(x) - \bar{d}(x)$. We may expect that further calculation [26] on the nucleon sea employing the light-cone baryon-meson fluctuation model may provide useful constraints on the value of $\alpha$, and thereby give more definite predictions on the sea quark content in the $\Sigma^+$ baryon.

In summary, besides the nucleon sea the studies on the sea quark content of the other members of the baryon octet are interesting and important since it is helpful to our understanding of both the structure of the octet baryons and non-perturbative QCD effects such as $SU(3)$ symmetry breaking and flavour asymmetry. We calculated the sea quark content of the $\Sigma^+$ baryon employing the light-cone baryon-meson fluctuation model. It was found that the $\Sigma^+$ sea is flavour asymmetric ($\bar{d} > \bar{u} > \bar{s}$) and quark-antiquark asymmetric ($u = \bar{u}$). Our prediction for the flavor asymmetry, $\bar{d} > \bar{u} > \bar{s}$, is significantly different from the $SU(3)$ prediction ($\bar{d} < \bar{u} < \bar{s}$), while our prediction for the $d-\bar{d}$ asymmetry is consistent with the $SU(3)$ prediction.
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References

Conformal invariance for non-relativistic field theory

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Abstract

Momentum space Ward identities are derived for the amputated $n$-point Green’s functions in 3+1 dimensional non-relativistic conformal field theory. For $n = 4$ and 6 the implications for scattering amplitudes (i.e. on-shell amputated Green’s functions) are considered. Any scale invariant 2-to-2 scattering amplitude is also conformally invariant. However, conformal invariance imposes constraints on off-shell Green’s functions and the three particle scattering amplitude which are not automatically satisfied if they are scale invariant. As an explicit example of a conformally invariant theory we consider non-relativistic particles in the infinite scattering length limit. © 2000 Published by Elsevier Science B.V. All rights reserved.

Poincaré invariant theories that are scale invariant usually have a larger symmetry group called the conformal group. A similar phenomena happens for 3+1 dimensional non-relativistic systems. These are invariant under the extended Galilean group, which consists of 10 generators: translations (4), rotations (3), and Galilean boosts (3). The largest space-time symmetry group of the free Schrödinger equation is called the Schrödinger or non-relativistic conformal group. This group has two additional generators corresponding to a scale transformation, and a one-dimensional special conformal transformation, sometimes called an “expansion”. The infinitesimal Galilean boost, scale and conformal transformations are

\begin{align}
\text{boosts:} & \quad x' = x + v t, \quad t' = t, \\
\text{scale:} & \quad x' = x + st, \quad t' = t + 2 s t, \\
\text{conformal:} & \quad x' = x - c t x, \quad t' = t - c t^2,
\end{align}

where $v, s$ and $c$ are the corresponding infinitesimal parameters. (The finite scale transformation is $x' = e^s x$, $t' = e^{2s} t$, and the finite conformal transformation is $x' = x/(1 + c t)$, $1/t' = 1/t + c$.)

In this letter we explore the implications of non-relativistic conformal invariance for 3+1 dimensional physical systems. In relativistic theories, conformal invariance can be used to constrain the functional form of $n$-point correlation functions [1], however, on-shell scattering amplitudes are typically ill-defined because of

1 Exceptions are known to exist, however, these theories suffer from pathologies, such as non-unitarity. A detailed discussion of scale and conformal invariance in relativistic theories can be found in Ref. [1].
infrared divergences associated with massless particles. In non-relativistic theories scattering amplitudes are well defined even in the conformal limit. We show how conformal invariance can be used to gain information about scattering amplitudes by deriving Ward identities for the amputated momentum space Green’s functions. While the off-shell Green’s functions can be changed by field redefinitions, the scattering amplitudes (on-shell Green’s functions) are physical quantities and are therefore unchanged. We find that any 2-to-2 (identical particle) scattering amplitude that satisfies the scale Ward identity automatically satisfies the conformal Ward identity. However, this is not the case for the corresponding off-shell Green’s function or for the 3-to-3 scattering amplitude. We construct a field theory that has a four point function which obeys the scale and conformal Ward identities and conjecture that the higher point functions in this theory also obey these Ward identities. On-shell it gives S-wave scattering with an infinite scattering length.

For the interaction of two nucleons, the scattering lengths in the \( ^1S_0 \) and \( ^3S_1 \) channels are large \( (d^{(1S_0)} = -23.7 \text{ fm} \) and \( d^{(3S_1)} = 5.4 \text{ fm} \) compared to the typical length scales in nuclear physics. In the limit that these scattering lengths go to infinity (and higher terms in the effective range expansion are neglected) we show that the four point Green’s function obeys the scale and conformal Ward identities. Thus, two body nuclear systems at low energies are approximately scale and conformal invariant. It is likely that in some spin-isospin channels the higher point functions will also obey these Ward identities. Whether this conformal invariance can lead to new predictions for many body nuclear physics is presently unclear, but seems worthy of further study.

The action for a free non-relativistic field \( N(x,t) \) is

\[
S_0 = \int d^3x \int_0^\infty \left[ i\partial_t + \frac{\nabla^2}{2M} \right] N ,
\]

where \( M \) is mass of the particle corresponding to the field \( N \). Under an infinitesimal Galilean transformation \( N'(x',t') = (1 + iM\mathbf{v} \cdot \mathbf{x}) N(x,t) \) or equivalently

\[
\delta_x N(x,t) = N'(x,t) - N(x,t) = D_x N(x,t) = \mathbf{v} \cdot (iM \mathbf{x} - \mathbf{i} \nabla) N(x,t) .
\]

The action in Eq. (2) is invariant under the infinitesimal scale transformation in Eq. (1) with \( N'(x',t') = (1 - 3s/2) N(x,t) \) or equivalently

\[
\delta_s N(x,t) = D_s N(x,t) = -s\left( \frac{1}{2} + \mathbf{x} \cdot \nabla + 2t\partial_t \right) N(x,t) ,
\]

and under the infinitesimal conformal transformation provided \( N'(x',t') = (1 + 3 ct/2 - 3M c x^2/2) N(x,t) \) or equivalently

\[
\delta_c N(x,t) = D_c N(x,t) = c\left( \frac{3t}{2} - iM x^2 \right) N(x,t) .
\]

Now consider adding interactions that preserve these invariances (an explicit example will be considered later). The position space Green’s functions for the interacting theory, \( G^{(2n)}(x_1,t_1) = G^{(2n)}(x_1,t_1,\ldots;x_{2n},t_{2n}) \), are defined by

\[
G^{(2n)}(x_1,t_1,\ldots;x_{2n},t_{2n}) = \langle \Omega | T\{ N(x_1,t_1) \cdot \cdots \cdot N(x_{2n},t_{2n}) N^\dagger(x_{n+1},t_{n+1}) \cdot \cdots \cdot N^\dagger(x_{2n},t_{2n}) \} | \Omega \rangle ,
\]

where \( | \Omega \rangle \) is the vacuum of the interacting theory and is assumed to be invariant under the Schrödinger group.

\[\text{In non-relativistic theories particle number is conserved so there must be the same number of } N \text{’s as } N^\dagger \text{’s.}\]
Under the infinitesimal transformations in Eq. (3)–(5)
\[
\delta_{(g,s,c)}^{(2n)}(x_i,t_i) = \langle \Omega | T \left\{ \delta_{(g,s,c)} N(x_1,t_1) N(x_2,t_2) \cdots N^\dagger(x_2n,t_{2n}) - \right. \\
+ \left. \delta_{(g,s,c)} N(x_1,t_1) \cdots N^\dagger(x_{2n-1},t_{2n-1}) \delta_{(g,s,c)} N^\dagger(x_{2n},t_{2n}) \right\} | \Omega \rangle 
\]
\[
= \left[ \sum_{k=1}^{n} D_{(g,s,c)}^k + \sum_{k=n+1}^{2n} D_{(g,s,c)}^k \right] \langle \Omega | T \left\{ N(x_1,t_1) \cdots N^\dagger(x_{2n},t_{2n}) \right\} | \Omega \rangle, 
\]
where $D_{(g,s,c)}^k$ is the differential operator for coordinates $(x_i,t_i)$. Invariance under Galilean boosts, scale, and conformal symmetry implies that
\[
\delta_{(g,s,c)}^{(2n)}(x_i,t_i) = 0. 
\]

The momentum space Green’s functions $G^{(2n)}(p_i,E_i) = G^{(2n)}(p_1,E_1;\ldots;p_{2n},E_{2n})$ are the Fourier transform of the position space Green’s functions
\[
G^{(2n)}(x_1,t_1;\ldots;x_{2n},t_{2n}) 
= \left[ \prod_{k=1}^{2n} \int \frac{d^4p_k}{(2\pi)^4} e^{-i(\eta_k(x_k-p_k,x_k))} \right] \langle \Omega | T \left\{ \sum_{k=1}^{2n} \eta_k E_k \delta^{(3)} \left[ \sum_{k=1}^{2n} \eta_k p_k \right] \right\} G^{(2n)}(E_1,p_1;\ldots;E_{2n},p_{2n}) \rangle, 
\]
where $\eta_k$ is 1 for incoming particles (subscripts $1,\ldots,n$) and $-1$ for outgoing particles (subscripts $n+1,\ldots,2n$). The delta functions in Eq. (9) arise due to translational invariance. Using Eq. (8) with $x_{2n} = 0$ and $t_{2n} = 0$ it is straightforward to show that invariance under Galilean boosts, scale transformations, and conformal transformations implies the Ward identities
\[
\mathcal{D}_{(g,s,c)}^{(2n)}(E_i,p_i;\ldots;E_{2n},p_{2n}) = 0, 
\]
where
\[
\mathcal{D}_g = \sum_{j=1}^{2n-1} \left( M \nabla_{p_j} + p_j \frac{\partial}{\partial E_j} \right), \quad \mathcal{D}_s = 7n-5 + \sum_{j=1}^{2n-1} \left( p_j \cdot \nabla_{p_j} + 2 E_j \frac{\partial}{\partial E_j} \right), 
\]
\[
\mathcal{D}_c = \sum_{j=1}^{2n-1} \eta_j \left( \frac{7}{2} \frac{\partial}{\partial E_j} + \frac{M}{2} \nabla_{p_j}^2 + E_j \frac{\partial^2}{\partial E_j} + p_j \cdot \nabla_{p_j} \frac{\partial}{\partial E_j} \right). 
\]
In deriving Eq. (10) we have integrated over the delta functions in Eq. (9) so that
\[
E_{2n} = \sum_{j=1}^{2n-1} \eta_j E_j, \quad p_{2n} = \sum_{j=1}^{2n-1} \eta_j p_j. 
\]
The S-matrix elements are related to the amputated Green’s functions $\mathcal{A}^{(2n)}(p_i,E_i) = \mathcal{A}^{(2n)}(p_1,E_1;\ldots;p_{2n},E_{2n})$ defined by
\[
\mathcal{A}^{(2n)}(E_i,p_i) = \left[ \prod_{j=1}^{2n} \left( E_j - \frac{p_j^2}{2M} \right) \right] G_{\text{con}}^{(2n)}(E_i,p_i), 
\]
\footnote{Neglecting relativistic corrections to $S_0$, Eq. (13) is exact because adding interactions to Eq. (2) does not effect the two point function since there is no pair creation in the non-relativistic theory.}
where \( E_{2n} \) and \( p_{2n} \) are given by Eq. (12). \( G_{\text{con}}^{(2n)} \) is the connected part of \( G^{(2n)} \) and also satisfies Eq. (10).

Applying the Galilean boost and scale Ward identities in Eq. (10) to Eq. (13) gives

\[
\tilde{\mathcal{G}}_{(x,y)} \mathcal{A}^{(2n)}(E_i, p_i) = 0,
\]

where \( \mathcal{G}_{x} = \mathcal{G} \) and

\[
\mathcal{G} = 3n - 5 + \sum_{j=1}^{2n-1} \left( p_j \cdot \nabla_{p_j} + 2E_j \frac{\partial}{\partial E_j} \right).
\]

(15)

Applying the conformal Ward identity in Eq. (10) to Eq. (13) gives

\[
\frac{1}{\sum_j \eta_i E_j - \left( \sum_j \eta_i p_j \right)^2 / (2M)} \left[ \frac{1}{M} \left( \sum_j \eta_j p_j \right) \cdot \mathcal{G} - \tilde{\mathcal{G}} \right] \mathcal{A}^{(2n)} = 0,
\]

where

\[
\mathcal{G} = 3n - 5 + \sum_{j=1}^{2n-1} \eta_j \left( \frac{3}{2} \frac{\partial}{\partial E_j} + \frac{M}{2} \nabla_{p_j}^2 + E_j \frac{\partial^2}{\partial E_j^2} + p_j \cdot \nabla_{p_j} \frac{\partial}{\partial E_j} \right).
\]

(17)

Therefore, amputated Green’s functions satisfying Eq. (14) also satisfy

\[
\tilde{\mathcal{G}}_{x} \mathcal{A}^{(2n)} = 0.
\]

(18)

The leading term in the effective field theory for non-relativistic nucleon-nucleon scattering corresponds to a scale invariant theory in the limit that the S-wave scattering lengths go to infinity (see for e.g. Ref. [3]). As we will see below, this limit corresponds to a fixed point of the renormalization group. Since in nature the S-wave scattering lengths are very large, it is the unusual scaling of operators at this non-trivial fixed point [4] that controls their importance in this effective field theory [5,6]. Motivated by this we add to Eq. (2) the interaction

\[
S_{1} = -\int dtd^3x C_0 \left( N^T P N \right)^\dagger \left( N^T P N \right),
\]

(19)

where \( N \) is now a two component spin-1/2 fermion field and \( P = i\sigma_2/2 \). Higher body non-derivative interaction terms are forbidden by Fermi statistics. The interaction in Eq. (19) only mediates spin singlet S-wave \( NN \) scattering. The \( NN \) scattering amplitude arises from the sum of bubble Feynman diagrams shown in Fig. 1.

The loop integration associated with a bubble has a linear ultraviolet divergence and consequently the values of the coefficients \( C_0 \) depend on the subtraction scheme adopted. In minimal subtraction, if \( p \gg 1/a \) where \( p \) is the center of mass momentum and \( a \) is the scattering length, then successive terms in the perturbative series represented by Fig. 1 get larger and larger. Subtraction schemes have been introduced where each diagram in Fig. 1 is of the same order as the sum. One such scheme is PDS [5], which subtracts not only poles at \( D = 4 \),

Fig. 1. Terms contributing to \( \mathcal{A}^{(1)} \) from the interaction in Eq. (19).
but also the poles at $D=3$ (which correspond to linear divergences). Another such scheme is the OS momentum subtraction scheme \cite{4,7}. In these schemes the coefficients are subtraction point dependent, $C_0 = C_0(\mu)$. Calculating the bubble sum in PDS or OS gives

$$\mathcal{A}^{(4)} = \frac{-C_0(\mu)}{1 + MC_0(\mu)} \left[ 2\mu - \sqrt{-4M(E_1 + E_2) + (p_1 + p_2)^2 - i\epsilon} \right]/(8\pi),$$

where

$$C_0(\mu) = -\frac{4\pi}{M} \frac{1}{\mu - 1/a}.$$  \hspace{1cm} (20)

Note that Eq. (20) holds in any frame and we have not imposed the condition that the external particles be on-shell. It is easy to see that the limit $a \to \pm \infty$ corresponds to a non-trivial ultraviolet fixed point in this scheme. If we define a rescaled coupling $\hat{C}_0 = M\mu C_0(\mu)/(4\pi)$, then

$$\mu \frac{d}{d\mu} \hat{C}_0(\mu) = \hat{C}_0(\mu) \left[ 1 + \hat{C}_0(\mu) \right].$$

The limit $a \to \pm \infty$ corresponds to the fixed point $\hat{C}_0 = -1$. At a fixed point one expects the theory to be scale invariant. In fact, it can be easily verified that in the $a \to \pm \infty$ limit

$$\mathcal{A}^{(4)} = \frac{8\pi}{M} \frac{1}{\sqrt{-4M(E_1 + E_2) + (p_1 + p_2)^2 - i\epsilon}}$$

satisfies both the scale and conformal Ward identities in Eqs. (14) and (18). In the case of $\mathcal{A}^{(4)}$ the conformal Ward identity gives non-trivial information about the off-shell amplitude. For instance the amplitude

$$\mathcal{A}^{(4)} = \frac{8\pi}{M} \frac{1}{\sqrt{-\left( p_1 - p_3 \right)^2 - \left( p_2 - p_3 \right)^2 - i\epsilon}}$$

is scale and Galilean invariant but not conformally invariant. The expressions for $\mathcal{A}^{(4)}$ in Eqs. (23) and (24) agree on-shell, where $E_i = p_i^2/(2M)$.

The interaction in Eq. (19) also induces non-trivial amputated Greens functions, $\mathcal{A}^{(2n)}$, for $n > 2$. (For $n = 3$ see Fig. 2.) It is believed that non-perturbatively the higher point functions are finite and we speculate that with $C_0$ at its critical fixed point the action $S_0 + S_1$ defines a non-relativistic conformal field theory.

We will now derive scale and conformal Ward identities for the on-shell amplitudes since these are the physical quantities of interest. Consider the four point function for a scalar field \cite{4}. After imposing translation invariance it is a function of 12 variables

$$\mathcal{A}^{(4)}(p_1, p_2, p_3, E_1, E_2, E_3).$$

The Ward identity $\tilde{\mathcal{G}}_4 \mathcal{A}^{(4)} = 0$ is solved by the function $\mathcal{A}^{(4)}(p_A, p_B, U_1, U_2, U_3)$ where

$$p_A = p_1 - p_3, \quad p_B = p_2 - p_3, \quad U_1 = ME_1 - p_1^2/2.$$  \hspace{1cm} (26)

\footnotetext[4]{Eq. (25) through (27) are valid for fermions, but when imposing rotational invariance we will assume the particles are scalars. For fermions $\mathcal{A}^{(4)}$ has spin singlet and spin triplet parts, and the expressions in Eq. (29) are valid for the spin singlet component.}
Therefore, using the Galilean boost invariance gives three constraints on $\mathcal{A}^{(4)}$ leaving 9 variables. For this function the scale and conformal identities are

$$\mathcal{G}_1 = 1 + p_A \cdot \nabla p_A + p_B \cdot \nabla p_B + 2 \sum_{j=1}^{3} U_j \frac{\partial}{\partial U_j},$$

$$\mathcal{G}_2 = M \left( -\nabla p_A \cdot \nabla p_B + \sum_{j=1}^{3} \eta_j \frac{\partial^2}{\partial U_j^2} \right).$$

Three more constraints are given by rotation invariance leaving a function of 6 variables, $\mathcal{A}^{(4)}(x, y, \gamma, U_1, U_2, U_3)$, where

$$x = p_A^2, \quad y = p_B^2, \quad \gamma = p_A \cdot p_B.$$  \hspace{1cm} (28)

In terms of these variables we have

$$\mathcal{G}_1 = 1 + 2x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} + 2\gamma \frac{\partial}{\partial \gamma} + 2 \sum_{j=1}^{3} U_j \frac{\partial}{\partial U_j},$$

$$\mathcal{G}_2 = -M \left( \frac{\partial}{\partial \gamma} \mathcal{G}_1 + 4\gamma \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 2 \sum_{j=1}^{3} U_j \frac{\partial^2}{\partial \gamma \partial U_j} - \sum_{j=1}^{3} \eta_j \frac{\partial^2}{\partial U_j^2} \right).$$ \hspace{1cm} (29)

On-shell the four point function has an additional four constraints $U_1 = U_2 = U_3 = 0$ and $\gamma = 0$, where the last condition follows because $U_i = U_1 + U_2 - U_3 - \gamma = 0$. The operator $\mathcal{G}_1$ can be defined consistently on-shell since all derivatives with respect to $U_1, U_2, U_3$ and $\gamma$ are multiplied by coefficients which vanish in the on-shell limit. In taking the on-shell limit we are assuming that derivatives of $\mathcal{A}^{(4)}$ with respect to the off-shell parameters are not singular. This is true of the explicit example in Eq. (23) as long as the momentum of the nucleons in the center of mass frame is nonzero. Finally, from Eq. (29) we see that on-shell a scale invariant $\mathcal{A}^{(4)}$ is automatically conformally invariant.

Solving $\mathcal{G}_1, \mathcal{A}^{(4)} = 0$, the most general scattering amplitude consistent with Schrödinger group invariance is

$$A^{(4)} = \frac{1}{\sqrt{x + y}} F \left( \frac{y - x}{y + x} \right) = \frac{1}{2p} F(\cos \theta),$$ \hspace{1cm} (30)

where $F$ is an arbitrary function, and $\theta$ is the scattering angle in the center of mass frame. Conformal invariance does not restrict the angular dependence of the scattering amplitude. Additional physical criteria can be used to provide further constraints. The condition that the S-wave scattering length goes to infinity corresponds to a fine tuning that produces a bound state at threshold. Assuming that this is the only fine tuning and that the interactions are short range the threshold behavior of the phase shift in the $\ell$th partial wave is $\delta_{\ell} \sim p^{2\ell+1} \text{ for } p > 0$. It is easy to see that the only partial wave obtained from Eq. (30) with acceptable threshold behavior is the S-wave, so $F$ can be replaced by a constant. In the limit $a \rightarrow \infty$ the interaction in Eq. (19) provides an explicit example of a scale invariant theory which has this behavior.
In the case of the 3-to-3 scattering amplitude, conformal invariance will provide a new constraint independent from that of scale invariance. We proceed exactly as in the case of the 2-to-2 scattering amplitude. After imposing energy and momentum conservation the 6 point function has 20 coordinates

\[ \mathscr{A}^{(6)}(p_1, \ldots, p_6, E_1, \ldots, E_6). \]  

(31)

Using the Galilean boost invariance leaves 17 coordinates

\[ \mathscr{A}^{(6)}(p, k, p', k', U_1, \ldots, U_6), \]  

(32)

where \( U_i = ME_i - p_i^2/2 \) and

\[ p = \frac{2p_3 - p_2 - p_1}{3}, \quad k = p_2 - p_1, \]

\[ p' = \frac{2(p_1 + p_2 + p_3)}{3} - p_4 - p_5, \quad k' = p_3 - p_4. \]  

(33)

In terms of these variables

\[ \tilde{\mathcal{G}} = 4 + p \cdot \nabla + k \cdot \nabla + p' \cdot \nabla + k' \cdot \nabla \]

\[ + 2 \sum_{j=1}^{5} U_j \frac{\partial}{\partial U_j}, \]

\[ \tilde{\mathcal{G}} = \frac{M}{3} \left[ \nabla^2 + 3 \mathbf{r}^2 - \mathbf{r}_p^2 - 3 \mathbf{r}_{p'}^2 \right] + M \sum_{j=1}^{5} \eta_j \mathbf{U}_j \frac{\partial^2}{\partial \mathbf{U}_j^2}. \]  

(34)

Next consider imposing rotational invariance. For simplicity we specialize to the case of a scalar field. Rotational invariance implies that \( \mathscr{A}^{(6)} \) should be a function of 14 variables. We have chosen

\[ \mathscr{A}^{(6)}(z_1, \ldots, z_8, \gamma, U_1, \ldots, U_5), \]  

(35)

where

\[ z_1 = p^2, \quad z_2 = k^2, \quad z_3 = p^{2'}, \quad z_4 = p \cdot k, \quad z_5 = p \cdot p', \quad z_6 = p \cdot k', \quad z_7 = k \cdot p', \quad z_8 = p' \cdot k', \]

\[ \gamma = k^2 - k^{2'} + 3p^2 - 3p'^2. \]  

(36)

The coordinates \( U_j \) and \( \gamma \) vanish on-shell since \( U_0 = \Sigma_{j=1}^{5} \eta_j U_j + \gamma/4 \). For the function in Eq. (35) the scale and conformal derivatives are

\[ \tilde{\mathcal{G}} = 4 + 2 \sum_{j=1}^{8} z_j \frac{\partial}{\partial z_j} + \ldots, \quad \tilde{\mathcal{G}} = 2M \left( \frac{\partial}{\partial z_1} + 3 \frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_3} \right) + \frac{M}{3} \sum_{j,k=1}^{8} A_{jk} \frac{\partial^2}{\partial z_j \partial z_k} + 4M \frac{\partial}{\partial \gamma} \tilde{\mathcal{G}} + \ldots \]  

(37)

The ellipses are terms with factors of \( U_j \) or \( \gamma \) and therefore vanish on-shell,

\[ A_{jk} = \begin{pmatrix}
4z_1 & 0 & 0 & 2z_2 & 2z_3 & 2z_4 & 0 & 0 \\
0 & 12z_2 & 0 & 6z_3 & 0 & 0 & 6z_7 & 0 \\
0 & 0 & -4z_3 & 0 & -2z_4 & 0 & -2z_7 & -2z_8 \\
2z_4 & 6z_2 & 0 & 3z_1 + z_2 & z_7 & z_9 & 3z_5 & 0 \\
2z_5 & 0 & -2z_5 & z_7 & -z_1 + z_3 & z_8 & -z_4 & -z_6 \\
2z_6 & 0 & 0 & z_9 & z_8 & z_2 - 3z_3 & 0 & -3z_5 \\
0 & 6z_2 & -2z_7 & 3z_5 & -z_4 & 0 & -z_5 + 3z_3 & -z_9 \\
0 & 0 & -2z_8 & 0 & -z_6 & -3z_5 & -z_9 & -3z_1 - z_2
\end{pmatrix}. \]  

(38)
and \( z_0 = \mathbf{k} \cdot \mathbf{k}' \). It is possible to express \( z_0 \) in terms of \( z_1, \ldots, z_s \). For scale invariant amputated Green's functions the conformal operator can be defined on-shell because terms that involve derivatives with respect to the off-shell parameters \((U, \gamma)\) have coefficients which vanish on-shell.

Even after demanding scale invariance the conformal Ward identity still imposes a nontrivial constraint on the amplitude. It is easy to find examples of boost and scale invariant functions which do not satisfy \( \mathcal{G}, \mathcal{N}(\mathcal{N}) = 0 \). Due to the complexity of Eq. (37) we have not attempted to find its general solution.

The effective field theory for the strong interactions of nucleons is more complicated than the toy model given by \( S_0 + S_\gamma \), because nucleons have isospin degrees of freedom. The inclusion of internal degrees of freedom does not change the Ward identities that correlations must satisfy to be Schrödinger invariant. However, isospin allows additional contact interactions to exist. There are two four-nucleon operators (\( ^1S_0 \) and \( ^3S_1 \)) and one six-nucleon operator that can be formed without using derivatives. With infinite spin singlet and spin triplet \( NN \) scattering lengths the four point functions are identical to Eq. (23) at leading order, and are therefore invariant under the Schrödinger group. For nucleons, the six point point functions can involve states with total spin 1/2 and 3/2. In the spin 1/2 channel a three body contact interaction with no derivatives exists and is needed to renormalize the integral equation for three body scattering [8]. This three body contact operator is expected to introduce a new scale and therefore break scale and conformal invariance. In the spin 3/2 channel [9], no three body operator is needed and this amplitude is expected to respect the constraints from scale and conformal invariance. Explicit verification of this would be interesting.

In this letter we derived Ward identities for amputated momentum space Green’s functions that follow from invariance under the Schrödinger group. We also examined implications of these constraints for 2-to-2 and 3-to-3 on-shell scattering amplitudes. Motivated by recent developments in nuclear theory, we considered a non-relativistic theory in the limit of infinite scattering length and found it gives rise to a four point function which satisfies the Ward identities which follow from Schrödinger invariance.

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Neutrino oscillations with three flavors in matter: Applications to neutrinos traversing the Earth

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Abstract

Analytic formulas are presented for three flavor neutrino oscillations in matter in the plane wave approximation. We calculate in particular the time evolution operator in both mass and flavor bases. We also find the transition probabilities expressed as functions of the vacuum mass squared differences, the vacuum mixing angles, and the matter density parameter. The application of this to neutrino oscillations for both atmospheric and long baseline neutrinos in a mantle-core-mantle step function model of the Earth’s matter density profile is discussed. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

As an accumulating amount of data on neutrino oscillations is becoming accessible, it is of interest to study three flavor neutrino oscillations. Here we would like to give analytic expressions for the neutrino oscillation probabilities in presence of matter expressed in the mixing matrix elements and the neutrino energies or masses, i.e., incorporating the so called Mikheyev–Smirnov–Wolfenstein (MSW) effect [1,2]. These probability formulas are of interest for the solar neutrino problem, the atmospheric neutrino problem, and the long baseline (LBL) neutrino experiments. We here apply the formalism to neutrinos traversing the Earth in a mantle-core-mantle step function model of the Earth’s matter density profile.

We will assume that CP nonconservation is negligible at the present level of experimental accuracy [3]. Thus, the mixing matrix for the neutrinos is real.

Previous work on models for three flavor neutrino oscillations in matter includes works of Barger et al. [4], Kim and Sze [5], and Zaglauer and Schwarzer [6]. Our method is different from all these approaches and their...
parameterizations also differ slightly from ours. In particular, we calculate the time evolution operator and do not use the auxiliary matter mixing angles.

Approximate solutions for three flavor neutrino oscillations in matter have been presented by Kuo and Pantaleone [7] and Joshipura and Murthy [8]. Approximate treatments have also been done by Toshev and Petcov [9]. D’Olivo and Oteo have made contributions by using an approximative Magnus expansion for the time evolution operator [10]. Extensive numerical investigations for matter enhanced three flavor oscillations have been made by Fogli et al. [11]. Studies of neutrino oscillations in the Earth has been performed by several authors [12–18].

2. The evolution operator

Let the flavor state basis and mass eigenstate basis be given by $\mathcal{H}_f = \{ |\nu_\alpha \rangle \}_{\alpha = e, \mu, \tau}$ and $\mathcal{H}_m = \{ |\nu_a \rangle \}_{a = 1,2,3}$ respectively. Then the flavor states $|\nu_\alpha \rangle \in \mathcal{H}_f$ can be obtained as a superpositions of the mass eigenstates $|\nu_a \rangle \in \mathcal{H}_m$, or vice versa. The bases $\mathcal{H}_f$ and $\mathcal{H}_m$ are of course just two different representations of the same Hilbert space $\mathcal{H}$.

In the present analysis, we will use the plane wave approximation to describe neutrino oscillations. In this approximation, a neutrino flavor state $|\nu_\alpha \rangle$ is a linear combination of neutrino mass eigenstates $|\nu_a \rangle$ such that

$$|\nu_\alpha \rangle = \sum_{a=1}^{3} U_{\alpha a}^* |\nu_a \rangle,$$

(1)

where $\alpha = e, \mu, \tau$, and $a = 1,2,3$. In what follows, we will use the short-hand notations $|\alpha \rangle \equiv |\nu_\alpha \rangle$ and $|a\rangle \equiv |\nu_a \rangle$ for the flavor states and the mass eigenstates, respectively.

For the components of a state $\psi$ in the flavor state basis and mass eigenstate basis, respectively, they are related to each other by

$$\psi_f = U \psi_m,$$

(2)

where

$$\psi_f = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix} \in \mathcal{H}_f \quad \text{and} \quad \psi_m = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \in \mathcal{H}_m.$$ 

A convenient parameterization for $U = U(\theta_1, \theta_2, \theta_3)$ is given by [20]

$$U = \begin{pmatrix} C_2 C_3 & S_1 C_2 & S_2 \\ -S_1 C_1 - S_2 S_3 & C_1 C_3 - S_1 S_2 S_3 & S_1 C_2 \\ S_1 S_3 - S_2 C_1 C_3 & -S_1 C_3 - S_2 S_3 C_1 & C_1 C_2 \end{pmatrix},$$

(3)

where $S_i \equiv \sin \theta_i$ and $C_i \equiv \cos \theta_i$ for $i = 1,2,3$. This is the standard representation of the mixing matrix. The quantities $\theta_i$, where $i = 1,2,3$, are the vacuum mixing angles. Since we have put the $CP$ phase equal to zero in the mixing matrix, this means that $U_{\alpha a} = U_{a \alpha}$ for $\alpha = e, \mu, \tau$ and $a = 1,2,3$.

In the mass eigenstate basis, the Hamiltonian $H$ for the propagation of the neutrinos in vacuum is diagonal and given by

$$H_m = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix},$$

(4)

where $E_a = \sqrt{m_a^2 + p^2}$, $a = 1,2,3$, are the energies of the neutrino mass eigenstates $|a\rangle$, $a = 1,2,3$ with masses $m_a$, $a = 1,2,3$. We will assume $p$ to be the same for all mass eigenstates.
When neutrinos propagate in matter, there is an additional term coming from the presence of electrons in matter [2]. This term is diagonal in the flavor state basis and is given by

\[ V_f = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

where

\[ A = \pm \sqrt{2} G_F N_e = \pm \frac{1}{\sqrt{2}} G_F \frac{1}{m_N} \rho \]

is the matter density parameter. Here \( G_F \) is the Fermi weak coupling constant, \( N_e \) is the electron density, \( m_N \) is the nucleon mass, and \( \rho \) is the matter density. The sign depends on whether we deal with neutrinos (+) or antineutrinos (−). We will assume that the electron density \( N_e \) (or the matter density \( \rho \)) is constant throughout the matter in which the neutrinos are propagating. In the mass basis, this piece of the Hamiltonian is \( V_m = U^{-1} V_f U \), where again \( U \) is the mixing matrix.

The unitary transformation that leads from the initial state \( \psi_i(0) \) in flavor basis at time \( t = 0 \) of production of the neutrino, to the state of the same neutrino \( \psi_f(t) \) at the detector at time \( t \) is given by the operator \( U_f(t) = U_f(t,0) \), where \( U_f(t_2,t_1) \) is the time evolution operator from time \( t_1 \) to time \( t_2 \) in flavor basis. This operator can be formally written as \( U_f(t) = e^{-i H_f t} \). When the neutrinos are propagating through vacuum, the Hamiltonian in flavor basis is \( H_f = U H_m U^{-1} \). In this case, the exponentiation of \( H_f \) can be performed easily: \( U_f(t) = e^{-i H_f t} = U e^{-i H_m U^{-1} t} \), and the result can be expressed in closed form. In the case when the neutrinos propagate through matter, the Hamiltonian is not diagonal in either the mass eigenstate basis or the flavor state basis, and we have to calculate the evolution operator \( U_f(t) \) or \( U_f(L) \equiv e^{-i \mathcal{H}_m L} = U e^{-i \mathcal{H}_m L} U^{-1} \) if we set \( t = L \).

To do so it is convenient to introduce the traceless matrix \( T \) defined by \( T = \mathcal{H}_m - (\text{tr} \mathcal{H}_m)I/3 \). The trace of the Hamiltonian in the mass basis \( \mathcal{H}_m = H_m + U^{-1} V_f U \) is \( \text{tr} \mathcal{H}_m = E_1 + E_2 + E_3 + A \), and the matrix \( T \) can then be written as

\[ T = \left( T_{ab} \right) = \begin{pmatrix} A U_{e1}^2 - \frac{1}{3} A + \frac{1}{3} (E_{12} + E_{13}) & A U_{e1} U_{e2} & A U_{e1} U_{e3} \\ A U_{e1} U_{e2} & A U_{e2}^2 - \frac{1}{3} A + \frac{1}{3} (E_{21} + E_{23}) & A U_{e2} U_{e3} \\ A U_{e1} U_{e3} & A U_{e2} U_{e3} & A U_{e3}^2 - \frac{1}{3} A + \frac{1}{3} (E_{31} + E_{32}) \end{pmatrix}, \]

where \( E_{ab} \equiv E_a - E_b \). Of the six quantities \( E_{ab} \), where \( a, b = 1,2,3 \) and \( a \neq b \), only two are linearly independent, since the \( E_{ab} \)'s fulfill the relations \( E_{ba} = -E_{ab} \) and \( E_{12} + E_{23} + E_{31} = 0 \). Using Eq. (6), the evolution operator in the mass basis can be written as

\[ U_m(L) = e^{-i \mathcal{H}_m L} = \phi e^{-i L T}, \]

where \( \phi = e^{-i \frac{1}{3} \text{tr} \mathcal{H}_m L} \).

The flavor states can be expressed either as linear combinations of the vacuum mass eigenstates \( A = 0 \) in the basis \( \mathcal{H}_m \), as in Eq. (2) or the matter mass eigenstates \( A \neq 0 \) in the basis \( \mathcal{H}_m' \). In the latter case, the corresponding components are related to each other as

\[ \psi_f = U^M \phi_M, \]

where \( U^M = U^M(\theta_1^M, \theta_2^M, \theta_3^M) \) is the unitary mixing matrix for matter and \( \theta_i^M, i = 1,2,3 \), are the (auxiliary) matter mixing angles.

\[ ^1 \text{Later, we will use the usual (vacuum) mass squared differences } \Delta m_{12}^2 \text{ and } \Delta m_{13}^2 \text{ instead of } E_{21} \text{ and } E_{32}, \text{ which are related to each other by } \Delta m_{21}^2 = 2E_1 E_{23} \text{ and } \Delta m_{31}^2 = 2E_2 E_{32}, \text{ where } E_n \text{ is the neutrino energy.} \]
Combining these expressions for the flavor state components, one obtains the following relation between the two different sets of mass eigenstate components

$$\psi_M = R \psi_m,$$  \hspace{1cm} (9)

where

$$R \equiv (U^M)^{-1} U.$$

The matrix $R$ is, of course, a unitary matrix (even orthogonal, since $U$ and $U^M$ are real). This means that the matter mixing matrix can be expressed in the vacuum mixing matrix as

$$U^M (\theta_1^M, \theta_2^M, \theta_3^M) = U(\theta_1, \theta_2, \theta_3) R^{-1}. \hspace{1cm} (10)$$

The relations between the different bases are depicted in the following diagram:

$$\begin{align*}
\psi_m & \in \mathcal{H}_m \quad U \quad \mathcal{H}_f \ni \psi_f \\
R & \\
\psi_M & \in \mathcal{H}_M \quad U^M
\end{align*}$$

From this diagram one readily obtains

$$\mathcal{H}_M \equiv R \mathcal{H}_m R^{-1} \quad \text{or} \quad \mathcal{H}_M \equiv RTR^{-1} + \frac{1}{3} (\text{tr} \mathcal{H}_m) I,$$

where $\mathcal{H}_M$ is the Hamiltonian in matter, which is diagonal in the basis $\mathcal{H}_M$.

Due to the invariance of the trace, we have

$$T_M \equiv \mathcal{H}_M - \frac{1}{3} (\text{tr} \mathcal{H}_m) I = RTR^{-1}. \hspace{1cm} (11)$$

and $T_M$ is a diagonal matrix with elements $\lambda_a$, $a = 1, 2, 3$, the eigenvalues of $T$. This implies that

$$e^{-iLT_m} = Re^{-iLT} R^{-1}. \hspace{1cm} (12)$$

Now, Cayley–Hamilton’s theorem implies that, since $T$ is a $3 \times 3$ matrix, the infinite series defining $e^{-iLT}$ can be written as a second order polynomial in $T$ [21]:

$$e^{-iLT} = a_0 I - iLT a_1 - L^2 T^2 a_2, \hspace{1cm} (13)$$

where $a_0$, $a_1$, and $a_2$ are coefficients to be determined. Inserting Eq. (13) into Eq. (12) and using Eq. (11) gives a linear system of three equations that will determine the coefficients $a_0$, $a_1$, and $a_2$:

$$\begin{align*}
e^{-iLA_1} & = a_0 - iLA_1 a_1 - L^2 a_1^2 a_2 \\
e^{-iLA_2} & = a_0 - iLA_2 a_1 - L^2 a_2^2 a_2 \\
e^{-iLA_3} & = a_0 - iLA_3 a_1 - L^2 a_3^2 a_2
\end{align*} \hspace{1cm} (14)$$

where $\lambda_a$, $a = 1, 2, 3$, are the diagonal elements of $T_M$, or, equivalently, the eigenvalues of $T$, i.e., the solutions to the characteristic equation

$$\lambda^3 + c_2 \lambda^2 + c_1 \lambda + c_0 = 0. \hspace{1cm} (15)$$
with

\[ c_2 = -\text{tr} T = 0, \quad (16) \]
\[ c_1 = \det T \text{tr} T^{-1}, \quad (17) \]
\[ c_0 = -\det T. \quad (18) \]

The coefficients \( c_0, c_1, \) and \( c_2 \) are all real and the eigenvalues \( \lambda_a, a = 1,2,3, \) can be expressed in closed form in terms of these \([21]\).

When the system of Eq. (14) is solved for the \( a_i \)'s, we obtain

\[ e^{-iLT} = \sum_{a=1}^{3} e^{-iL \lambda_a} \frac{1}{3\lambda_a^2 + c_1} \left[ (\lambda_a^2 + c_1) I + \lambda_a T + T^2 \right]. \quad (19) \]

From Eqs. (7) and (19) we then obtain

\[ U_a(L) = e^{-i\mathcal{W}_a L} = \phi e^{-iLT} = \phi \sum_{a=1}^{3} e^{-iL \lambda_a} \frac{1}{3\lambda_a^2 + c_1} \left[ (\lambda_a^2 + c_1) I + \lambda_a T + T^2 \right]. \quad (20) \]

The evolution operator for the neutrinos in flavor basis is thus given by

\[ U_f(L) = e^{-i\mathcal{W}_f L} = U e^{-i\mathcal{W}_w L} U^{-1} = \phi \sum_{a=1}^{3} e^{-iL \lambda_a} \frac{1}{3\lambda_a^2 + c_1} \left[ (\lambda_a^2 + c_1) I + \lambda_a \tilde{T} + \tilde{T}^2 \right], \quad (21) \]

where \( \tilde{T} = U T U^{-1} \). Formula (21) is the final expression for the evolution operator. It expresses the time (or \( L \)) evolution directly in term of the mass squared differences and the vacuum mixing angles without introducing the auxiliary matter mixing angles.

3. Transition probabilities

The probability amplitude is defined as

\[ A_{\alpha\beta} \equiv \langle \beta | U_f(L) | \alpha \rangle, \quad \alpha, \beta = e, \mu, \tau. \quad (22) \]

Inserting Eq. (21) into Eq. (22) gives

\[ A_{\alpha\beta} = \phi \sum_{a=1}^{3} e^{-iL \lambda_a} \left( \lambda_a^2 + c_1 \right) \delta_{\alpha\beta} + \lambda_a \tilde{T}_{\alpha\beta} + (\tilde{T}^2)_{\alpha\beta}, \quad (23) \]

where \( \delta_{\alpha\beta} \) is Kronecker's delta. Note that \( \tilde{T}_{\alpha\beta} = \tilde{T}_{\beta\alpha} \) and \((\tilde{T}^2)_{\alpha\beta} = (\tilde{T}^2)_{\beta\alpha}\).

Inserting \( L = 0 \) into Eq. (21) yields

\[ \delta_{\alpha\beta} = \sum_{a=1}^{3} \left( \lambda_a^2 + c_1 \right) \delta_{\alpha\beta} + \lambda_a \tilde{T}_{\alpha\beta} + (\tilde{T}^2)_{\alpha\beta}, \quad (24) \]
Hence, the transition probabilities in matter are

$$
P_{\alpha\beta} = |A_{\alpha\beta}|^2 = \delta_{\alpha\beta} - 4 \sum_{a=1}^{3} \sum_{b=1}^{3} \left( \left( \lambda_b^2 + c_1 \right) \delta_{\alpha\beta} + \lambda_a \overline{T}_{a\beta} + (\overline{T}^2)_{a\beta} \right) \frac{3\lambda_a^2 + c_1}{3\lambda_b^2 + c_1} \sin^2 x_{ab}, \quad \alpha, \beta = e, \mu, \tau,
$$

(25)

where $x_{ab} \equiv (\lambda_a - \lambda_b)L/2$.

From unitarity, there are only three independent transition probabilities, since the other three can be obtained from them, i.e., from the equations

$$
P_{ee} + P_{e\mu} + P_{e\tau} = 1, \quad (26)
$$

$$
P_{\mu e} + P_{\mu\mu} + P_{\mu\tau} = 1, \quad (27)
$$

$$
P_{e\tau} + P_{\mu\tau} + P_{ee} = 1. \quad (28)
$$

Note that $P_{ee} = P_{\mu e}, \quad P_{e\tau} = P_{\tau e}$, and $P_{\mu \tau} = P_{\tau \mu}$. If we choose $P_{e\mu}, \quad P_{e\tau}, \quad$ and $P_{\mu \tau}$ as the three independent ones, we thus have for $\alpha \neq \beta$

$$
P_{\alpha\beta} = -4 \Im \text{op} \sum_{a=1}^{3} \sum_{b=1}^{3} \frac{\lambda_a \overline{T}_{a\beta} + (\overline{T}^2)_{a\beta} \lambda_b \overline{T}_{b\alpha} + (\overline{T}^2)_{b\alpha}}{3\lambda_a^2 + c_1} \frac{3\lambda_b^2 + c_1}{3\lambda_a^2 + c_1} \sin^2 x_{ab}.
$$

(29)

4. Applications and discussion

The main results of our analysis is given by the time evolution operator for the neutrinos when passing through matter with constant matter density in Eq. (21) and the expression for the transition probabilities in Eqs. (25) and (29), expressed as finite sums of elementary functions in the elements of $\overline{T}$.

In our treatment the auxiliary mixing angles in matter, $\theta^m$, play no independent role and are not really needed.

When the neutrinos travel through a series of matter densities with matter density parameters $A_1, \ldots, A_n$ and thicknesses $L_1, \ldots, L_n$, the total evolution operator is simply given by

$$
U(L) = \prod_{i=1}^{n} U_j(L_i) = U_j(L_n) \ldots U_j(L_1),
$$

(30)

where $L = \sum_{i=1}^{n} L_i$, and $U_j(L_i)$ is calculated for $A = A_i$. Eq. (30) gives a simple and fast algorithm to obtain the total evolution operator instead of using numerical integration.

As an application to show the usefulness of our derived formulas, we have calculated the transition probabilities $P_{\alpha\beta}$ for neutrino oscillations in a mantle-core-mantle step function model simulating the Earth’s matter density profile. Let $R = 6371$ km be the radius of the Earth and $r = 3486$ km be the radius of the core.

The thickness of the mantle is then $R - r = 2885$ km, with matter density parameter $A_1 = 1.70 \cdot 10^{-13}$ eV ($\rho_1 = 4.5$ g/cm$^3$), whereas the matter density parameter of the core is $A_2 = 4.35 \cdot 10^{-13}$ eV ($\rho_2 = 11.5$ g/cm$^3$).

Neutrinos traversing the Earth towards a detector close to the surface of the Earth, pass through the matter densities $A_1, A_2, A_1$ of thicknesses $L_1, L_2, L_3$ where the distances $L_i, i = 1, 2, 3$ are functions of the nadir angle $h$, where $h = 180^\circ - \theta$; $\theta$ being the zenith angle. As $h$ varies from 0 to 90$^\circ$, the chord $L = L(h)$ of the neutrino passage through the Earth becomes shorter and shorter. At an angle larger than $h = \arcsin(r/R) = 33.17^\circ$, the distance $L_2 = 0$, and the neutrinos no longer traverse the core.
For \( 0 \leq h \leq h_0 \) the distances \( L_1 \) and \( L_2 \) are given by

\[
L_1 = R \left( \cos h - \sqrt{\left( \frac{r}{R} \right)^2 - \sin^2 h} \right),
\]

\[
L_2 = 2 R \sqrt{\left( \frac{r}{R} \right)^2 - \sin^2 h},
\]

\[
L = 2 L_1 + L_2.
\]

For \( h_0 \leq h \leq 90^\circ \):

\[
L = 2 R \cos h.
\]

The mass squared differences (\( \Delta M^2 = \Delta m^2_{12} \) and \( \Delta m^2 = \Delta m^2_{21} \)) and the vacuum mixing angles (\( \theta_1, \theta_2, \theta_3 \)) used here are chosen to correspond to those obtained from analyses of various neutrino oscillation data. For our illustration we have taken the following parameter values

\[
\Delta M^2 = 3.2 \times 10^{-3} \text{ eV}^2, \quad \Delta m^2 = 0, \quad \Delta M^2 / 10, \quad \theta_1 = 45^\circ, \quad \theta_2 = 5^\circ, \quad \theta_3 = 45^\circ.
\]

The values of \( \Delta M^2 \) and \( \theta_1 \) are governed by atmospheric neutrino data [22] and the values of \( \Delta m^2 \) and \( \theta_3 \) by solar neutrino data [23]. The value of \( \theta_2 \) is below the CHOOZ upper bound, which is \( \sin^2 2 \theta_2 = 0.10 \) [24]. These choices are the most optimistic ones for obtaining any effects in LBL experiments from the sub-leading \( \Delta m^2 \) scale [25]. We should mention though, that these data are taken from two flavor model analyses.

The probabilities corresponding to various situations in this scenario are illustrated in Figs. 1–4. These figures correspond to the physically measurable quantities. From the figures one can see that there are several resonance phenomena superimposed on each other.

Fig. 1a–1d show the results for the case of \( h = 0 \), one mass squared difference equal to \( 3.2 \times 10^{-3} \text{ eV}^2 \) [22], and the other one equal to 0. In Fig. 1a we have included for comparison the corresponding result of a two flavor model. Fig. 1b–1d show the probabilities \( P_{ee}, P_{e \mu}, \) and \( P_{\mu \tau} \), respectively, in the three flavor model. The other probabilities can be obtained by unitarity from these three using Eq. (26)–(28). We observe that although the survival probability \( P_{ee} \) is the same for both cases, the transition probability \( P_{e \mu} \) is only half of

---

**Fig. 1.** Transition probabilities as functions of the neutrino energy \( E_\nu \) for the parameter values \( h = 0, \theta_1 = \theta_3 = 45^\circ \) (bimaximal mixing), \( \theta_2 = 5^\circ, \Delta M^2 = 3.2 \times 10^{-3} \text{ eV}^2, \) and \( \Delta m^2 = 0 \). a) \( P_{ee} \) (two flavors; parameters values: \( \theta = 5^\circ \) and \( \Delta m^2 = 3.2 \times 10^{-3} \text{ eV}^2 \)), b) \( P_{ee} \) (three flavors), c) \( P_{e \mu} \) (three flavors), and d) \( P_{\mu \tau} \) (three flavors).
that in the two flavor model. In a two flavor model, only one transition probability is needed, since if $P_{ee}$ is given and $P_{\mu e} = P_{e\mu}$, the other two follow from $P_{e\mu} = 1 - P_{ee}$ and $P_{\mu \mu} = 1 - P_{e\mu} = P_{ee}$.

In Fig. 2a–d) the second mass squared difference is chosen to be $1/10$ of the first one. This affects the disappearance rate for low energies in the three flavor case, as can be seen from Fig. 2b). It also modifies the appearance rates $P_{\mu e}$ and $P_{e\mu}$ in Fig. 2c) and 2d), respectively. Fig. 2a) is again the same as Fig. 1a).

When the small mass squared difference is much smaller than the large mass squared difference, then the two flavor model coincides with the three flavor model. This is not the case when the small mass squared difference is of comparable size to the large mass squared difference. See Fig. 1a), 1b), 2a), and 2b) for a comparison.

Fig. 3 shows a contour plot of $P_{ee}$ as a function of the matter density parameter of the core $A_{core}$ (as it changes from the value of the mantle $A_1$ up to its full value $A_2$) and the neutrino energy $E_\nu$. The dark areas

![Figure 3](image URL)

**Fig. 3.** The survival probability $P_{ee}$ as a function of the core matter density parameter $A_{core} \in [A_1, A_2]$ and the neutrino energy $E_\nu$, using the same parameter values as for Fig. 2.
correspond to regions of large conversion probability for the assumed parameter values. When the small mass squared difference decreases, the structure below the main conversion valley goes away. Furthermore, the conversion regions above the main conversion valley are obviously due to the core (either purely to the core or to interference effects between the core and the mantle).

Similarly, Fig. 4 shows a contour plot of $P_{\nu_e}$ as a function of the nadir angle $h$ and the neutrino energy $E_\nu$. As in the previous figure, the dark areas correspond to regions of large conversion probability for the assumed parameter values. At $h = h_0$, the neutrinos no longer traverse the core, but only the mantle. Again, when the small mass squared difference drops, the oscillations in the bottom of the plot (sub-GeV region) go away and no conversion takes place, i.e., $P_{\nu_e} = 1$ in this region. In the limit when the nadir angle goes to $90^\circ$, the survival probability $P_{\nu_e}$ becomes 1, because the traveling path length $L$ of the neutrinos approaches 0.

The results and the applications presented here are relevant to both the atmospheric neutrino experiments (e.g. Super-Kamiokande) and the LBL neutrino experiments. The present treatment assumes that in the results of the atmospheric and LBL experiments, the scattering and absorption of the neutrinos in matter is taken care of in the Monte Carlo programs that are used to analyze the data from these experiments. In a realistic analysis, we should also introduce damping due to uncertainties in the energy $E_\nu$ of the neutrinos and their path length $L$.

Examining some of the future LBL neutrino experiments, we find their nadir angles to be $h = 86.7^\circ$, $h = 86.7^\circ$, and $h = 88.9^\circ$ for CERN-NGS ($L = 743$ km), MINOS ($L = 732$ km), and K2K ($L = 252$ km), respectively [25,26]. See Fig. 4 for how the survival probability $P_{\nu_e}$ depends on the neutrino energy for these fixed nadir angles. Note that for these three experiments the neutrinos only traverse the mantle and not the core.

There are, at the moment, no planned LBL experiments in which the neutrinos also traverse the core. The baseline has in that case to be longer than $L = 10670$ km.

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References

Deeply virtual Compton scattering in next-to-leading order

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Abstract

We study the amplitude of deeply virtual Compton scattering in next-to-leading order of perturbation theory including the two-loop evolution effects for different sets of skewed parton distributions (SPDs). It turns out that in the minimal subtraction scheme the relative radiative corrections are typically of order 20–50%. In special cases they can even be much larger. We analyze the dependence of our predictions on the choice of SPD, that will allow to discriminate between possible models of SPDs from future high precision experimental data, and discuss shortly theoretical uncertainties induced by radiative corrections. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Virtual Compton scattering (VCS) in a light cone dominated region, usually referred to as the deeply virtual Compton scattering (DVCS) [1–3], is a favourable process to get access to the so-called skewed parton distributions (SPDs) [1–6]. The latter can be viewed as a generalization of the conventional parton densities as measured in deep inelastic scattering (DIS). They contain complementary information about the internal structure of hadrons, e.g. the total angular momentum carried by quarks and gluons [2].

The study of SPDs promises to become one of the main issues of hadronic physics for the next decade. This marks general trends in the development of experimental techniques towards exclusive reactions at the facilities like CERN and DESY. In general, it will be a delicate task to measure DVCS, since it is contaminated by the Bethe–Heitler (BH) process. However, it is rather encouraging that a first DVCS signal has already been detected by the ZEUS collaboration [7]. In addition, azimuthal, spin and charge asymmetries [8] allow to get access to the interference term between DVCS and BH processes and this should allow to separate real and imaginary parts of the DVCS amplitude. Such measurements would pin down plausible shapes of SPDs.

The theoretical goal is, therefore, to push our understanding of SPDs to the level reached nowadays for the usual parton distribution functions. One of the most important questions is the $Q^2$-evolution of SPDs. In earlier papers we gave a complete solution at the next-to-leading order (NLO) level [9].
In the present paper we study the rôle of the NLO radiative corrections for the DVCS amplitude quantitatively in the region \( x_{Bj} > 10^{-2} \) including the effects of scaling violations. Our aim is to decide whether an extension to NNLO is necessary to obtain a stable interpretation of a given set of DVCS data. For a first rough comparison with experimental measurements NLO is probably fine but to judge the potential of SPDs measurement the precise size of the corrections as well as the uncertainties involved have to be known.

In Section 2 we set up our definitions of the DVCS amplitude and point out the underlying conformal structure of the hard-scattering amplitude. The NLO corrections to these amplitudes are presented in a simple form. To study the model dependence of the predictions, we introduce in Section 3 two different models for the SPD. We present then our numerical results for the real as well as the imaginary part of DVCS amplitude. The theoretical uncertainties induced by radiative corrections are shortly discussed. Finally, we give our conclusions and an outlook.

2. Perturbative QCD results from conformal symmetry

The VCS amplitude is defined in terms of the time ordered product of two electromagnetic currents

\[
T_{\mu\nu}(P, q_1, q_2) = i\int dx e^{i q \cdot x} \langle P_2, S_2 | T[j_\mu(x/2) j_\nu(-x/2)] | P_1, S_1 \rangle.
\]

Here \( P_1 \) and \( P_2 \) are the momenta of the initial and the final hadrons, respectively. The incoming photon with momentum \( q_1 \) has a large virtuality. The scaling variables, \( \xi \) and \( \eta \), which allow to describe different 'two-photon' processes in the light-cone dominated region are introduced as follows [1]:

\[
\xi = \frac{Q^2}{P q}, \quad \eta = \frac{\Delta q}{P q}, \quad \text{where}
\]

\[
Q^2 = -q^2 = -\frac{1}{4}(q_1 + q_2)^2, \quad P = P_1 + P_2, \quad \Delta = P_1 - P_2 = q_2 - q_1.
\]

In DVCS kinematics \(-q_1^2 \gg m_{\text{hadron}}^2, q_2^2 = 0\) and we have in fact only one scaling variable \( \xi \) since \( \eta = \xi (1 - \frac{\Delta^2}{2q_1^2}) \approx \xi \). From the experimental point of view it is more appropriate to work with the variables \(-q_1^2\) and \(x_{Bj} \equiv -q_1^2/2(P_1 q_1)\), which are related to the variables (2) by

\[
Q^2 = -\frac{1}{2} q_1^2 \left(1 - \frac{\Delta^2}{2q_1^2}\right), \quad \xi = \frac{x_{Bj}(1 - \Delta^2/2q_1^2)}{2 - x_{Bj}(1 + \Delta^2/q_1^2)} \approx \frac{x_{Bj}}{2 - x_{Bj}}.
\]

In the kinematical region we are interested in there are two leading twist contributions

\[
T_{\mu\nu}(P, q_1, q_2) = -\tilde{g}_{\mu\nu}^T \mathcal{F}(\xi, \eta = \xi, Q^2, \Delta^2) + i e_{\mu\nu q p} \frac{1}{P q} \mathcal{G}(\xi, \eta = \xi, Q^2, \Delta^2) + \ldots,
\]

where the ellipsis stand for a leading twist-two contribution coming from longitudinally polarized photons which, however, does not appear in the DVCS cross section, as well as higher twist functions. The transverse part of the metric tensor, denoted by \( \tilde{g}_{\mu\nu}^{T} \) and the \( e \)-tensor are contracted with the projection operators \( \mathcal{P}_{\alpha\beta} = \delta_{\alpha\beta} - q_{1\alpha} q_{2\beta}/(q_1 q_2) \), \( \tilde{g}_{\mu\nu} = \mathcal{P}_{\mu\nu} \), \( \mathcal{G}_{\mu\nu q p} = \mathcal{P}_{\mu\nu q p} \mathcal{G}_{\alpha\beta} \). Therefore, current conservation is manifest. Our definitions are chosen in such a way that in the forward case, i.e. \( \Delta = 0 \), the usual structure functions measured in DIS are

\[
F_i(x_{Bj}, Q^2) = \frac{1}{2\pi} \text{Im} \mathcal{F}_i(\xi = x_{Bj}, Q^2, 0),
\]

\[
g_i(x_{Bj}, Q^2) = \frac{1}{2\pi} \text{Im} \mathcal{G}_i(\xi = x_{Bj}, Q^2, 0).
\]

Since the virtuality of the incoming photon is deep in the Euclidean domain the hadron is probed almost on the light cone \( x^2 \approx 0 \). Hence, the amplitude can be straightforwardly treated by a non-local version of the light-cone operator product expansion [10]. More recently, it has been proven that assuming a smooth SPD the collinear singularities are indeed factorizable at leading twist to all orders of perturba-
tion theory [6,11,12]. Therefore, the amplitudes $\mathcal{F}_1$ and $\mathcal{F}_2$ factorize in a perturbative hard scattering amplitude and a SPD $q(t, \eta, \Delta^2, \mu^2)$ as

$$\mathcal{F}^a = \sum_0 \epsilon^2 \int_0^1 \frac{dt}{N}\left[q^a(t, \eta, \Delta^2, \mu^2) = \begin{cases} q^a(t, \eta, \Delta^2, \mu^2) \times q^a(t, \eta, \Delta^2, \mu^2) + \alpha(t \to -t), \end{cases}ight.$$}

where the $'-'$ ($'+'$) sign corresponds to the $V(A)$ channel. The tree level coefficient functions are

$$q^e(t) = \frac{1}{1 - t/\xi - t/\eta}, \quad q^o(t) = 0. \quad (10)$$

The hard scattering amplitude $T^{(1)}$ can be calculated by making use of standard methods of perturbative QCD [14]. At the same time there exists an interesting possibility to predict [13] these quantities with the help of the conformal operator product expansion (COPE). It has been introduced more than two decades ago in Ref. [15] and consequently applied to exclusive processes at leading order [16]. The main advantage of the COPE is that under the assumption of conformal covariance it predicts the Wilson coefficients of local conformal operators with a given conformal spin entering the expansion of the product of two currents up to a normalization constant. The latter is fixed by the known Wilson coefficients of forward DIS. Since the COPE is valid only for $|\xi| > 1$ we perform a summation over all conformal partial waves and obtain a representation of the hard scattering amplitudes as a convolution of kernels. E.g. for the non-singlet case we have

$$T(\xi, \eta, t, Q^2, \mu^2) = \frac{1}{\xi} F \left(\frac{\xi}{\eta}, \frac{t}{\mu^2}, \frac{Q^2}{\eta} \right) \otimes C \left(\frac{t}{\eta}, \frac{\mu^2}{\eta}, \alpha_s \right), \quad (11)$$

where the convolution is defined for any test functions $H, E$ and $H, \bar{E}$ for the parity even and odd sectors, respectively, as introduced in Ref. [2].

Using the spatial parity and scaling properties evident from the convolution-type formulae of Ref. [13] we write the perturbative expansion of $T$ as

$$\xi T^{a} = T^{a(0)} \left(\frac{\xi}{\eta}, \frac{t}{\eta} \right) + \frac{\alpha(t \eta^2 \mu^2)}{2 \pi} T^{a(1)} \left(\frac{\xi}{\eta}, \frac{t}{\eta}, \frac{Q^2}{\eta} \right) + \theta(\alpha_s^2) \otimes (t \to -t) \quad (9)$$

and their eigenvalues $\gamma(\alpha_s)$ and $c(\alpha_s)$ coincide with anomalous dimensions and Wilson coefficients, respectively, which appear in DIS. A closed form of
the function $F$ can be deduced order by order in $\alpha_s$ from the sum

$$F(x,y) = \sum_{j=0}^{\infty} \frac{2x}{1+x} \frac{B(j+1,j+2)}{(1+x)^{j/2-2}} \times F_{x,j}(1+j+y/2,2j+y/2) \times \Gamma_j(y).$$

The expressions (11)–(13) which are exact in QCD up to NLO define the so-called conformal subtraction (CS) scheme. To obtain the MS results one has to perform a scheme transformation which is governed by a conformal anomaly appearing in special conformal Ward identities [18]. Finally, the NLO coefficient functions read

$$Q^{\mathcal{T}(0)} = \frac{1}{1-t}, \quad Q^{\mathcal{T}(1)} = Q^{\mathcal{T}(1)} - \frac{C_F}{1-t} \ln \frac{1-t}{2},$$

$$Q^{\mathcal{T}(2)} = \frac{C_F}{2(1-t)} \left[ 2 \ln \frac{1-t}{2} + 3 \left( \ln \frac{-q_1^2}{\mu^2} + \frac{1}{2} \ln \frac{1-t}{2} - \frac{1}{2} \right) \right]$$

$$+ \ln \frac{1-t}{2} - 2 \left( \ln \frac{1-t}{2} - \frac{1}{2} \right) + \ln \frac{1-t}{2} - 2 \left( \ln \frac{1-t}{2} - 2 \right),$$

$$Q^{\mathcal{T}(3)} = -Q^{\mathcal{T}(1)} + \frac{N_f}{2} \left[ \frac{1}{1-t} \left( \ln \frac{-q_1^2}{\mu^2} + \frac{1}{2} \ln \frac{1-t}{2} - 2 \right) \right].$$

We have used here the scaling property mentioned above and the fact that the hard scattering amplitude for DVCS, i.e. $\xi = \eta$, is a simple analytical continuation of the ones for the production of a scalar and pseudo-scalar mesons, respectively, in the collision of a real and a highly virtual photon.

To the same accuracy we have to include the two-loop evolution of the SPDs. Based on the knowledge of NLO anomalous dimensions constructed from conformal constraints [18] we will use the methods of Ref. [9] where we have already dealt with scaling violations in NLO.

3. Model dependent estimates

For numerical estimates of the DVCS amplitude we expand the hard scattering amplitude in terms of Legendre polynomials and reexpress the expansion coefficients in terms of conformal moments. This allows us to include easily the NLO evolution of the SPDs as described in Ref. [9]. Unfortunately, this method is reliable only for moderate values of $x_{ Bj}$ since the convergence of the series of polynomials is not under control for very low $x_{ Bj}$. Already for $x_{ Bj} \sim 0.05$ one needs about 180 polynomials.

To model the SPD one can first explore the simplest possibility of equating it to the usual forward parton density. We will designate this as FPD-model. For small $\eta$, where the so-called DGLAP region with $|t| > |\eta|$ dominates, this may be justified. However, it is an open problem how the exclusive region $|t| < |\eta|$ corresponding to the production or absorption of a meson-like state looks like. To get a more realistic model one can use the relation of

$$g^{\mathcal{T}(m)} = \frac{N_f}{2} \left[ \frac{1}{1-t} \left( \ln \frac{1-t}{2} + \frac{1}{2} \ln \frac{1-t}{2} - 2 \right) \right].$$
SPDs to the so-called double distributions (DDs) \( f(z_-, z_+) \), introduced in [1] and rediscovered in [6]

\[
q(t, \eta, Q^2) = \int_{-1}^{1} dz_- \int_{-1+|z_-|}^{1-|z_-|} dz_+ \delta(z_+ - \eta z_- - t) 
\times f(z_-, z_+, Q^2).
\]

The latter, according to Ref. [19], is given by the product of a forward distribution \( f(z) \) (more precisely \( q(z) \)) for quarks and \( zg(z) \) for gluons) with a profile function \( \pi \)

\[
f(z_-, z_+) = \pi(z_-, z_+) f(z_+),
\]

where \( \pi \) for quarks and gluons is given by

\[
\bar{G}_\pi(z_-, z_+) = \frac{3}{4} \frac{[1 - |z_-|^2] - z_-^2}{[1 - |z_+|^3]},
\]

\[
G_\pi(z_-, z_+) = \frac{15}{16} \left( \frac{[1 - |z_-|^2] - z_-^2}{[1 - |z_+|^3]} \right)^2,
\]

respectively. We will refer to this prescription as the DD-model. Note, that parton distributions used here have the support \([-1 \leq z \leq 1\] and, therefore, the contributions \(-q(-z), +\Delta q(-z)\) with \( z \geq 0 \) have the usual interpretation as antiquarks.

Now we are in a position to study the NLO amplitudes in the \( \overline{\text{MS}} \) scheme for the models specified above. In what follows we use the convention \( \sigma^2 \equiv -q^2 \) and equate the factorization scale with \(-q^2\). The value of \( \Lambda_{\overline{\text{MS}}} \) is \( \Lambda_{\overline{\text{MS}}} = 246 \text{ MeV} \). Since we will rely on available parametrizations of the parton densities which are defined at different scales the DD-models will differ as well. Given a forward distribution at an input scale e.g. \( \sigma^2 = 4 \text{ GeV}^2 \), as e.g. for the MRS fit [20], it can be viewed as evolved only according to the DGLAP equation from a lower scale and then folded with a profile to form the DD-model. If the former is defined at an input of \( \sigma^2 = 0.4 \text{ GeV}^2 \), like e.g. the GRV densities [21], to confront both models we have to evolve GRV-based SPDs with non-forward evolution equations. We demonstrate these features of the DD-model in Fig. 1 where we have compared the MRSS0 [20] with the

GRV parametrization [21] according to the procedure sketched above.

In general we have found that the amplitudes calculated from the models mentioned above are similar in shape. However, in the parity even sector the DVCS amplitude is very sensitive to the \( z \to 0 \) behaviour of the sea quarks. Thus, different models will produce qualitatively different predictions. This is demonstrated in Fig. 2(a) for \( \mathcal{F}_1 \), where even the sign of the amplitude for small \( x_{Bj} \) differs for the FPD- and DD-model. Note that the sign of the FPD will change for very low \( x_{Bj} \) and that a slightly different parametrization gives a prediction similar to the one for a DD-model. We see from this figure that the NLO corrections are as large as 50% for the FPD-model and quite larger in the case of the DD-model with MRS [20] parametrization. Evidently, the imaginary part is more sensitive to the shape of the model distribution than the real part. Their ratio is shown in Fig. 2(b). In Fig. 2(c) we compare the predictions of the DD-model for the MRS and the GRV parametrizations taken at different input scales as explained above. This affects the size of both the real and imaginary part, so that the ratio changes less with the scale as shown in Fig. 2(d). The ratio is also less sensitive to the radiative corrections in the coefficient functions. Note that the evolution will suppress the DGLAP region, so that the asymptotic distributions are concentrated in the region \( |t| \leq \eta \) and are given by the terms with the lowest conformal
spin in the conformal expansion of SPD. Thus for asymptotically large $\mathcal{Q}^2$ the imaginary part has to vanish and the radiative corrections to the real part are determined by the lowest Wilson coefficient in the COPE.

In Fig. 3 we show the predictions for $F_1$ with the GSA parametrization [22] and investigate the size of radiative corrections in detail. In this case we have found a similar model dependence as in the previous case: predictions are sensitive to the sea quark parametrization which turn on at $x_{\text{Bj}} \approx 0.1$ and also may cause very substantial radiative corrections. Since the polarized sea quark distributions are not well known, we simply neglect them. The real and imaginary parts for the DD-model are shown in Fig. 3(a,b), respectively, at the input scale 4 GeV$^2$ and evolved upwards to the scale 10 GeV$^2$. Not surprisingly, the size of the radiative corrections is decreasing with increasing $\mathcal{Q}^2$. Of course, for this considered region the effects are small. Note that for large $x_{\text{Bj}}$ the ratio of real to imaginary part provides again a useful information about the SPD as discussed above. The size of radiative corrections for $F_1$ and $F_2$ are similar. In Fig. 3(c,d) we demonstrate the factorization scale dependence, where we took in the hard scattering amplitude and SPD the scale to be $\mathcal{Q}^2 = (\mathcal{Q}^2/2, \mathcal{Q}^2, 2\mathcal{Q}^2)$. We have also studied the perturbative corrections in the conformal subtraction scheme, however, we have found that in the DVCS kinematics this scheme only slightly reduces the NLO corrections.

4. Conclusions

To summarize, we have studied the DVCS amplitude in perturbative QCD for the region $x_{\text{Bj}} \geq 0.05$. It turns out that a separate measurement of real and imaginary part in this kinematic domain would provide us with information about the SPDs, which is indispensable to discriminate between different models. Moreover, the amplitude at $x_{\text{Bj}} \leq 0.1$ depends in a crucial way on both the model and the used parametrization of the forward parton distributions,
especially, of the sea distribution. Generally, we found that the radiative correction can be as large as 50% and even more, depending on $x_{Bj}$ and the chosen SPD-model. Fortunately, the ratio of real to imaginary part is less sensitive to these radiative corrections and is only mildly scheme dependent. Because of this property it can give us more insights into the structure of SPDs.

The kinematical situation relevant for HERA experiments requires an extension of our analysis downwards to very low $x_{Bj}$. In this case the polynomial method used here will not be the most appropriate tool. Fortunately, a direct numerical convolution of the hard scattering part and skewed parton distributions can be carried out without major difficulties. The remaining problem is to evolve the SPD in this kinematics, which may be solved by numerical integration of the evolution equation. Recently, the evolution kernels at two-loop order have been completely constructed [23] from the knowledge of conformal anomalies and splitting functions and extensive use of supersymmetric constraints. This provides in future the opportunity to study evolution effects also at very low $x_{Bj}$ which is a very important task in view of ongoing experiments on diffractive meson production at HERA.

To decrease the theoretical uncertainties due to radiative corrections, it is necessary to go beyond NLO. In a first step one can include only the hard-scattering amplitude to two-loop order accuracy. Of course, a direct calculation will be very cumbersome. Fortunately, however, in the conformal limit of the theory, when the QCD Gell-Mann-Low function formally equals zero, we can rely directly on the COPE and it seems feasible to extend this technology to NNLO. Conformal symmetry breaking effects can be perturbatively calculated and one piece of information is already available [24]. From the practical point of view it should be stressed that all high hopes connected to SPDs rely on the assumption that the higher order terms are controllable. We want to emphasize that our approach allows to study efficiently the structure of perturbative corrections and might be crucial to establish the same rigour of treatment as for the forward parton densities.

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References

U(2) flavor physics without U(2) symmetry

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Abstract

We present a model of fermion masses based on a minimal, non-Abelian discrete symmetry that reproduces the Yukawa matrices usually associated with U(2) theories of flavor. Mass and mixing angle relations that follow from the simple form of the quark and charged lepton Yukawa textures are therefore common to both theories. We show that the differing representation structure of our horizontal symmetry allows for new solutions to the solar and atmospheric neutrino problems that do not involve modification of the original charged fermion Yukawa textures, or the introduction of sterile neutrinos.

1. Introduction

One path toward understanding the observed hierarchy of fermion masses and mixing angles is to assert that at some high energy scale all Yukawa couplings, except that of the top quark, are forbidden by a new symmetry $G_f$ that acts horizontally across the three standard model generations. As this symmetry is spontaneously broken to smaller subgroups at successively lower energy scales, a hierarchy of Yukawa couplings can be generated. The light fermion Yukawa couplings originate from higher-dimension operators involving the standard model matter fields and a set of ‘flavon’ fields $\phi$, which are responsible for spontaneously breaking $G_f$. The higher-dimension operators are suppressed by a flavor scale $M_f$, which is the ultraviolet cut-off of the effective theory; ratios of flavon vacuum expectation values (vevs) to the flavor scale, $\langle \phi \rangle / M_f$, provide a set of small symmetry-breaking parameters that may be included systematically in the low-energy effective theory. Many models of this type have been proposed, with $G_f$ either gauged or global, continuous or discrete, Abelian or non-Abelian, or some appropriate combination thereof [1]. Non-Abelian symmetries are particularly interesting in the context of supersymmetric theories, where flavor-changing neutral current (FCNC) processes mediated by superparticle exchange can be phenomenologically unacceptable [2]. If the three generations of any given standard model matter field are placed in $2 \oplus 1$ representations of some non-Abelian horizontal symmetry group, it is possible to achieve an exact degeneracy between superparticles of the first two generations when $G_f$ is unbroken. In the low-energy theory, this degeneracy is lifted by the same small
symmetry-breaking parameters that determine the light fermion Yukawa couplings, so that FCNC effects remain adequately suppressed, even with superparticle masses less than a TeV.

A particularly elegant model of this type considered in the literature assumes the continuous, global symmetry $G_f = U(2)$ [3–5]. Quarks and leptons are assigned to $2 \oplus 1$ representations, so that in tensor notation, one may represent the three generations of any matter field by $F^n + F^3$, where $a$ is a U(2) index, and $F$ is $Q, U, D, L, N$. A set of flavons is introduced consisting of $\phi_a, S_{ab}$, and $A_{ab}$, where $\phi$ is a U(2) doublet, and $S(A)$ is a symmetric (antisymmetric) U(2) triplet (singlet). If one assumes the pattern of vevs

$$\left\langle \phi \right\rangle = \begin{pmatrix} 0 \\ \epsilon \end{pmatrix}, \quad \left\langle S \right\rangle = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon \end{pmatrix},$$

and

$$\left\langle A \right\rangle = \begin{pmatrix} 0 & \epsilon' \\ -\epsilon' & 0 \end{pmatrix},$$

(1.1)

which follows from the sequential breaking

$$U(2) \xrightarrow{\epsilon} U(1) \xrightarrow{\epsilon'} \text{nothing},$$

(1.2)

then all fermion masses and Cabibbo-Kobayashi-Maskawa (CKM) mixing angles can be reproduced. More specifically, the pattern of vevs in Eq. (1.1) yields a Yukawa texture for the down quarks of the form

$$Y_D \sim \begin{pmatrix} 0 & d_1 \epsilon' & 0 \\ -d_1 \epsilon' & d_2 \epsilon & d_3 \epsilon \\ 0 & d_3 \epsilon & 1 \end{pmatrix},$$

(1.3)

where $\epsilon \approx 0.02$, $\epsilon' \approx 0.004$, and $d_1, \ldots, d_3$ are $O(1)$ coefficients that can be determined from Ref. [5]. Differences between hierarchies in $Y_D$ and $Y_U$ can be obtained by embedding the model in a grand unified theory [5]. For example, in an SU(5) GUT, one obtains differing powers of $\epsilon$ and $\epsilon'$ in the up quark Yukawa matrix by assuming that $S_{ab}$ transforms as a $7S$; combined GUT and flavor symmetries prevent $A_{ab}$ and $S_{ab}$ from coupling to the up and charm quark fields, unless an additional flavor singlet field $\Sigma$ is introduced that transforms as an SU(5) adjoint.

With $\langle \Sigma \rangle / M_f \sim \epsilon$, it is possible to explain why $m_\nu : m_e : m_b = \lambda^4 : \lambda^2 : 1$, while $m_\mu : m_e : m_t = \lambda^3 : \lambda : 1$, where $\lambda = 0.22$ is the Cabibbo angle. The ratio $m_t / m_b$ is assumed to be unrelated to U(2) symmetry breaking, and is put into the low-energy theory by hand.

In this letter we show that the properties of the U(2) model leading to the successful Yukawa textures described above are also properties of smaller discrete symmetry groups. To reproduce all of the phenomenological successes of the U(2) model, we require a candidate discrete symmetry group to have the following properties:

- 1, 2, and 3 dimensional representations.
- The multiplication rule $2 \otimes 2 = 3 \oplus 1$.
- A subgroup $H_f$ such that the breaking pattern $G_f \rightarrow H_f \rightarrow \text{nothing}$ reproduces the canonical U(2) texture given in Eq. (1.3). This implies that an unbroken $H_f$-symmetry forbids all Yukawa entries with $O(\epsilon^4)$ vevs, but not those with $O(\epsilon)$ vevs versus

In the next section we show that the smallest group satisfying these conditions is a product of the double tetrahedral group $T'$ and an additional $Z_3$ factor. Since U(2) is isomorphic to SU(2) $\times$ U(1), it is not surprising that our candidate symmetry involves the product of a discrete subgroup of SU(2), $T'$, and a discrete subgroup of U(1), $Z_3$. At this point, the reader who is unfamiliar with discrete group theory may feel somewhat uneasy. We stress that the group $T'$ is in fact a very simple discrete symmetry, a spino
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4 For a review of basic terms, see Ref. [11].
One of the virtues of the model discussed in this letter is that it allows for elegant extensions that explain the solar and atmospheric neutrino deficits, while maintaining the original quark and charged lepton Yukawa textures. This distinguishes our model from the modified version of the $U(2)$ model presented in Ref. [7]. Preserving the $U(2)$ charged fermion textures is desirable since they lead to successful mass and mixing angle relations such as $|V_{ub}/V_{cd}| = \sqrt{m_u/m_c}$, which are 'exact' in the sense that they contain no unknown $O(1)$ multiplicative factors. Since we succeed in explaining solar and atmospheric neutrino oscillations without sacrificing the predictivity of the original model, we need not introduce sterile neutrinos, as in Ref. [8]. However, we do not try to explain simultaneously the more controversial LSND results [9] in this paper. We will consider versions of our model that include sterile neutrinos in a longer publication [10].

2. The symmetry

We seek a non-Abelian candidate group $G_f$ that provides the $2 \oplus 1$ representation (rep) structure for the matter fields described in the previous section. In order for the breaking of $G_f$ to reproduce the $U(2)$ charged fermion Yukawa texture in Eq. (1.3), one must have flavons that perform the same roles as charged fermion Yukawa textures $S_f$ and $A_{ab}$ in the $U(2)$ model. Since these are doublet, triplet, and nontrivial singlet reps, respectively, we require $G_f$ to have reps of the same dimensions. Nontrivial singlets appear in all discrete groups of order $< 32$ [12], so we seek groups $G_f$ with doublet and triplet representations.

The order 12 tetrahedral group $T$, the group of proper symmetries of a regular tetrahedron (which is also the alternating group $A_4$, consisting of even permutations of four objects), is the smallest containing a triplet rep, but has no doublet reps. A number of groups with orders $< 24$ possess either doublet or triplet reps, but not both (See, for example, [12]).

It turns out that two groups of order 24 possess both doublet and triplet reps. One is the symmetric group $S_4$ of permutations on four objects, which is isomorphic to the group $O$ of proper symmetries of a cube as well as the group $T_d$ of all proper and improper symmetries of a regular tetrahedron. $S_4$ possesses two triplets $3^e$, two singlets $1^\pm$, and one doublet $2$. However, in this case one encounters another difficulty: The combination rule for doublets in $S_4$ is $2 \oplus 2 = 2 \oplus 1^\pm \oplus 1^\pm$, which implies that the triplet flavon cannot connect two doublet fields such as those of the first two generations of $Q$ and $U$. Thus, $S_4$ is not suitable for our purposes.

The unique group of order $< 32$ with the combination rule $2 \oplus 2 \oplus 3$ is the double tetrahedral group $T'$, which is order 24. The character table, from which one may readily generate explicit representation matrices, is presented in Table 1. Geometrically, $T'$ is the group of symmetries of a regular tetrahedron under proper rotations (Fig. 1). These symmetries consist of 1) rotations by $2\pi/3$ about an axis connecting a vertex and the opposite face ($C_3$), 2) rotations by $\pi$ about an axis connecting the midpoints of two non-intersecting edges ($C_2$), and 3) the rotation $R$ by $2\pi$ about any axis, which produces a factor $-1$ in the even-dimensional reps, exactly as in $SU(2)$. Indeed, this feature is a consequence of $T \subset SU(2)$, and the rotations $C_3$ and $C_2$ are actually of orders 6 and 4, respectively. Also, $T'$ is isomorphic to the group $SL_2(F_3)$, which consists of $2 \times 2$ unimodular matrices whose elements are added and multiplied as integers modulo 3.

$T'$ has three singlets $1^0$ and $1^\pm$, three doublets, $2^0$ and $2^\pm$, and one triplet, $3$. The triline superscript describes in a concise way the rules for combining these reps: With the identification of $\pm$ as $\pm 1$, the

<table>
<thead>
<tr>
<th>Sample element</th>
<th>$E$</th>
<th>$R$</th>
<th>$C_3$, $C_2$, $R$</th>
<th>$C_3$, $C_2$, $R^2$</th>
<th>$C_3$, $C_2$, $R^3$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1</td>
<td>6</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Order of element</td>
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<td>2</td>
<td>4</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>1$^0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1$^\pm$</td>
<td>1</td>
<td>1</td>
<td>$\eta$</td>
<td>$\eta^2$</td>
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<tr>
<td>1$^\pm$</td>
<td>1</td>
<td>1</td>
<td>$\eta^2$</td>
<td>$\eta$</td>
<td>$\eta^2$</td>
</tr>
<tr>
<td>2$^0$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>$-1$</td>
</tr>
<tr>
<td>2$^\pm$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>$\eta$</td>
<td>$-\eta^2$</td>
</tr>
<tr>
<td>2$^\pm$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>$\eta^2$</td>
<td>$-\eta$</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>-1</td>
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<td>0</td>
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</table>
One must now determine whether it is possible to place a sequence of vevs hierarchically in the desired elements of the Yukawa matrices. Notice if \( G_f \) is broken to a subgroup \( H_f \) that rotates the first generation matter fields by a common nontrivial phase, then \( H_f \) symmetry forbids all entries with \( O(\epsilon') \) vevs in Eq. (1.3). Therefore, we require that the elements of \( G_f \) defining this subgroup have two-dimensional rep matrices of the form \( \text{diag}(\rho,1) \), with \( \rho = \exp(2\pi i n/N) \) for some \( N \) that divides the order of \( G_f \) and some integer \( n \) relatively prime with respect to \( N \). This form for \( \rho \) follows because reps of finite groups may be chosen unitary, and must give the identity when raised to the power of the order of \( G_f \). Such elements generate a subgroup \( H_f \subset Z_N^f \) of \( G_f \). Whether such elements exist in \( G_f \) can be determined since the rep of any element can be brought to diagonal form by a basis transformation, while the eigenvalues \( \rho, 1 \) are invariant under such basis changes.

Even if a given element \( C \in G_f \) has the diagonal form \( \text{diag}(\rho_1,\rho_2) \), \( \rho = \exp(2\pi i n_j/N) \) (and thus generates a subgroup, \( Z_N^f \), of \( G_f \)), a phase rotation of the form \( \text{diag}(\rho,1) \) can be achieved if the original \( G_f \) is extended by forming a direct product with an additional factor \( Z_N^f \). We then identify \( H_f \) as a subgroup of \( Z_N^f \times Z_N^f \). We choose one element of the additional \( Z_N^f \) to compensate the phase of the 22 element of \( C \), and similarly for the other elements of the \( Z_N^f \). The element corresponding to \( C \) in \( G_f \times Z_N^f \) then effectively acts upon the doublet as diag(\( \exp(2\pi i (n_1 - n_2)/N) \), 1), and the remaining symmetry is \( Z_N^f \times Z_N^f \). In the case that \( |n_1 - n_2| \) and \( N \) are relatively prime, this reduction amounts to forming the diagonal subgroup \( Z_N^f \) of \( Z_N^f \times Z_N^f \). Similar arguments apply to the singlet and triplet reps.

In the particular case of \( G_f = T' \), one finds elements \( C \) that generate either \( Z_2 \) or \( Z_3 \) subgroups. By introducing an additional \( Z_n \) (with \( n = 2 \) or 3) one can arrange for a \( Z_n \) subgroup that affects only the first generation fields. In the case of \( Z_2 \), the nontrivial element of the diagonal subgroup is of the form \( \text{diag}(-1,1) \), which leaves the 11 and 22 entries of the Yukawa matrices invariant. The incorrect relation \( m_1 = m_2 \) then follows. On the other hand, \( Z_3 \) prevents an invariant 11 entry, so we are led to adopt

\[
G_f = T' \times Z_3. 
\]

The reps of \( G_f \) are named by extending the notation for \( T' \) to include a superscript indicating the \( Z_3 \) rep. These are the trivial rep 0, which takes all elements to the identity, and two complex-conjugate reps \(+\) and \(-\). Like the trialities, these indices combine via addition modulo 3. We adopt the convention that the \( T' \times Z_3 \) reps \( 1^0, 1^+, 1^- \), \( 2^0, 2^+ \) and \( 2^- \) are special, in that these singlet reps and the second component of the doublets remain invariant under \( Z_3^0 \). Thus any \( 2 \oplus 1 \) combination of these reps is potentially interesting for model building.

3. A minimal model

The minimal model has the three generations of matter fields transforming as \( 2^0 \oplus 1^0 \) under \( G_f = T' \times Z_3 \). The Higgs fields \( H_{UD} \) are pure singlets of \( G_f \) and transform as \( 1^0 \). Given these assignments, it
is easy to obtain the transformation properties of the
Yukawa matrices,

\[ Y_{\nu,\rho,L} \sim \left( \begin{array}{c|c|c} [3^- + 1^0] & [2^0] & [1^0] \\ \hline [2^0] & [1^0] & [1^0] \end{array} \right) . \] (3.1)

Eq. (3.1) indicates the reps of the flavon fields
needed to construct the fermion mass matrices. They
are \( 1^0 \), \( 2^0 + \), and \( 3^0 \), which we call \( A, \phi, \) and \( S \),
respectively. Once these flavons acquire vevs, the
flavor group is broken. We are interested in a two-
step breaking controlled by two small parameters \( \epsilon \),
and \( \epsilon' \), where

\[ T' \otimes Z_3 \rightarrow Z_3^0 \rightarrow \epsilon' \rightarrow \text{nothing} . \] (3.2)

Since we have chosen a doublet rep for the first two
generations that transforms as \( \text{diag}(\rho,1) \) under \( Z_3^0 \),
only the 22, 23, and 32 entries of the Yukawa
matrices may develop vevs of \( O(\epsilon') \), which we
assume originate from vevs in \( S \) and \( \phi \). The symmetry
\( Z_3^0 \) is then broken by a \( 1^0 \) vev of \( O(\epsilon') \). The
Clebsch-Gordan coefficients that couple a \( 1^0 \) to
two \( 2^0 + \) doublets is proportional to \( \sigma_3 \), so the \( \epsilon' \)
appears in an antisymmetric matrix. We therefore
produce the U(2) texture of Eq. (1.3). Since the \( 1^0 \)
and \( 3^0 \) flavon vevs appear as antisymmetric and
symmetric matrices, respectively, all features of the
grand unified extension of the U(2) model apply
here, assuming the same GUT transformation proper-
ties are assigned to \( \phi, S, \) and \( A \). One can also show
readily that the squark and slepton mass squared
matrices are the same as in the U(2) model.

It is worth noting that we could construct com-
pletely equivalent theories had we chosen to place the
matter fields in reps like \( 2^0 + \otimes 1^0 \) or \( 2^{-0} \otimes 1^{00} \),
which have the same transformation properties under \( Z_3^0 \) as our original choice. The reps \( 2^0 + \otimes 1^0 \) are
desirable in that they fill the complete SU(2) repre-
sentations \( 2 \otimes 1 \), if we were to embed \( T' \) in SU(2).
Since anomaly diagrams linear in this SU(2) vanish
(and hence the linear Ibañez-Ross condition is satis-
fied [13]), we conclude that \( T' \) is a consistent dis-
crete gauge symmetry [14]. The additional \( Z_3 \) may
also be considered a discrete gauge symmetry, pro-
viding its anomalies are cancelled by the Green-
Schwarz mechanism.

4. Neutrinos

In this section, we show that the model presented in
Section 3 can be extended to describe the ob-
erved deficit in solar and atmospheric neutrinos. We
consider two cases:

Case I: Here we do not assume grand unification,
so that all flavons are SU(5) singlets. This case is of
interest, for example, if one is only concerned with
explaining flavor physics of the lepton sector. We
choose

\[ \nu_R \sim 1^0 . \] (4.1)

Note that the only difference from the other matter
fields is the representation choice for the third gen-
eration field. The neutrino Dirac and Majorana mass
matrices then have different textures from the charged
fermion mass matrices. Their transformation proper-
ties are given by

\[ M_{LR} \sim \left( \begin{array}{c|c|c} [3^- + 1^0] & [2^{+0}] & [1^{+0}] \\ \hline [2^{+0}] & [1^{+0}] & [1^{+0}] \end{array} \right) , \]

\[ M_{RR} \sim \left( \begin{array}{c|c|c} [3^-] & [2^{+0}] & [1^{+0}] \\ \hline [2^{+0}] & [1^{+0}] & [1^{+0}] \end{array} \right) . \] (4.2)

Note that we obtain the same triplet and nontrivial
singlet in the upper \( 2 \times 2 \) block as in the charged
fermion mass matrices, as well as one of the same
flavor doublets, the \( 2^{+0} \), the rep \( 1^{0+} \) is not present in \( M_{RR} \), since Majorana mass matrices are symmet-
ic. In addition we obtain the reps \( 2^{+0}, 1^{+0}, \) and \( 1^{+0} \), which did not appear in Eq. (3.1). New flavon
fields can now be introduced with these transforma-
tion properties, and their effects on the neutrino
physics can be explored. Let us consider introducing
a single \(^5\) new flavon \( \phi_r \) transforming as a \( 2^{+0} \)
and with a vev

\[ \langle \phi_r \rangle \sim \sigma_3 \left( \begin{array}{c} \epsilon' \\ \epsilon \end{array} \right) , \] (4.3)

where \( \sigma_3 \) is the Clebsch that couples the two
doublets to \( 1^{0+} \). The introduction of this new flavon
is the only extension we make to the model in order to
describe the neutrino phenomenology. After intro-

\(^5\)Assuming more than one \( \phi_r \) leads to the same qualitative
results.
Reducing $\phi_\nu$, the neutrino Dirac and Majorana mass matrices read

$$M_{LR} = \begin{pmatrix} 0 & l_1 e' & l_1 r_2 e' \\ -l_1 e' & l_2 e & l_1 r_1 e \\ 0 & l_2 e & 0 \end{pmatrix} \langle H_D \rangle,$$

$$M_{RR} = \begin{pmatrix} r_x r_2^2 e'^2 & r_y r_1 r_2 e' & r_z e' \\ r_y r_1 r_2 e' & r_z e & r_1 e \\ r_z e' & r_1 e & 0 \end{pmatrix} \Lambda_R,$$  \hspace{1cm} (4.4)

where $\Lambda_R$ is the right-handed neutrino mass scale, and we have parameterized the $O(1)$ coefficients. Furthermore, the charged lepton Yukawa matrix including $O(1)$ coefficients reads

$$Y_L \sim \begin{pmatrix} 0 & c_1 e' & 0 \\ -c_1 e' & 3 c_2 e & c_3 e \\ 0 & c_4 e & 1 \end{pmatrix}.$$  \hspace{1cm} (4.5)

The factor of 3 in the 22 entry is simply assumed at present, but originates from the Georgi-Jarlskog mechanism [15] in the grand unified case considered later.

The left-handed Majorana mass matrix $M_{LL}$ follows from the seesaw mechanism

$$M_{LL} \approx M_{LR} M_{RR}^{-1} M_{LR}^\dagger,$$  \hspace{1cm} (4.6)

which yields

$$M_{LL} \sim \begin{pmatrix} (e' / e)^2 & e'/e & e'/e \\ e'/e & 1 & 1 \\ e'/e & 1 & 1 \end{pmatrix} \langle H_D \rangle \chi_\nu \Lambda_R,$$  \hspace{1cm} (4.7)

where we have suppressed the $O(1)$ coefficients. We naturally obtain large mixing between second- and third-generation neutrinos, while the 12 and 13 mixing angles are $O(e'/e)$. However, taking into account the diagonalization of $Y_L$, the relative 12 mixing angle can be made smaller, as we discuss below. Explanation of the observed atmospheric neutrino fluxes by $\nu^\mu \rightarrow \nu^\tau$ mixing suggests $\sin^2 \theta_{23} \geq 0.8$ and $10^{-3} \leq \Delta m_{23}^2 \leq 10^{-2}$, while the solar neutrino deficit may be accommodated assuming the small-angle MSW solution $2 \times 10^{-3} \leq \sin^2 \theta_{12} \leq 10^{-2}$ for $4 \times 10^{-6} \leq \Delta m_{12}^2 \leq 10^{-5}$, where all squared masses are given in eV$^2$. We display below an explicit choice of the $O(1)$ parameters that yields both solutions simultaneously; a more systematic global fit will be presented in Ref. [10].

If $M_{LL}$ and $Y_L$ are diagonalized by $M_{LL} = VM_{LL}^V$, $Y_L = U_L Y_L U_L^\dagger$, then the neutrino CKM matrix is given by

$$V_\nu = U_L^\dagger V.$$  \hspace{1cm} (4.8)

We aim to reproduce the 12 and 23 mixing angles, as well as the ratio $10^2 \leq \Delta m_{23}^2 / \Delta m_{12}^2 \leq 2.5 \times 10^3$ suggested by the data. Obtaining this ratio is sufficient since $A_y$ is not determined by symmetry considerations and may be chosen freely. Assuming the previous values $e = 0.02$ and $e' = 0.004$ and the parameter set $(l_1, \ldots, l_4, r_1, \ldots, r_4, c_1, \ldots, c_4) = (0.5, 1.0, -1.2, 2.3, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0)$, we find:

$$\frac{\Delta m_{23}^2}{\Delta m_{12}^2} = 105, \quad \sin^2 2 \theta_{12} = 5 \times 10^{-3},$$

$$\sin^2 2 \theta_{23} = 0.9,$$  \hspace{1cm} (4.9)

which fall in the desired ranges. While all our coefficients are of natural size, we have arranged for an $O(15\%)$ cancellation between 12 mixing angles in $U_L$ and $V$ to reduce the size of $\sin^2 2 \theta_{12}$ to the desired value.

Case II: Here we assume that the flavons transform nontrivially under an SU(5) GUT group, namely $A \sim 1$, $S \sim 75$, $\phi \sim 1$, and $\Sigma \sim 24$. Note that since $H \sim \tilde{5}$, the products $SH$ and $\Delta H$ transform as a $\overline{45}$ and $\tilde{5}$, respectively, ultimately providing a factor of 3 enhancement in the 22 entry of $Y_L$ (the Georgi-Jarlskog mechanism). In addition, two $2^+0$ doublets are introduced, $\phi_{1+}$ and $\phi_{2+}$, since the texture obtained for the neutrino masses by adding only one extra doublet is not viable. Both doublets $\phi_\nu$ have vevs of the form displayed in Eq. (4.3). Crucially, the presence of these two new doublets does not alter the form of any charged fermion Yukawa texture.
The neutrino Dirac and Majorana mass matrices now take the form

$$M_{LR} = \begin{pmatrix} 0 & l_1 e' & l_5 r_2 e' \\
-l_1 e' & l_1 e^2 & l_5 r_1 e \\
0 & l_4 e & 0 \end{pmatrix} \langle H_D \rangle,$$

$$M_{RR} = \begin{pmatrix} r_3 e'^2 & r_4 e e' & r_2 e' \\
r_4 e e' & r_3 e & r_4 r_1 e \\
r_2 e' & r_4 r_1 e & 0 \end{pmatrix} \Lambda_R,$$  \hspace{1cm} (4.10)

while the charged fermion mass matrix is the same as in Eq. (4.5). Using Eq. (4.6) one obtains the texture:

$$M_{LL} \approx \begin{pmatrix} (e'/e)^2 & e'/e & e'/e \\
e'/e & 1 & 1 \\
e'/e & 1 & 1 \end{pmatrix} \langle H_D \rangle^2 \Lambda_R.$$  \hspace{1cm} (4.11)

If we now choose \((l_1, \ldots, l_5, r_1, \ldots, r_5, c_1, \ldots, c_5) = (-1.0, 1.0, 1.0, 0.5, 1.0, 1.0, 1.0, 1.0, -2.0, 1.0, 1.0, 1.0, 1.0),\) we find

$$\frac{\Delta m_{23}^2}{\Delta m_{12}^2} = 282, \quad \sin^2 2\theta_{12} = 6 \times 10^{-3},$$

$$\sin^2 2\theta_{23} = 0.995.$$  \hspace{1cm} (4.12)

Again these values fall in the desired ranges to explain the atmospheric and solar neutrino deficits, assuming an appropriate choice for \(\Lambda_R.\)

5. Conclusions

In this letter we have shown how to reproduce the quark and charged lepton Yukawa textures of the U(2) model in their entirety, using a minimal non-Abelian discrete symmetry group. We showed that the representation structure of \(T \times Z_3,\) in particular the existence of three distinct 2-dimensional irreducible representations, allows for solutions to the solar and atmospheric neutrino problems that require neither a modification of the simple charged fermion Yukawa textures of the U(2) model nor the introduction of singlet neutrinos. The simplicity of the symmetry structure of our model suggests that a more comprehensive investigation of the space of possible models is justified. Work on alternative neutrino sectors as well as a more detailed phenomenological analysis of the models described here will be presented elsewhere [10].

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References


First evidence for center dominance in SU(3) lattice gauge theory

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Abstract

The dominance of center degrees of freedom is observed in SU(3) lattice gauge theory in maximal center gauge. The full asymptotic string tension is reproduced, after center projection, by the center elements alone. This provides further evidence for the role played by center vortices in the mechanism of color confinement in quantum chromodynamics, but more extensive simulations with a better gauge-fixing procedure are still needed. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The idea that center vortices play a decisive role in the mechanism of color confinement in quantum chromodynamics was proposed more than 20 years ago by ’t Hooft [1] and other authors [2]. Recently, the center-vortex picture of confinement has found remarkable confirmation in numerical simulations of the SU(2) lattice gauge theory [3-12]. Our group has proposed a technique for locating center vortices in thermalized lattice configurations based on fixing to the so-called maximal center gauge, followed by center projection [3-5].

In SU(2) lattice gauge theory, the maximal center gauge is a gauge in which the quantity

\[ R = \sum_x \sum_\mu \left| \text{Tr} [U_\mu(x)] \right|^2 \]  

(1)

reaches a maximum. This gauge condition forces each link variable to be as close as possible, on average, to a $Z_2$ center element, while preserving a residual $Z_2$ gauge invariance. Center projection is a
mapping of each SU(2) link variable to the closest $Z_2$ center element:

$$U_\mu(x) \rightarrow Z_2(\cdot) \equiv \text{signTr}[U_\mu(x)]. \quad (2)$$

The excitations on the projected $Z_2$ lattice are point-like, line-like, or surface-like objects, in $D = 2, 3$, or 4 dimensions respectively, called “P-vortices.” These are thin objects, one lattice spacing across. There is substantial numerical evidence that thin P-vortices locate the middle of thick center vortices on the unprojected lattice. The string tension computed on center projected configurations reproduces the entire asymptotic SU(2) string tension [5]. It has also been demonstrated recently that removal of center vortices not only removes the asymptotic string tension, but restores chiral symmetry as well, and the SU(2) lattice is then brought to trivial topology [7].

Putting the above and other pieces of evidence together, it seems clear that our procedure of maximal-center-gauge fixing and center projection identifies physical objects that play a crucial role in the mechanism of color confinement. However, the gauge group of QCD is color SU(3), not SU(2), and it is of utmost importance to demonstrate that the observed phenomena are not specific to the SU(2) gauge group only. Some preliminary results for SU(3) were presented in Section 5 of Ref. [5]. They came from simulations on very small lattices and at strong couplings. It was shown that center-projected Wilson loops reproduce results of the strong-coupling expansion of the full theory up to $\beta = 4$.

The purpose of the present letter is to present further evidence on center dominance in SU(3) lattice gauge theory, very similar to the results that arose from SU(2) simulations. Though not as convincing as the SU(2) data, the first SU(3) results support the view that the vortex mechanism works in SU(3) in the same way as in SU(2).

2. Maximal center gauge in SU(3)

The maximal center gauge in SU(3) gauge theory is defined as the gauge which brings link variables $U$ as close as possible to elements of its center $Z_3$. This can be achieved as in SU(2) by maximizing a “mesonic” quantity

$$R = \sum_x \sum_\mu | \text{Tr} U_\mu(x) |^2, \quad (3)$$

or, alternatively, a “baryonic” one

$$R' = \sum_x \sum_\mu \text{Re} \left( \left[ \text{Tr} U_\mu(x) \right]^3 \right). \quad (4)$$

The latter was the choice of Ref. [5], where we used the method of simulated annealing for iterative maximization procedure. The convergence to the maximum was rather slow and forced us to restrict simulations to small lattices and strong couplings. The results, that will be presented below, were obtained in a gauge defined by the “mesonic” condition (3). The maximization procedure for this case is inspired by the Cabibbo–Marinari–Okawa SU(3) heat bath method [14,15]. The idea of the method is as follows: In the maximization procedure we update link variables to locally maximize the quantity (3) with respect to a chosen link. At each site we thus need to find a gauge-transformation matrix $\Omega(x)$ which maximizes a local quantity

$$R(x) = \sum_\mu \left( | \text{Tr} \left[ \Omega(x) U_\mu(x) \right] |^2 \right. \left. + | \text{Tr} \left[ U_\mu(x - \hat{\mu}) \Omega^\dagger(x) \right] |^2 \right). \quad (5)$$

Instead of trying to find the optimal matrix $\Omega(x)$, we take an SU(2) matrix $g(x)$ and embed it into one of the three diagonal SU(2) subgroups of SU(3). The expression (5) is then maximized with respect to $g$, with the constraint of $g$ being an SU(2) matrix. This reduces to an algebraic problem (plus a solution of a

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*A similar approach was applied for SU(3) cooling by Hoek et al. [16].*
non-linear equation). Once we obtain the matrix \( g(x) \), we update link variables touching the site \( x \), and repeat the procedure for all three subgroups of SU(3) and for all lattice sites. This constitutes one center gauge fixing sweep. We made up to 1200 sweeps for each configuration. Center projection is then done by replacing the link matrix by the closest element of \( Z_3 \).

The above iterative procedure was independently developed by Montero, and described with full details in his recent publication [17]. Montero, building on the work of Ref. [18], has constructed classical SU(3) center vortex solutions on a periodic lattice. He has found that P-vortex plaquettes accurately locate the middle of the classical vortex, which is evidence of the ability of maximal center gauge to properly find vortex locations.

3. Center dominance in SU(3) lattice gauge theory

The effect of creating a center vortex linked to a given Wilson loop in SU(3) lattice gauge theory is to multiply the Wilson loop by an element of the gauge group center, i.e.

\[
W(C) \rightarrow e^{\pm 2\pi i/3} W(C).
\]  

Quantum fluctuations in the number of vortices linked to a Wilson loop can be shown to lead to effects of lattice falloff; the simplest, but urgent question is whether center disorder is sufficient to produce the whole asymptotic string tension of full, unprojected lattice configurations.

We have computed Wilson loops and Creutz ratios at various values of the coupling \( \beta \) on a \( 12^4 \) lattice, from full lattice configurations and from center-projected link configurations in maximal center gauge. Fig. 1 shows a typical plot at \( \beta = 5.6 \). It is obvious that center elements themselves produce a value of the string tension which is close to the asymptotic value of the full theory.

Recently we have argued that center dominance by itself does not prove the role of center degrees of freedom in QCD dynamics [20]; some sort of central dominance exists also without any gauge fixing and can hardly be attributed to center vortices. Distinctive features of center-projected configurations in maximal center gauge in SU(2), besides center dominance, were that: 1. Creutz ratios were approximately constant starting from small distances (this we called “precocious linearity”), 2. the vortex density scaled with \( \beta \) exactly as expected for a physical quantity with dimensions of inverse area.

Precocious linearity, the absence of the Coulomb part of the potential on the center-projected lattice at short distances, can be quite clearly seen from Fig. 1. One observes some decrease of the Creutz ratios at intermediate distances. A similar effect is present also at other values of \( \beta \). It is not clear to us whether this decrease is of any physical relevance, or whether it should be attributed to imperfect fixing to the maximal center gauge.

The issue of scaling is addressed in Fig. 2. Here values of various Creutz ratios are shown as a function of \( \beta \) and compared to those quoted in Ref. [21]. All values for a given \( \beta \) lie close to eachother (precocious linearity once again) and are in reasonable agreement with asymptotic values obtained in time-consuming SU(3) pure gauge theory simulations. The plot in Fig. 2 is at the same time a hint that the P-vortex density also scales properly. The density is approximately proportional to the value of \( \chi(1) \) in center-projected configurations, and \( \chi(1) \) follows the same scaling curve as Creutz ratios obtained from larger Wilson loops.
A closer look at Fig. 2 reveals that there is no perfect scaling, similar to the SU(2) case, in our SU(3) data. Broken lines connecting the data points tend to bend at higher values of $\beta$. In our opinion, this is a finite-volume effect and should disappear for larger lattices.

4. Conclusion

We have presented evidence for center dominance, precocious linearity, and scaling of center-projected Creutz ratios and of P-vortex density from simulations of the SU(3) lattice gauge theory. Our data – and conclusions that can be drawn from them – look quite similar to the case of SU(2). However, the SU(3) data at present are not as convincing and unambiguous as those of SU(2). The reason essentially is that the gauge-fixing maximization for SU(3) is very time consuming, either with simulated annealing or by the Cabibbo–Marinari–Okawa-like method used in the present investigation \(^7\). Moreover, the maximal center gauge is known to suffer from the Gribov problem, which makes gauge fixing notoriously difficult (in this context see also Refs. \([22,23]\)). A better alternative is badly needed, and may be provided by the recent proposal of de Forcrand et al. \([24]\) based on fixing to the so-called Laplacian center gauge. Their first SU(2) results are promising, and the method can readily be extended to the case of SU(3).

It is encouraging that none of the pieces of data, which we have accumulated in SU(3) lattice gauge theory until now, contradicts conclusions drawn from earlier SU(2) results. If future extensive simulations with a more suitable, Gribov-copy free center-gauge fixing method confirm the evidence obtained in our exploratory investigation, center vortices will have a very strong claim to be the true mechanism of color confinement in QCD.

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\(^7\) Typically thousands of iterations were needed for gauge fixing also in the investigation of Montero \([17]\).
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Near-threshold $\eta$ meson production in proton–proton collisions

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Abstract

The production of $\eta$ mesons has been measured in the proton–proton interaction close to the reaction threshold using the COSY-11 internal facility at the cooler synchrotron COSY. Total cross sections were determined for eight different excess energies ($\epsilon$) in the range from $\epsilon = 0.5$ MeV to $\epsilon = 5.4$ MeV. The energy dependence of the total cross section is well described by the available phase-space volume weighted by FSI factors for the proton–proton and proton–$\eta$ pairs. © 2000 Elsevier Science B.V. All rights reserved.
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1. Introduction

Over the last few years, creation of mesons near threshold in the elementary nucleon–nucleon collision has become an important field for studies of meson production mechanisms as well as of meson–nucleon interactions. Measurements at the new generation of medium energy proton accelerators, storage rings with phase-space cooling of the beam as the IUCF-ring, CELSIUS and COSY, delivered high precision values of cross sections for the production of various mesons in the mass region up to 1 GeV/c². The experimental information gained so far is consistent with approximately constant production matrix elements when the final state interaction (FSI) is factored out. The pion production cross sections are described very precisely including only the proton–proton FSI, since the pion–proton interaction is comparatively weak close to threshold. Contrary, in the $\eta$ meson production in proton–pro-
ton collisions the $\eta$–proton FSI can essentially influence the energy dependence of the total cross section. Effects of $\eta$–proton FSI have been seen in Dalitz plots from investigations at CELSIUS [1]. Inclusive measurements at CELSIUS [2] indicate additional contributions from partial waves of higher order than s-wave even at an excess energy as low as 36 MeV, which again change the energy dependence of the cross section according to their relative strength.

2. Experiment

Existing data on the $pp \rightarrow pp\eta$ reaction near threshold, originating from measurements at SATURNEN using the spectrometers SPES-3 [3] and PINOT [4] and at CELSIUS [1] with the PROMICE/WASA detection system, still leave enough freedom for interpreting the energy dependence of the cross sections. Therefore, further data of the $pp$ induced $\eta$ production very close to threshold were needed. Measurements were performed at the COoler SYnchrotron COSY [5] in Jülich with the use of the COSY-11 facility, shown schematically in Fig. 1, in the range of excess energies below $\epsilon = 6$ MeV.

![Fig. 1. Schematic view of the COSY-11 facility. The particle trajectories are measured by means of hits in two sets of drift chambers D1 and D2. The scintillation hodoscopes S1 and S2 are used as start detectors and S3 as the corresponding stop detector for time of flight measurements.](image)

The COSY-11 facility, described in detail in Ref. [6], uses an internal hydrogen cluster target [7], installed in front of a COSY accelerator dipole magnet. Due to their lower momenta, the two outgoing protons of the reaction $pp \rightarrow pp\eta$ are separated from the beam in the magnetic field of the C-shaped dipole and are diverted towards the direction of the centre of the synchrotron into the COSY-11 detector arrangement. Their trajectories are measured by means of hits in a set of two drift chambers (marked D1 and D2 in Fig. 1), allowing the momentum to be determined by ray tracing back through the precisely known magnetic field to the target position. Identification of particles (here protons) is performed by additionally measuring the time-of-flight over a distance of $\approx 9.4$ m between start and stop scintillator hodoscopes (S1 and S3). The uncharged $\eta$ mesons are not registered exclusively but are identified using the missing mass method.

Measurements were performed with the beam momentum varied continuously in the range from 9.6 MeV/c below to 20.4 MeV/c above the threshold momentum which is equal to 1981.6 MeV/c. For the data analysis, this range was grouped into 2
MeV/c intervals. In the following, the central momenta for these intervals and not their limits are quoted.

For different excess energies examples of missing mass distributions are shown in Fig. 2, each of them being dominated by a clear peak due to the $\eta$ meson production, except for the case where the beam momentum is below the reaction threshold. The $\eta$ meson production broadens with increasing excess energy, which is a kinematical effect. The square of the missing mass $MM$ is determined by the square of the four-momentum vector evaluated when subtracting the sum of the two exit proton four momentum vectors ($P_1 + P_2$) from the one of the proton–proton entrance channel ($P_0$):

$$MM^2 = (P_0 - (P_1 + P_2))^2.$$  \hspace{1cm} (1)

For the limit of $\epsilon \ll m_{\eta}$ it can be shown [8] that the resolution of the square of the missing mass is proportional to the experimental momentum resolution of the two protons measured (which is supposed to be constant for all excess energies in the range discussed here) times $\sqrt{\epsilon}$:

$$\Delta(MM^2) = a \cdot \sqrt{\epsilon}.$$  \hspace{1cm} (2)

Fitting the width distribution ($\Delta(MM^2)$) of the $\eta$ meson missing mass peak as function of the excess energy in the present experiment at proton beam momenta around 2.0 GeV/c, as shown in Fig. 3, a value of $a_{\eta} = (390 \pm 20) \sqrt{\text{MeV}^2/c^4}$ was extracted.

3. Data evaluation

The number of events corresponding to the $\eta$ meson production was derived from the missing mass spectra. The background underneath the $\eta$ peak, being due to the production of two to four pions, was determined using measurements below threshold. For this reason, a smoothed background measured below threshold has been shifted according to the kinematical limit of the missing mass spectra, shown as a dotted line in Fig. 2, and was scaled according to the ratio of luminosities for the beam momenta above and below threshold. The background is low enough that the approximation of the smoothed background shown by the dashed line in Fig. 2 can hardly be seen.

Due to the rapid variation of near-threshold cross sections as function of beam momentum, a high precision knowledge of the absolute value of the beam momentum is extremely crucial for the present measurements. The present “nominal” beam momenta in the range around 2 GeV/c, calculated from the synchrotron frequency and the beam orbit length, are known at COSY with an accuracy of $Dp = 10^{-3}$ [5]. The corresponding uncertainty of the total cross section amounts to values as large as $Ds = 50\%$ at $\epsilon = 2$ MeV.

When evaluating the missing mass spectra with the nominal COSY beam momenta the average of the $\eta$ meson missing mass is shifted by about $+0.66$ MeV/c compared to the $\eta$ meson mass known from literature [9], as indicated in Fig. 2 by an arrow. This discrepancy might partly be due to a systematic uncertainty in the detection system as incorrect assumptions of the magnetic fringe field or of the positions of the drift chambers ($\leq 0.28$ MeV/c²) [10]. However, the corresponding correction of the beam momentum of $\Delta p = (-1.88 \pm 0.80)$ MeV/c is in accordance with the $\Delta p/p = 10^{-3}$ uncertainty of the nominal beam momentum.

The relative uncertainty of the corrected beam momentum of $\delta p/p = 0.4 \cdot 10^{-3}$ is by a factor of...
two and a half smaller than the uncertainty of the beam momentum determined from the beam orbit
length and the frequency of the synchrotron.

In the experiment proton–proton elastic scattering was measured simultaneously. The luminosity was
determined by comparing differential counting rates with data obtained by the EDDA collaboration [11].

4. Determination of the excess energy

Above, the best value for the true beam momenta at the eight different momentum intervals was deter-
mined by shifting the extracted η meson mass to its value known from the literature [9].

In the following we present a second method for a determination of the beam momenta, where the mea-
sured dependence of the \( pp \rightarrow pp \eta \) counting rate on the beam momentum can be used to evaluate the beam momentum with high precision. The applied method is analogous to the one used by the COSY-11 collaboration for the beam energy determination in measurements of the \( pp \rightarrow pK^+\Lambda \) reaction [12] and is largely independent of systematical uncertainties due to the magnetic fringe field or drift chamber positions. However, it assumes a phase space dependence modified by final state interactions of the total cross section close to threshold. The measured yield of the \( pp \rightarrow pp \eta \) events \( N \) normalized to the integrated luminosity \( L \) is extrapolated as a function of the excess energy towards zero. The corresponding offset of the excess energy was used to correct the nominal value of the excess energy:

\[
N/L = C \cdot A(e - \Delta e) \cdot \sigma(e - \Delta e) ,
\]

where \( C \) is a normalization factor, \( A \) is the acceptance of the detection system, \( \sigma \) is the total cross section and \( \Delta e \) is the searched correction of the nominal excess energy. The values of \( C \) and \( \Delta e \) are adjusted by the fitting procedure. The efficiency \( A \) is calculated using Monte Carlo simulations of the experiment assuming a uniform phase-space distribution of reaction products modified by the \( pp \) FSI. The influence of the \( \eta p \) FSI on the acceptance is negligible.

The geometrical acceptance of the COSY-11 detection system is limited especially in the vertical direction due to the narrow opening of the dipole gap. The calculated efficiency includes also the inefficiency of detecting two close tracks due to limited double track resolution in the drift chambers equal to 3 mm.

The overall efficiency decreases from 31% to 4.4% in the range between the lowest and the highest excess energy measured. The energy dependence of the total cross section \( \sigma \) was assumed to be determined by the phase-space volume weighted by the proton–proton FSI factor, which was calculated as the squared absolute value of the complex amplitude of the proton–proton scattering amplitude in the effective range approximation with included Coulomb barrier penetration factor [13].

As shown in Fig. 4, the experimental data are well described by the applied function. The obtained correction to the excess energy is \( \Delta e = (-0.66 \pm 0.27) \text{ MeV} \), which means that the real excess energy is by that value smaller than the nominal one. The indicated error contains contributions due to final statistics of the data \( \pm 0.06 \text{ MeV} \), due to the uncertainty of the detector acceptance \( \pm 0.09 \text{ MeV} \), and due to the uncertainty of the mass of the η meson \( 547.30 \pm 0.12 \text{ MeV}/c^2 \) [9], which influences the present result via the threshold energy. The correc-

![Fig. 4. Determination of the absolute value of the beam momentum by extrapolating the \( pp \rightarrow pp \eta \) counting rate towards the threshold. The curve is given by Eq. (3), the fit results in \( \chi^2 = 1.5 \) per degree of freedom. The arrow corresponds to the resulting correction of the excess energy of \( e = (-0.66 \pm 0.27) \text{ MeV} \).](image)
tion coincidences with the value found from the shift of the missing mass peak, the latter, as a model-free measure of the true excess energy, is used in the following.

5. Total cross sections

The values of the total cross sections are given in Table 1 and are depicted in Fig. 5. The indicated vertical error bars denote the statistical uncertainty only. The overall systematical error amounts to 15%, where 10% originate from the determination of the detection efficiency and 5% from the luminosity determination. Data from measurements at SAT-URNE with the spectrometers SPES-3 [3] and PINOT [4] and at CELSIUS [1] using the PROMICE/WASA system are added in Fig. 5 to the present results. The data are consistent and determine rather precisely the excitation function in the full range of the considered excess energies up to 40 MeV.

In order to describe the shape of the energy dependence of the cross sections one can assume that it is dominated by the available phase-space weighted by the $pp$ FSI factor. The $pp\eta$ FSI can be factorized into $pp$ ($f_{pp}$) and $p\eta$ ($f_{p\eta}$) factors and integrated over the available phase-space volume $p$:

$$\sigma(\varepsilon) \sim \int f_{pp}(q_{pp}) \cdot f_{p\eta}(q_{p\eta}) \cdot f_{p\eta}(q_{p\eta}) d\rho,$$

(4)

where $q_{pp}$ is the relative momentum of two protons and $q_{p\eta}$ and $q_{p\eta}$ are the relative momenta of the $\eta$ meson with respect to first and second proton, respectively. The enhancement factors for the $\eta$–proton FSI were calculated in the effective range approximation with the complex $\eta$–proton scattering length $a_{np} = 0.717 + i0.263$ taken from Ref. [14] and the complex $\eta$–proton effective range parameter $r_{np} = -1.50 - i0.24$ from Ref. [15]. The calculations shown by the solid line in Fig. 5 describe the experimental data in the whole range of excess energy. Omission of the $\eta$–proton FSI leads to discrepancies with the data as shown by the dashed curve. At the same time, a calculation neglecting the proton–proton Coulomb interaction (dotted curve) fails to reproduce the energy dependence of the data within the limits of the relative uncertainty in the true beam energy.

6. Conclusions

The total cross section of the reaction $pp \to pp\eta$ was measured close to the kinematical threshold. The
present results together with other available data determine the energy dependence of the cross section in a wide excess energy range above threshold. This dependence is well reproduced by the phase-space integral, weighted by the full $pp\eta$ final-state interaction. In particular, inclusion of the $\eta$–proton FSI as well as the proton–proton Coulomb interaction is essential for the description of the data.

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References

Decay $\phi \to \pi^+\pi^-$


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Abstract

The process $e^+e^- \to \pi^+\pi^-$ has been studied with the SND detector at VEPP-2M $e^+e^-$ collider in the vicinity of $\phi(1020)$ resonance. From the analysis of the energy dependence of measured cross section the branching ratio $B(\phi \to \pi^+\pi^-) = (7.1 \pm 1.1 \pm 0.9) \times 10^{-3}$ and the phase $\psi_\phi = -(34 \pm 4 \pm 3)\degree$ of interference with the non-resonant $\pi^+\pi^-$ production amplitude were obtained. © 2000 Published by Elsevier Science B.V. All rights reserved.

PACS: 13.25.-k; 13.65. + i; 14.40.-n

Keywords: $e^+e^-$ collisions; Vector meson; Hadronic decay; Detector

1. Introduction

The decay $\phi \to \pi^+\pi^-$ reveals itself as an interference pattern in the energy dependence of the cross section of the process $e^+e^- \to \pi^+\pi^-$ in the region close to $\phi$ peak. The $\phi \to \pi^+\pi^-$ decay was previously studied at VEPP-2M collider [1,2] and current PDG value $B(\phi \to \pi^+\pi^-) = (8.3 \pm 3) \times 10^{-5}$ [3] is based on these results.

The decay $\phi \to \pi^+\pi^-$ violates both OZI rule and G-parity conservation. The decay amplitude in Vector Dominance Model (VDM) was calculated in [4]. The main contribution into the amplitude of the $\phi \to \pi^+\pi^-$ decay in this work comes from the electromagnetic $\phi - \rho$ mixing. The contribution of the $\phi - \rho$ transitions through the $\omega$ meson and other intermediate states such as $K\bar{K}$, $\eta\gamma$, etc. is estimated to be $\sim 20\%$ of electromagnetic $\phi - \rho$ mixing. The value of the branching ratio of the decay $\phi \to \pi^+\pi^-$ calculated from the decay amplitude obtained in the work [4] is almost 2 times higher than current PDG value [3]. Different $\phi - \omega$ mixing models were scrutinized in respect to this decay in [5]. The branching ratio calculated in this work is lower than that in [4], but discrepancy between the experimental results, especially [2], and the theoretical prediction [5] still

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exists. Possible mechanisms, which could decrease the theoretical branching ratio, are discussed in [5]. One of them is the existence of direct decay $\phi \rightarrow \pi^+ \pi^-$. 

2. Experiment

The experiments with SND detector (Fig. 1) at VEPP-2M $e^+e^-$ collider are being conducted since 1995. SND is a general purpose non-magnetic detector [6]. The main part of the SND is a 3-layer spherical electromagnetic calorimeter, consisting of 1632 NaI(Tl) crystals [7]. The solid angle of the calorimeter is $\sim$ 90% of $4\pi$ steradian. The angles of charged particles are measured by two cylindrical drift chambers covering 95% of full solid angle. The important part of the detector for the process under study is the outer muon system, consisting of streamer tubes and plastic scintillation counters.

The 1998 experiment was carried out in the energy range $2E_b = 984$–1060 MeV in 16 energy points and consisted of 2 data taking runs [8]: PHI_9801, PHI_9802. The total integrated luminosity $\Delta L = 8.6 \text{ pb}^{-1}$ collected in these runs corresponds to $13.2 \times 10^6$ produced $\phi$ mesons. The integrated luminosity was measured using $e^+e^- \rightarrow e^+e^-$ events selected in the same acceptance angle as the events of the process under study. The interference term in the $e^+e^- \rightarrow e^+e^-$ cross section due to $\phi \rightarrow e^+e^-$ decay was also taken into account. The systematic error of the luminosity measurement was estimated to be 2%.

3. Event selection

The energy dependence of the cross section of the process

$$e^+e^- \rightarrow \pi^+\pi^-$$

was studied in the vicinity of $\phi$ meson. Events containing two collinear charged particles and no photons were selected for analysis. The following cuts on angles of acollinearity of the charged particles in azimuthal and polar directions were imposed: $|\Delta \varphi| < 10^\circ$, $|\Delta \theta| < 25^\circ$. To suppress the beam background the production point of charged particles was required to be within 0.5 cm from the interaction point in the azimuthal plane and $\pm 7.5 \text{ cm}$ along the beam direction (the longitudinal size of the interaction region $\sigma_z$ is about 2 cm). The polar angles of the charged particles were required to be in the range $45^\circ < \theta < 135^\circ$, determined by acceptance angle of the muon system.

The main sources of background are cosmic muons and the following processes:

$$e^+e^- \rightarrow e^+e^-, \quad e^+e^- \rightarrow \mu^+\mu^-, \quad e^+e^- \rightarrow \pi^+\pi^-\pi^0, \quad e^+e^- \rightarrow K_S K_L.$$ 

To suppress the background from the process (2) a procedure of $e/\pi$ separation was used. It utilizes the difference in the longitudinal energy deposition profiles in the calorimeter for electrons and pions. The separation parameter was calculated for each charged particle in an event:

$$K = \log \left\{ \frac{\mathcal{P}_e(E_1,E_2,E_3,E_4)}{\mathcal{P}_\pi(E_1,E_2,E_3,E_4)} \right\},$$

where $\mathcal{P}_e(\pi)$ – the probability for an electron (pion) with the energy $E_{i}(\pi)$ to deposit the energy $E_i$ in the $i$-th calorimeter layer. $E_{i}(\pi)$ in our case is equal to the beam energy. The separation parameters distribution for both particles in collinear events with no hits

![Fig. 1. Detector SND – section across the beam; 1 – beam pipe, 2 – drift chambers, 3 – inner scintillation counters, 4 – NaI(Tl) counters, 5 – vacuum phototriodes, 6 – iron absorber, 7 – streamer tubes, 8 – outer scintillation counters](image-url)
in the muon system, is shown in Fig. 2. This distribution is asymmetric because the particles are ordered according to their energy depositions in the calorimeter. To select the events of the process (1) the cut $K_1 + K_2 < 0$ was imposed. The background from the process (2) was suppressed by a factor of $\sim 3000$, while only $7\%$ of the events of the process under study were lost. Remaining background from the process $e^+e^- \rightarrow e^+e^-$ was about $1.5\%$.

The events of the process (3) and cosmic muons can be efficiently suppressed by the muon system. We required no hits in the scintillation counters of the muon system. The efficiency of these counters was estimated using cosmic muons selected by special cuts. Due to possible admixture of beam events which actually produce no hits in the muon counters only the lower boundary of the efficiency was obtained: $99.8\%$. Thus estimated contribution of cosmic muons does not exceed $0.7\%$ of the total number of events of the process (1) and was neglected.

The energy dependences of the probabilities for muons and pions to produce hits in outer scintillation counters were obtained from the experimental data. With energy increasing from $492 \text{ MeV}$ up to $530 \text{ MeV}$ these probabilities rise from $84\%$ up to $94\%$ for muons and from $0.5\%$ to $11\%$ for pions. In the final selection of the process (1) the background from the process (3) was about $15\%$.

To suppress the resonant background from the processes (4) and (5) the following cuts on energy depositions in the calorimeter were applied:

1. the energy deposition in the first calorimeter layer of the most energetic particle in an event is less than $75 \text{ MeV}$;
2. the energy deposition in the third calorimeter layer of the least energetic particle in an event is more than $50 \text{ MeV}$.

In the events of the process (4), which satisfy the geometrical cuts, the energetic photon from $\pi^0$ decay propagates along the direction of a charged pion producing unusually large energy deposition in the first calorimeter layer for this pion. Such events are suppressed by the first cut. The pions from process (5) are relatively soft with a maximum energy of about $300 \text{ MeV}$ and low probability of significant energy deposition in the third calorimeter layer. The second cut is crucial for the rejection of the process (5). The residual cross sections of resonant background processes were estimated by Monte Carlo (MC) simulation: $0.06 \text{ nb}$ for the process (4), $0.09 \text{ nb}$ for the process (5).

To determine the remaining resonant background more accurately the selected events were divided into two data samples using the parameter $\Delta \varphi$: $|\Delta \varphi| < 5^\circ$ and $|\Delta \varphi| > 5^\circ$. The resolution in $\Delta \varphi$ is about $1^\circ$. The main part of the events of the process (1) is contained in the first sample. Due to the emission of hard photons by initial or final particles and errors in the reconstruction of the particle angles some events of the process (1) can migrate into the second sample. The level of the resonant cross section $\sigma^{\text{res}}_{1}$, determined in the second sample, was used to estimate the resonant background in the first sample: $\sigma^{\text{res}}_{1} = k \sigma^{\text{res}}_{2}$. The coefficient $k = 1.5 \pm 0.3$ was obtained by MC simulation of the processes (4) and (5), its error is determined by accuracy of simulation of energy depositions of pions in the calorimeter. Because the level of the resonant background is low, the error in $k$ does not give significant contribution into the errors of the interference parameters. The cut $|\Delta \varphi| < 5^\circ$ reduces the level of resonant background down to as low as $0.09 \text{ nb}$. This value is less than $1\%$ of the process (1) detection cross section.

Fig. 2. Distribution of the parameters $K_1$ and $K_2$ for two electrons and pions from the processes (2) and (1). Electrons occupy the top right corner while pions concentrate in the bottom left one.
The pion polar angle distribution for the process (1) at beam energy higher than 520 MeV is shown in Fig. 3. At this energy the cross sections of the resonant processes (4) and (5) are small. The additional cut on the total energy deposition in the calorimeter $E_{\text{tot}} > 400$ MeV rejects the events of the process (3). A good agreement between experimental distribution and the simulation of the process (1) shows that selected pion sample is quite pure and the level of QED background is low.

4. Data analysis

The fitting of the detection cross sections for the first and the second samples were performed simultaneously (Figs. 4 and 5). To describe the cross sections the following formulae were used:

$$
\sigma_{\text{vis}}(E) = \sigma_{\text{vis}}^1(E) + \sigma_{\text{vis}}^\mu(E) + \sigma_{\text{vis}}^e(E),
$$

$$
\sigma_{\text{vis}}^1(E) = C + D \cdot (E - m_\phi) + \sigma_{\text{res}}^1(E),
$$

$$
\sigma_{\text{vis}}^\mu(E) = k \sigma_{\text{vis}}^\mu(E),
$$

$$
\sigma_{\text{vis}}^e(E) = \varepsilon_{\text{res}} \cdot \left( 0.39 \cdot \sigma_{\phi, \rho, \omega}^\text{vis}(E) + 0.61 \cdot \sigma_{\phi, \rho, \omega}^\text{vis}(E) \right),
$$

where $E$ is the CM energy; $\sigma_{\text{vis}}^\text{vis}(E)$ – the detection cross section of the process (1); $\sigma_{\text{vis}}^\mu(E)$ – the contribution of the process (3) (this process was studied in our work [9]); $\sigma_{\text{vis}}^e(E) = 0.2(\text{nb}) \cdot (m_\phi/E)^2$ – the contribution of the process (2). The ratio 0.39:0.61 between the processes (4) and (5) was taken from the simulation. The coefficients $C$, $D$ and $\varepsilon_{\text{res}}$ were free fit parameters.

The following expression was used for $\sigma_{\text{vis}}^\text{vis}$:

$$
\sigma_{\text{vis}}^\text{vis}(E) = \sigma_0(E) \cdot R(E) \left[ 1 - \frac{m_\phi \Gamma_\phi}{\Delta_\phi(E)} \right] \beta^3(E) |F_\phi(E)|^2,
$$

$$
\sigma_0(E) = \frac{\alpha \beta^3(E) |F_\phi(E)|^2}{3 \cdot E^2},
$$

where $\alpha$ is the fine structure constant; $\beta(E) = (1 - 4 \cdot m_\phi^2/E^2)^{1/2}$; $m_\phi$, $\Gamma_\phi$, $\Delta_\phi(E) = m_\phi^2 - E^2 - iE\Gamma(E)$ – $\phi$-meson mass, width and propagator respectively; $\sigma_0(E)$ – the Born cross section of the process $e^+e^- \to \pi^+\pi^-$; $Z_\phi$ – complex parameter characterizing strength of the interference. Two representations of $Z_\phi$ are used in different works: $Z_\phi = Q_\phi e^{i\psi_\phi} = \text{Re}Z_\phi + i\text{Im}Z_\phi$. $F_\phi(E)$ is the pion form factor without $\phi$-meson contribution:

$$
|F_\phi(E)|^2 = |F_\phi^0|^2 \cdot \left[ 1 + A \cdot (E - m_\phi) + B \cdot (E - m_\phi)^2 \right],
$$

with $F_\phi^0$ as the pion form factor at the maximum of $\phi$ resonance. $Q_\phi$, $\psi_\phi$, $A$, $B$ and $|F_\phi^0|^2$ are free
fitting parameters. \( R(E) \) is a factor taking into account detection efficiency and radiative corrections:

\[
R(E) = \frac{\sigma_{\pi\pi}(E)}{\sigma(E)} \left[ 1 - Q_s e^{i\psi_s} \frac{m_\phi \Gamma_\phi}{\Delta_E(E)} \right]^2
\]  

(11)

\( \sigma_{\pi\pi} \) is the result of MC integration of differential cross section of the process (1) with all geometrical restrictions [10]. Since the probability for pions to hit the outer scintillation counters depends on energy, it was taken into account during \( \sigma_{\pi\pi} \) calculation. The remaining contributions into the detection efficiency do not depend on CM energy and pions energies and were included into \( \varepsilon_s \). The value \( \varepsilon_s = 0.234 \) was obtained using MC simulation and experimental data.

It is mainly determined by the cuts on energy depositions. Its independence of the pions energy was checked in the range 430–530 MeV using the pions from the process (4) with energies up to 450 MeV and pions from the process (1) at the beam energy 530 MeV. The geometrical cuts and the requirement on no hits in the outer scintillation counters led to 50% efficiency losses, so the total detection efficiency of the process (1) was approximately 12% at \( E = m_\phi \).

The \( R(E) \) was calculated by iteration method. As a first approximation the interference parameters \( Q_s \) and \( \psi_s \) from [2] were used. The pion form factor was taken from [11] while calculating \( \sigma_s(E) \). After fitting \( R(E) \) was recalculated with corrected \( Q_s \) and \( \psi_s \). This procedure was repeated until convergence was reached.

The branching ratio \( B(\phi \rightarrow \pi^+ \pi^-) \) is related to the interference parameters by the following formula:

\[
B(\phi \rightarrow \pi^+ \pi^-) = \frac{Q_s^2 \alpha^2 \beta^3 (m_\phi) |F_s|^2}{36 \cdot B(\phi \rightarrow e^+ e^-)}
\]  

(12)

where \( B(\phi \rightarrow e^+ e^-) = (2.99 \pm 0.08) \cdot 10^{-4} \) [3].

The fitting has been performed for each experimental run separately. The results are listed in Table 1. The fit parameters for two runs are in statistical agreement, therefore combined fit was performed to obtain the final results also listed in Table 1. The observed level of resonant background 0.07 nb is in a good agreement with the MC estimation of 0.09 nb. The fitted values of the coefficients \( A \) and \( B \) from the Eq. (10) are \( A = -(8.5 \pm 0.3) \cdot 10^{-3} \text{MeV}^{-1} \) and \( B = (4.9 \pm 1.0) \cdot 10^{-3} \text{MeV}^{-2} \).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>PHI_9801</th>
<th>PHI_9802</th>
<th>Combined</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi^2 / \text{NDF} )</td>
<td>15.3/26</td>
<td>27.5/26</td>
<td>48.4/58</td>
</tr>
<tr>
<td>(</td>
<td>F_s</td>
<td>^2</td>
<td>)</td>
</tr>
<tr>
<td>( Q_s )</td>
<td>0.069 ± 0.008</td>
<td>0.078 ± 0.007</td>
<td>0.073 ± 0.005</td>
</tr>
<tr>
<td>( \psi_s )</td>
<td>–35 ± 10°</td>
<td>–35 ± 6°</td>
<td>–34 ± 4°</td>
</tr>
<tr>
<td>( B(\phi \rightarrow \pi^+ \pi^-) )</td>
<td>((6.2 \pm 1.4) \cdot 10^{-5})</td>
<td>((8.2 \pm 1.8) \cdot 10^{-5})</td>
<td>((7.1 \pm 1.1) \cdot 10^{-5})</td>
</tr>
</tbody>
</table>
To check the accuracy of the process (3) background subtraction, the fit to the data with more stringent event selection cuts has been done. The additional requirement that the total energy deposition in the calorimeter is higher than 400 MeV significantly reduced the muon background. The obtained interference parameters:

\[ Q_e = 0.073 \pm 0.006, \quad \psi_e = -(32 \pm 5)^\circ \]

agrees well with the results from the Table 1.

The additional contribution into the shape of interference pattern may come from the process

\[ e^+ e^- \rightarrow \phi \rightarrow f_0, \gamma \rightarrow \pi^+ \pi^- \gamma \, . \quad (13) \]

The process (13) interferes with the process (1) when soft photon is emitted by pions. This contribution estimated using CMD-2 analysis of the process

\[ e^+ e^- \rightarrow \pi^+ \pi^- \gamma \, \text{in the vicinity of } \phi \, \text{resonance} \, [12] \]

does not exceed 1.5% of the interference under study. This value was included into the systematic error.

The representation \[ Z_\phi = \text{Re} Z_\phi + i \text{Im} Z_\phi \] is suitable to present the different contributions into the systematic error of the interference parameters:

1. the calculation of the radiative corrections: \[ \text{Re} Z_\phi - 5\%, \quad \text{Im} Z_\phi - 3\%; \]
2. the subtraction of the non-resonant background: \[ \text{Re} Z_\phi - 0.8\%, \quad \text{Im} Z_\phi - 0.6\%; \]
3. the contribution of the process (13): \[ \text{Re} Z_\phi - 1.5\%, \quad \text{Im} Z_\phi - 1.5\%; \]
4. the model dependence on the choice of the function approximating the pion form factor: \[ \text{Re} Z_\phi - 1\%, \quad \text{Im} Z_\phi - 8\%; \]
5. the subtraction of the resonant background: \[ \text{Im} Z_\phi - 3\% . \]

The systematic error of \[ |F_\phi|^2 \] is determined by the error of the detection efficiency \( e_\phi \) (\( \sim 5\% \)) and the accuracy of luminosity determination (2%).

The final results are the following:

\[ |F_\phi|^2 = 2.98 \pm 0.02 \pm 0.16, \]
\[ Q_e = 0.073 \pm 0.005 \pm 0.004, \]
\[ \psi_e = -(34 \pm 4 \pm 3)^\circ. \]

(14) For another representation of \( Z_\phi \) we obtained:

\[ \text{Re} Z_\phi = 0.061 \pm 0.005 \pm 0.003, \]
\[ \text{Im} Z_\phi = -0.041 \pm 0.006 \pm 0.004. \]

5. Discussion

The obtained value of the branching ratio

\[ B(\phi \rightarrow \pi^+ \pi^-) = (7.1 \pm 1.1 \pm 0.9) \cdot 10^{-5} \]

agrees well with the world average value \[ B(\phi \rightarrow \pi^+ \pi^-) = (8^{+1}_{-1}^{+3}) \cdot 10^{-5} \] [3] and has a 3 times higher accuracy. However there is a discrepancy between our result and the preliminary result of CMD-2 experiment [13]: \[ B(\phi \rightarrow \pi^+ \pi^-) = (18.1 \pm 2.5 \pm 1.9) \cdot 10^{-5}. \]

The measured value \[ \text{Im} Z_\phi = -0.041 \pm 0.006 \pm 0.004 \] agrees with the theoretical predictions [4] while the value \[ \text{Re} Z_\phi = 0.061 \pm 0.005 \pm 0.003 \] is 2.5 times lower than the expected value. The different models of the \( \phi - \omega \) mixing were examined in the work [5].

The lowest value \[ \text{Re} Z_\phi^{\text{th}} = 0.12 \] from this work also contradicts our results. This disagreement could be understood if the direct decay \( \phi \rightarrow \pi^+ \pi^- \) exists or/and in case of nonstandard \( \rho - \omega - \phi \) mixing. One can notice that the measured branching ratio of another rare decay \( \phi \rightarrow \omega \pi^0 \) [14], which violates OZI rule and G-parity, disagrees with theoretical predictions.

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Direct observation of longitudinally polarised W ± bosons

L3 Collaboration


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Abstract

The three different helicity states of $W^\pm$ bosons, produced in the reaction $e^- e^- \rightarrow W^+ W^- \rightarrow \nu \bar{\nu} q \bar{q}$ are studied using leptonic and hadronic $W$ decays at $\sqrt{s} = 183$ GeV and 189 GeV. The $W$ polarisation is also measured as a function of the scattering angle between the $W^-$ and the direction of the $e^-$ beam. The analysis demonstrates that $W$ bosons are produced with all three helicities, the longitudinal and the two transverse states. Combining the results from the two center-of-mass energies and with leptonic and hadronic $W$ decays, the fraction of longitudinally polarised $W^\pm$ bosons is measured to be $0.261 \pm 0.051_{\text{stat.}} \pm 0.016_{\text{syst.}}$ in agreement with the expectation from the Standard Model.

1. Introduction

Previous measurements of $W^+ W^-$ production at LEP have concentrated on measurements of the $W$ mass, the $W$ branching ratios, the differential and total cross sections and the anomalous couplings [1–9]. These measurements show, using the differential cross sections with respect to the $W$ production and decay angles, good agreement with theoretical calculations within the Standard Model [10–14]. This good agreement with the Standard Model indicates indirectly that $W$ bosons with all three helicities are produced in the reaction $e^- e^- \rightarrow W^+ W^-$. The primary goal of the measurement described in this paper is a quantitative and model independent analysis of all three $W$ helicity states and in particular, the direct observation of longitudinally polarised $W^\pm$ bosons. Measurements of longitudinally polarised $W$ bosons have previously been reported in the reaction $e^- e^- \rightarrow W^+ W^-$ [9] and in top decays [15]. At center-of-mass energies close to 190 GeV and within the Standard Model, one expects that about one quarter of all $W$ bosons should be longitudinally polarised [16]. Furthermore, the production of $W$ bosons with different helicities depends strongly on the $W^-$ scattering angle $\theta_{W^-}$ with respect to the $e^-$ beam direction. For example one expects for $\theta_{W^-}$ larger than 90 degrees that almost 40% of the events...
contain at least one longitudinally polarised W boson. In contrast, for $\theta_{W^-}$ between 20 and 70 degrees, the cross section is dominated by the neutrino-exchange diagram and the $W^+W^-$ should be produced dominantly with transverse polarisation. The fractions of the $W^+$ helicity states should thus also be measured as a function of $\theta_{W^-}$.

The measurement is performed with the L3 detector at LEP, using data samples of 55.5 pb$^{-1}$ and 176.4 pb$^{-1}$ collected at average center-of-mass energies of 183 GeV and 189 GeV, respectively. A detailed description of the L3 detector and its performance is given in Ref. [17–23]. The L3 detector response for $W^+W^-$ events from the KORALW [24,25] and the EEWW [26] Monte Carlo programs is simulated with the GEANT-based L3 detector simulation program [27].

2. Analysis strategy

The different $W$ helicity states result in different angular distributions of the $W$ decay products. The decay angle $\theta^*$ in the $W$ rest frame between the left-handed negatively charged lepton and the $W^-$ has a $(1 + \cos \theta^*)^2$ distribution for a $W^-$ with helicity $\mp 1$. The right-handed positively charged lepton has a $(1 + \cos \theta^*)^2$ distribution for a $W^+$ with helicity $\pm 1$. Longitudinally polarised $W$ bosons (helicity 0) result in a symmetric distribution of the decay products, proportional to $\sin^2 \theta^*$. To simplify the description of the helicity fractions, we refer in the following text only to the fractions $f_-, f_0$ and $f_+$ of the $W^-$ helicities, which includes the corresponding $W^+$ states with $f_+, f_0$ and $f_-$, respectively.

In order to study the $W$ polarisation, we use events of the type $e^+e^-\rightarrow W^+W^-\rightarrow \ell\nu q\bar{q}$ with $\ell\nu$ being either $e\nu$ or $\mu\nu$. The neutrino four-momentum vector is reconstructed from the total missing momentum vector of the event. These event samples are essentially background free and allow a measurement with good accuracy of the $W^\pm$ momentum vector, the $W$ charge and the decay angle $\theta^*$ in the $W$ rest frame.

In contrast to leptonic $W$ decays, where the decay angle $\theta^*$ of the $\ell\nu$ is well defined, the corresponding $\theta^*$ for quarks in $W$ decays has to be calculated from the hadronic decay products. To approximate the quark decay angle in the $W$ rest frame, we proceed in the following way. First, all particles besides the charged lepton and the missing neutrino in the event are associated with the hadronic decay of the $W$. We then calculate their associated four-vectors in the rest frame of the $W$ and determine the corresponding thrust axis in this rest frame. The angle $\theta_{\text{thrust}}$ of this thrust axis with respect to the $W$ momentum vector in the laboratory frame is used to describe the quark decay angle $\theta^*$ in the $W$ rest frame.

After correcting for efficiencies, the contributions from different $W$ polarisation states are obtained from a fit to the $\cos \theta^*$ distributions. For the leptonic $W$ decays the fractions $f_-, f_+$ and $f_0$ of the three $W$ helicity states are obtained from:

$$
\frac{1}{N} \frac{dN}{d\cos \theta^*} = f_- \frac{3}{8} (1 + \cos \theta^*)^2 + f_+ \frac{3}{8} (1 - \cos \theta^*)^2 + f_0 \frac{3}{8} \sin^2 \theta^*.
$$

(1)

For hadronic $W$ decays, without quark charge identification, one measures only the absolute value of the $W$ hadronic decay angle $|\cos \theta^*|$. However, this distribution can still be used to measure the fractions for the sum of the two transverse helicity states $f_0 = f_- + f_+$ and $f_0$ using:

$$
\frac{1}{N} \frac{dN}{d|\cos \theta^*|} = f_0 \frac{3}{8} (1 + \cos^2 \theta^*) + f_0 \frac{3}{8} \sin^2 \theta^*.
$$

(2)

The predictions for the compositions of $W$ helicity states as a function of the $W^-$ scattering angle $\theta_{W^-}$, following the formalism of Hagiwara et al. [16] and its implementation in the KORALW Monte Carlo program [24,25], are used as the Standard Model prediction for our analysis. The helicity composition of the total $W$ sample is extracted from a fit to the distribution of the simulated decay angles. From a fit to a KORALW Monte Carlo event sample at $\sqrt{s} = 189$ GeV, with a size 100 times larger than the data sample, the Standard Model predictions for inclusive $W$ helicity fractions $f_-, f_+$ and $f_0$ are obtained to be 56.3%, 18.0% and 25.7%, respectively. The statistical errors are smaller than 0.5%.

Within the statistical errors, the same fractions are found from a WW event sample generated with the EEWW Monte Carlo program [26] which uses the
zero total W width approximation and assigns the W helicities on an event by event basis. The W helicity fractions obtained from the fit to the decay angle distributions agree, within statistical errors smaller than 0.9%, with the generated W helicity fractions. This shows that the Born level formulæ (1) and (2) are applicable after radiative corrections.

3. Selection of $W^+W^-\rightarrow e(\mu)\nu q\bar{q}$ events

The selection of $W^+W^-\rightarrow e(\mu)\nu q\bar{q}$ events is similar to the selections described in our previous publications on WW final states [1,2]. However, in order to assure well measured W production and decay angles, more restrictive criteria are used. Charged leptons are identified using their characteristic signatures. Electrons are identified as isolated energy depositions in the electromagnetic calorimeter with electromagnetic shower shape which are matched in azimuth to a track reconstructed in the central tracking chamber. The energy and direction of electrons are measured using the electromagnetic calorimeter, while the charge is obtained from the associated track. Muons are identified and measured with tracks reconstructed in the muon chambers which point back to the interaction vertex. All other energy depositions in the calorimeters are assumed to originate from the hadronic W decay. The neutrino momentum vector is set equal to the total missing momentum vector of the event. In addition the following criteria are used for the selection of $W^+W^-\rightarrow e(\mu)\nu q\bar{q}$ events:

- The reconstructed momentum should be greater than 20 GeV for electrons and 15 GeV for muons.
- The momentum of the neutrino should be greater than 10 GeV and its polar angle, $\theta_\nu$, has to satisfy $|\cos\theta_\nu| < 0.95$.
- The invariant mass of the $e\nu$ system should be greater than 60 GeV.
- The invariant mass of the hadronic system should be between 50 and 110 GeV.

Using these criteria, 81 and 288 events of the type $W^+W^-\rightarrow e\nu q\bar{q}$ are selected at center-of-mass energies of 183 GeV and 189 GeV, respectively. The corresponding event numbers for $\mu\nu q\bar{q}$ are 67 and 262 events. Adding the electron and muon event samples together, we find 68 and 280 $e^\pm$ events and 80 and 270 $\mu^\pm$ events, respectively, in the 183 GeV and 189 GeV data samples. These samples have a purity of 96%, where the background from $W^+W^-\rightarrow \tau\nu q\bar{q}$ with leptonic $\tau$ decays and the background from $e^+e^-\rightarrow$ hadrons contribute each about 2%.

The measured $\cos\theta_\nu$ distribution is found to be in good agreement with the MC expectations, as shown in Fig. 1 for events with electrons and with muons for the 189 GeV data sample. About 5% of the accepted events with electron candidates have a wrongly assigned charge. Charge confusion is insignificant for events with muons. The charge confusion depends on the reconstructed W scattering angle and is largest for W bosons with small scattering angle with respect to the beam direction. This results in a small misassignment between W bosons with helicity $+1$ and -1 but has negligible effects for the fraction of longitudinally polarised W bosons, which is essentially independent of the charge assignment.

4. Analysis of the W helicity states

After subtracting the backgrounds from the data, the fractions of the W helicity states are measured from the distributions $dN/d\cos\theta_\nu$ and $dN/d|\theta_\nu|$. 

Fig. 1. The $\cos\theta_\nu$ distribution for (a) $W^+W^-\rightarrow e\nu q\bar{q}$ and (b) $W^+W^-\rightarrow \mu\nu q\bar{q}$ events from the $\sqrt{s} = 189$ GeV data (points) and the KORALW Monte Carlo prediction (histogram).
\[ \cos \theta_{\text{Thrus}} \] for the leptonic and hadronic W decay angle and as a function of the scattering angle \( \theta_{yy} \).

To extract the W helicity fractions, the observed distributions are corrected for the selection efficiencies which are obtained as a function of \( \cos \theta^* \). To take into account possible deviations between the helicity fractions in the data and Monte Carlo as a function of \( \cos \theta_{yy} \), the data are corrected differentially using 9 bins of the \( \cos \theta_{yy} \) scattering angle. For each \( \cos \theta_{yy} \) bin, the efficiency is obtained as a function of \( \cos \theta^* \) using the ratio of the reconstructed and the generated \( \cos \theta^* \) distributions for the leptonic and hadronic W decays. The measured \( \cos \theta^* \) distributions for the corresponding \( \cos \theta_{yy} \) bins in the data are corrected and combined.

The efficiency corrections are obtained from large samples of fully simulated KORALW Monte Carlo events. Using these Monte Carlo events we have studied the accuracy with which we reconstruct the \( \theta^* \) decay angles. The study shows that \( \theta^* \) is reconstructed with a standard deviation of 9.2 degrees and a small shift of \( +3.2 \) degrees for the leptonic W decays. For hadronic W decays one finds that \( \theta^* \) is reconstructed with a standard deviation of 12.0 degrees and a shift of \( +3.3 \) degrees.

The bias and sensitivity loss due to the efficiency corrections and the \( \theta^* \) resolution has been determined with fully simulated and reconstructed Monte Carlo events where the generated W helicity fractions have been varied over a large range. This was done both with the EEW\( \mathcal{W} \) Monte Carlo program, where the generated W helicities are known on an event by event basis and with the KORALW Monte Carlo using a weighting method to assign the W helicities on a statistical basis, ignoring W spin correlations.

Averaging both Monte Carlo estimates one finds that leptonic W decays with 100% helicity \( -1 \) states would be measured to consist of 94% of helicity \( -1 \) and 6% helicity 0 states while a W sample with 100% helicity \( +1 \) would be reconstructed to consist of more than 99% of helicity \( +1 \) states. Similar numbers are found if one starts with 100% helicity 0, which would be measured with 92% helicity 0, 3% helicity \( -1 \) and 5% helicity \( +1 \). The corresponding numbers for hadronic W decays are that 94% of W bosons with helicity \( \pm 1 \) and 85% of W bosons with helicity 0 are correctly reconstructed. The study has been repeated as a function of \( \cos \theta_{yy} \) and within the statistical errors the results are the same as the ones from the total W sample. To obtain a correction function for the bias and the efficiency loss, the fraction \( f_0 \) has been varied between 0 and 100%. A linear relation between the generated and the fitted W helicity fractions is found.

### 4.1. Results and systematics

These efficiency corrected \( \cos \theta^* \) distributions are used to extract the W \( \pm \) helicity fractions. The results of the binned \( \chi^2 \) fits to these distributions for leptonic and hadronic W decays from the \( \sqrt{s} = 189 \) GeV data are shown in Fig. 2. No constraint on the total cross section is applied and one finds that the data are well described only if all three W helicity states are used in the fit. Fits which include only \( -1 \) and \( \pm 1 \) helicities, as also shown in Fig. 2, fail to describe the data. For leptonic W decays one finds that the \( \chi^2 \) increases from 7.1 for seven degrees of freedom if all three W helicity states are included to 17.8 for eight degrees of freedom if only helicity \( -1 \) and \( +1 \) are used to describe the data. For hadronic W decays the \( \chi^2 \) increases from 9.8 for eight de-
The fraction of longitudinally polarised W bosons in the $\sqrt{s} = 189$ GeV data is measured to be $0.220 \pm 0.077$ for the leptonic decays and $0.285 \pm 0.084$ for hadronic decays. The fractions for the different W helicity states, together with the Standard Model Monte Carlo expectations, are given in Table 1 for the $\sqrt{s} = 189$ GeV and $\sqrt{s} = 183$ GeV data. The observed fractions of longitudinally polarised W bosons measured with leptonic and hadronic W decays agree with each other and with the Standard Model expectation of 0.26 and differ from zero by several standard deviations.

Systematic studies have been performed to verify the stability of the fit results with respect to the fraction of longitudinally polarised W bosons. We have investigated (1) uncertainties due to backgrounds, (2) efficiencies and selection criteria, (3) the hadron energy response functions of the electromagnetic and hadronic calorimeters, (4) the difference between the differential and overall efficiency corrections and (5) a method where the fraction $f_0$ has been obtained directly from a fit to the measured $\cos \theta^*$ distributions using the Monte Carlo shape from the different W helicity states after the reconstruction.

The analysis has been repeated assuming large relative background uncertainties of $\pm 50\%$ from either the hadronic background or from misidentified $W \rightarrow t\bar{t} \nu$ decays. Using these modifications the measured fractions of longitudinally polarised W bosons is found to vary by at most 0.012 for leptonic W decays and by 0.004 for the hadronic W decays. The hadron energy measurement is obtained from a combination of the energy deposited in the electromagnetic and hadron calorimeter multiplied by calibration constants which take the average calorimeter $e^+/\text{hadron}$ response function into account. These calibration constants have been varied over a wide range while demanding that the average of the recon-

<table>
<thead>
<tr>
<th>$\sqrt{s}$ (GeV)</th>
<th>Helicity W $\rightarrow e\nu$</th>
<th>Helicity W $\rightarrow$ hadrons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$-1$</td>
<td>$+1$</td>
</tr>
<tr>
<td></td>
<td>$\pm 1$</td>
<td>$0$</td>
</tr>
<tr>
<td>189 Data</td>
<td>0.568 $\pm$ 0.071</td>
<td>0.212 $\pm$ 0.046</td>
</tr>
<tr>
<td>MC</td>
<td>0.56</td>
<td>0.18</td>
</tr>
<tr>
<td>183 Data</td>
<td>0.56 $\pm$ 0.14</td>
<td>0.10 $\pm$ 0.08</td>
</tr>
<tr>
<td>MC</td>
<td>0.53</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Table 2
Measurements of the fraction of longitudinally polarised W bosons for leptonic and hadronic W decays from the $\sqrt{s} = 189$ GeV data sample investigating various sources of systematics.

<table>
<thead>
<tr>
<th></th>
<th>W $\rightarrow e\nu$</th>
<th>W $\rightarrow$ hadrons</th>
<th>average</th>
</tr>
</thead>
<tbody>
<tr>
<td>standard method</td>
<td>0.220 $\pm$ 0.077</td>
<td>0.285 $\pm$ 0.084</td>
<td>0.252 $\pm$ 0.057</td>
</tr>
<tr>
<td>background corrections</td>
<td>0.209–0.232</td>
<td>0.282–0.286</td>
<td>0.241–0.258</td>
</tr>
<tr>
<td>efficiency uncertainty</td>
<td>0.214</td>
<td>0.279</td>
<td>0.247</td>
</tr>
<tr>
<td>calorimeter calibration (hadrons)</td>
<td>0.197–0.215</td>
<td>0.282–0.300</td>
<td>0.244–0.254</td>
</tr>
<tr>
<td>integrated efficiency correction</td>
<td>0.233</td>
<td>0.268</td>
<td>0.250</td>
</tr>
<tr>
<td>analysis method</td>
<td>0.237</td>
<td>0.279</td>
<td>0.258</td>
</tr>
</tbody>
</table>
structured masses for leptonic and hadronic $W$ decays agree within better than $\pm 3$ GeV with an average $W$ mass of 80.4 GeV. Since the neutrino momentum vector is reconstructed from the observed missing momentum vector, correlations exist between the reconstructed decay angles in the hadronic $W$ system and the corresponding leptonic $W$ system. For example, a particular choice of the energy calibration constants reduces the fraction of longitudinally polarised $W$ bosons by 0.024 as seen with the leptonic $W$ decays but increases the corresponding fraction for the hadronic decays by 0.015.

Similar variations in the fraction of longitudinally polarised $W$ bosons have been seen with the other systematic studies, as summarised in Table 2. Assuming that the variations given in Table 2 are all due to systematics and adding them in quadrature, a systematic error of $\pm 0.034$, $\pm 0.024$ and $\pm 0.016$ is assigned to the fraction of longitudinally polarised $W$ bosons measured with leptonic, hadronic decays and for the combined measurement, respectively.

Combining the results from the $\sqrt{s} = 183$ GeV and 189 GeV, ignoring the slight energy dependence of the $W$ helicity fractions expected from the Standard Model, the fraction of longitudinally polarised $W$ bosons is measured to be

$$f_0 = 0.261 \pm 0.051(\text{stat.}) \pm 0.016(\text{syst.})$$

and agrees with expectation from the Standard Model of 0.26.
Table 3
Same as Table 1, except in this case the helicity fractions are given as a function of \( \cos \theta_{W^-} \) and combining the \( \sqrt{s} = 183 \) GeV and \( \sqrt{s} = 189 \) GeV data and Monte Carlo.

<table>
<thead>
<tr>
<th>( \sqrt{s} ) (GeV)</th>
<th>( \cos \theta_{W^-} )</th>
<th>Helicity ( W \rightarrow \bar{\nu} )</th>
<th>Helicity ( W \rightarrow \text{hadrons} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(-1.0 \pm 0.4)</td>
<td>(-0.4 \pm 0.3)</td>
<td>(0.3 \pm 1.0)</td>
</tr>
<tr>
<td>data</td>
<td>0.27 \pm 0.12</td>
<td>0.40 \pm 0.09</td>
<td>0.66 \pm 0.08</td>
</tr>
<tr>
<td>(183 + 189)</td>
<td>0.45 \pm 0.22</td>
<td>0.23 \pm 0.08</td>
<td>0.08 \pm 0.04</td>
</tr>
<tr>
<td>KORALW MC</td>
<td>0.28 \pm 0.23</td>
<td>0.57 \pm 0.12</td>
<td>0.26 \pm 0.08</td>
</tr>
<tr>
<td>data</td>
<td>(0.87 \pm 0.28)</td>
<td>(0.94 \pm 0.16)</td>
<td>(0.75 \pm 0.11)</td>
</tr>
<tr>
<td>(183 + 189)</td>
<td>(0.13 \pm 0.42)</td>
<td>(0.42 \pm 0.29)</td>
<td>(0.23 \pm 0.09)</td>
</tr>
</tbody>
</table>

As mentioned in the introduction, it is interesting to measure the \( W \) helicity fractions as a function of the \( W^- \) scattering angle \( \theta_{W^-} \). Thus the fits are repeated for different ranges of \( \cos \theta_{W^-} \). The \( \cos \theta_{W^-} \) ranges are selected such that the contributions from the transversely polarised \( W \) bosons should be either suppressed or enhanced as shown in Figs. 3 and 4.

To obtain quantitative numbers for the \( W \) helicity fractions as a function of \( \cos \theta_{W^-} \) the data from the two different center-of-mass energies are combined and the helicity fractions are measured for three bins of \( \cos \theta_{W^-} \). The bins are chosen such that large variations of the different helicity fractions are expected [16] yet keeping a sufficient statistical significance. The results, given in Table 3, agree with the Standard Model expectations and demonstrate that the fraction of \( W \) bosons with helicity \(-1\) depends on the \( W \) scattering angle as shown in Fig. 5.

In summary, all three \( W \) boson helicity states, the two transverse as well as the longitudinal ones are observed with fractions in agreement with Standard Model expectations. The production of longitudinally polarised \( W \) bosons is thus directly observed with a significance of five standard deviations.

Acknowledgements

We would like to thank F. Jegerlehner, Z. Kunszt, Z. Wgs and D. Zeppenfeld for interesting discussions about the physics of \( W \) production.

We also wish to express our gratitude to the CERN accelerator divisions for the excellent performance of the LEP machine. We acknowledge the contributions of the engineers and technicians who have participated in the construction and maintenance of this experiment.

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\[ \Lambda_b \] polarization in $Z^0$ decays at LEP

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Abstract

The longitudinal polarization of the $A_b$ baryon is measured at the LEP $e^+e^-$ collider by DELPHI. It is determined from the charged lepton and neutrino energy spectra in 249 $\pm$ 19 $A_b$ semileptonic decays reconstructed in = 3.5 million hadronic $Z^0$ decays using $A_b^0$-lepton correlations. The measured polarization is: $P_{A_b} = -0.49^{+0.17}_{-0.30}$ (stat. $\pm$ 0.17 (syst.) © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

A measurement of the $A_b$ baryon polarization at the LEP $e^+e^-$ collider is presented using the hadronic $Z^0$ decays collected by DELPHI in the years 1992–1995. Semileptonic $A_b$ decays are reconstructed inclusively looking for the $L_0l$ energy in 249 $L_0l$ semileptonic decays reconstructed in $f_{3.5}$ million hadronic $b$ decays using $L_0$-lepton correlations. The measured polarization is: $P_{A_b} = -0.49^{+0.17}_{-0.30}$ (stat. $\pm$ 0.17 (syst.) © 2000 Elsevier Science B.V. All rights reserved.

A large longitudinal polarization of the $A_b$ is a direct consequence of the polarization of primary $b$ quark coming from a $Z^0$ decay. The polarization of fermions produced in the reaction $e^+e^- \rightarrow Z^0 \rightarrow ff$ is precisely predicted in the framework of the SM (Standard Model). In the case of unpolarized $e^+e^-$ beams the average longitudinal polarization of a $b$ quark is predicted to be [1,2]:

$$\langle P_b \rangle = -0.94$$

(1)

Neither quark nor photon radiation from the final state are predicted to degrade this high polarization significantly. One-loop QCD mass effects reduce it by an amount of 3% only [3,4]. The first possibility of altering the primary $b$ quark spin state arises during and after hadronization.

In the Heavy Quark Effective Theory approximation (HQET) [5] the spin degrees of freedom of a heavy quark are decoupled from a spin-zero light diquark. Therefore in the heavy quark limit a $b$ quark hadronizing directly to a $A_b$ should pass its complete initial polarization to the baryon and then conserve it throughout the whole $A_b$ lifetime.

However, $b$ quark fragmentation into $\Sigma_b$ and $\Sigma_b^*$ states which subsequently decay strongly into a $A_b$ can lead to a substantial depolarization of the heavy quark if the two $\Sigma_b^{(*)}$ states live long enough to form distinct narrow resonances. A detailed discussion of different scenarios of the indirect hadronization is given in [6,7]. Fig. 1 shows the prediction for the effective $A_b$ polarization as a function of the fraction of $A_b$’s produced indirectly through $\Sigma_b$ and $\Sigma_b^*$ states ($f_{\Sigma_b}$). The prediction holds only if $\Sigma_b$ and $\Sigma_b^*$ are distinct and narrow resonances.

Fig. 1. Theoretical prediction for the effective $A_b$ polarization as a function of the fraction of $A_b$’s produced indirectly through $\Sigma_b$ and $\Sigma_b^*$ states ($f_{\Sigma_b}$). The prediction holds only if $\Sigma_b$ and $\Sigma_b^*$ are distinct and narrow resonances.

In the Born approximation of the free quark semileptonic decay $b \rightarrow c + l + \bar{\nu}$ the matrix element exhibits a factorisation of the spin direction component [11,12]:

$$|\mathcal{M}|^2 \equiv |\mathcal{M}_{\text{pol}}|^2 (1 + P \cos \theta)$$

(2)
where \( P \) denotes the \( b \) polarization and \( \theta \) is the angle between the neutrino three-momentum and the spin quantization axis in the \( b \) rest frame. \( |\mathcal{M}_{\text{unpol}}|^2 \) is the decay matrix element of the unpolarized \( b \). QCD correction terms violate the factorisation (2) only at the percent level \([11–13]\). Being very small compared to the present experimental accuracy they were considered negligible. It can be also argued that when going to real heavy baryon decays, the dynamics of the reaction \( A_b \rightarrow \Lambda_c^+ l^- \nu \) remains identical with the free quark case discussed above \([14]\). This approximation is derived from the leading order of the HQET and the remaining mass corrections are negligibly small \([15,16]\).

The \( \bar{b} \) quarks are produced with polarization opposite to that of the \( b \) quarks (\( \bar{b} \) has positive polarization). However from the CP invariance of the weak decay \( A_b \rightarrow \Lambda_c^+ l^- \nu \) the final differential distributions of the charged lepton and neutrino look the same after the charge conjugation operation. Consequently, both \( A_b \) and \( \Lambda_b \) semileptonic decays lead to the same momentum distributions of the decay products. Hereafter, the antiparticles are always implied.

The paper is organised as follows. The experimental method is explained in Section 2. Section 3 briefly describes the DELPHI spectrometer. Section 4 contains a description of the analysis procedure: the \( A_b \) signal selection (Section 4.1), (Section 4.2), (Section 4.3), the possible background sources (Section 4.4) and the reconstruction of the neutrino energy which carries most sensitivity to the \( A_b \) polarization (Section 4.5). The results of the measurement and a discussion of the systematic uncertainties are presented in Section 5. This section also presents the result of the analogous polarization measurement performed on \( B \) mesons, serving as a consistency check of the analysis. The conclusions are given in Section 6.

2. Principles of the measurement

\( A_b \) polarization is studied in its semileptonic decays with a \( \Lambda^0 \) reconstructed in the final state. These decays have the following properties: the lepton is highly energetic and has high transverse momentum relative to the jet axis and the \( \Lambda^0 \) has a harder momentum spectrum than the \( \Lambda^0 \) baryons produced from fragmentation. Moreover, \( \Lambda^0 l^\pm \) pairs originating from a \( b \) baryon cascade have a well defined correlation between the lepton charge and the \( \Lambda^0 \) baryonic number. For brevity it will be called charge correlation. The \( b \) baryon signal is uniquely related to \( \Lambda^0 l^- \) (or \( \bar{\Lambda}^0 l^- \)) correlations, hereafter called right-sign (R.S.). \( \Lambda^0 l^+ \) (or \( \bar{\Lambda}^0 l^+ \)) correlations, hereafter called wrong-sign (W.S.), have a purely background origin. As will be shown in Section 4.4, the great majority of background events have no physically preferred charge correlation and therefore are equally distributed among the two classes. Hence, the excess of right-sign events over wrong-sign ones is attributed to the semileptonic decays of the \( A_b \) baryon.

Neither the \( A_b \) four-momentum nor the neutrino four-momentum can be fully reconstructed in the experiment. However, \( A_b \) baryons produced at LEP are highly boosted in the laboratory frame. In such a case the forward-backward asymmetry of a decay product can be directly expressed in terms of a shift in the average value of its energy. The charged lepton also carries a residual sensitivity to the \( A_b \) polarization. It is not explicit in formula (2) but arises as a reflection of the neutrino dependence from the four-momentum conservation. It follows that the average energies of the charged lepton, \( \langle E_l \rangle \), and the neutrino, \( \langle E_\nu \rangle \), are respectively anti-correlated and correlated with the polarization. Hence, the quantity defined as:

\[
y = \frac{\langle E_l \rangle}{\langle E_\nu \rangle}
\]

is highly sensitive to \( A_b \) polarization and is explicitly independent of fragmentation uncertainties \([17]\).

In reality the observed energy spectra undergo several deformations because of detector response and selection cuts. To correct for these effects the variable \( y \) obtained from the data was normalised to the one extracted from a sample of unpolarized simulated events. Therefore, the final variable is defined:

\[
R = \frac{y_{\text{DATA}}}{y_{\text{MC}}}
\]
Plots in Fig. 2 show the actual experimental response of polarization observables to the $\Lambda_b$ polarization after the entire reconstruction as obtained from the background-free simulation described in Section 4. Plots (a) and (b) were fitted with linear functions, as expected from the theory. Because of the constraint ($f(0) = 1$) there was only one free parameter in each fit. The curve in Fig. 2c representing the ratio of (a) and (b) resulted from a fit to the $y_{p=0}$ points with the function $f(x) = \frac{P_y}{P_x + P_T}$. This calibration curve will be used to extract the polarization of $\Lambda_b$ after determining the value of $R_y$ from data.

Other polarization observables proposed in [18] ($y_2 = \langle E_r^2 \rangle / \langle E_r^2 \rangle$ and $y_3 = \langle E_r / E_\nu \rangle$) were also investigated. No improvement in the sensitivity to polarization was observed. None of these discriminating variables is a priori guaranteed to be well reproduced in the simulation and hence they can be a potential source of systematic uncertainty. Only systematics related to the chosen $y$ variable were studied in detail. In addition to all systematic uncertainties present already in $y$, the $y_2$ variable exhibits dependence on the energy spectra widths and $y_3$ is sensitive to event-by-event lepton neutrino correlations.

All proposed approaches require a good knowledge of the escaping neutrino energy $E_\nu$. In the LEP environment such a determination is achievable using the hemisphere missing energy method described in detail in Section 4.5.

Polarization of the $A^0$ from the cascade although experimentally accessible does not have a direct simple connection to the $\Lambda_b$ polarization. Correlation between the two polarizations depends strongly on the $A_s$ decay channel as well as on the possible existence of heavier baryonic resonances in the decay cascade. Therefore, information coming from the $A^0$ polarization was not used in this measurement.

![Fig. 2](image-url)  
Fig. 2. Dependence of the polarization observables on the $\Lambda_b$ polarization as reconstructed in the simulation after the whole analysis procedure. The quantities are normalised to the unpolarized case. (a) charged lepton energy $\langle E_r \rangle / \langle E_r \rangle_{p=0}$; (b) neutrino energy $\langle E_\nu \rangle / \langle E_\nu \rangle_{p=0}$; (c) $y$ variable $y / y_{p=0}$. The dashed lines are fits to the simulation points as described in the text. The strong dependence on polarization of the neutrino mean energy is a direct consequence of Eq. (2). The residual sensitivity of the charged lepton comes from the four-momentum conservation in the $l\nu\Lambda_b$ system. It is therefore diluted by the three-body decay kinematics.
3. The DELPHI spectrometer

A complete description of the DELPHI spectrometer and its performance can be found in [19] and [20]. In this section only the characteristics most relevant for this analysis are summarised.

The detector elements used for tracking were the Vertex Detector (VD), the Inner Detector (ID), the Time Projection Chamber (TPC) and the Outer Detector (OD). In this central region, a highly uniform magnetic field of 1.23 T parallel to the $e^+e^-$ beam direction was provided by the superconducting solenoid. Charged particle tracks were reconstructed with a precision $\sigma_p/p < 2.0 \times 10^{-3} (p \text{ in GeV}/c)$ in the polar angle region $25^\circ < \theta < 155^\circ$. In the forward region there were two additional tracking devices: Forward Chambers A (FCA) and Forward Chambers B (FCB). The sensitive area of these drift chambers covered polar angles $11^\circ \leq \theta \leq 36^\circ$ and $144^\circ \leq \theta \leq 169^\circ$.

Calorimeters detected photons and neutral hadrons by the total absorption of their energy. Electromagnetic calorimeters served also as the main devices for lepton identification (see Section 4.1). The electromagnetic calorimetry system of DELPHI was composed of a barrel calorimeter, the HPC, covering the polar angle region $46^\circ < \theta < 134^\circ$, and forward calorimeters, the FEMC, for polar angles $8^\circ < \theta < 35^\circ$ and $145^\circ < \theta < 172^\circ$. The relative precision on the measured energy $E$ was parametrised as $\sigma_E/E = 0.32/\sqrt{E} \oplus 0.043$ (E in GeV) in the barrel, and $\sigma_E/E = 0.12/\sqrt{E} \oplus 0.03$ (E in GeV) in the forward region. The hadron calorimeter, HCAL, was installed in the return yoke of the DELPHI solenoid. In the barrel region, the energy was reconstructed with a precision of $\sigma_E/E = 1.12/\sqrt{E} \oplus 0.21$ (E in GeV).

Muon identification was provided by the muon chambers. In the barrel region they consisted of three layers covering the polar angle regions $53^\circ < \theta < 88.5^\circ$ and $91.5^\circ < \theta < 127^\circ$. The first layer contained three planes of chambers and was inside the return yoke of the magnet after 90 cm of iron, while the other two, with two chamber planes each, were mounted outside the yoke behind a further 20 cm of iron. In the endcaps there were two layers of muon chambers mounted one outside and one just inside the return yoke of the magnet. Each consisted of two planes of active chambers covering the polar angle regions $20^\circ < \theta < 42^\circ$ and $138^\circ < \theta < 160^\circ$ where the charged particle tracking was efficient.

4. Analysis procedure

The analysis was based on $3.5 \times 10^6$ hadronic $Z^0$ decays collected by the DELPHI detector in the 1992 to 1995 data-taking periods. A large sample of background-free $b \rightarrow L_{\mu 3}$ simulated events was used to determine the calibration curve of Fig. 2c and to evaluate the reference value $y_{MC}$ of the quantity $y$ (see formulae (3) and (4)). From this sample over 2000 candidates for the cascade decay $L_{\mu 3} \rightarrow l\nu\ell^0X$ remain after the whole reconstruction and the complete analysis selection. To cross-check the signal selection and result extraction, a sample of $5.5 \times 10^6$ simulated hadronic $Z^0$ events (unbiased $q\bar{q}$) was used. In both cases events were generated using the JETSET [10] generator with parton shower option and the DELPHI tuning [21]. The $L_{\mu 3}$ semileptonic decays were generated explicitly unpolarized and without QCD corrections, i.e. according to: $|\mathcal{M}|^2 = (\mathbf{e}\ell\mathbf{b}\nu)$. The $L_{\mu 3}$ polarization in the background-free signal sample was then simulated by reweighting events according to the approximation of Eq. (2).

4.1. Lepton identification

Lepton identification in the DELPHI detector was based on the electromagnetic calorimeters (for electrons) and the muon chambers (for muons). Therefore, the angular coverage of the identification was limited by the acceptance of the above devices (see Section 3). Only particles with momentum larger than 3 GeV/c were considered as possible lepton candidates.

The $\chi^2$ of the match between the track extrapolation to the muon chambers and the observed hits gave the probability of the lepton candidate being a muon. With the selections applied, inside the angular acceptance of the muon chambers the muon identification efficiency was $(95 \pm 1)\%$ and the hadron misidentification probability $(1.5 \pm 0.1)\%$.

The probability of a lepton candidate being an electron was calculated using a comparison between its momentum reconstructed in the tracking devices and the energy of associated electromagnetic shower.
reconstructed in the HPC or FEMC. In the HPC a fit to the longitudinal profile of the electromagnetic shower was performed as well. An independent \(dE/dX\) measurement in the TPC leads to additional \(e - \pi\) separation. With the selections applied and inside the angular acceptance of the HPC and FEMC, the electron identification efficiency was found to be \((55 \pm 1)\%\) and the hadron misidentification probability \(0.4\%\).

Lepton candidates selected using the identification tools described above had to satisfy further quality requirements: track length > 30 cm, relative error on momentum < 25\%, impact parameter to the interaction point < 4 cm in the plane transverse to the beam and < 10 cm in the beam direction. In addition, electron candidates were rejected in cases where they matched the photon conversion hypothesis tagged by a successful vertex fit with an oppositely charged electron candidate giving an invariant mass below 20 MeV/\(c^2\).

4.2. \(\Lambda^0\) reconstruction

\(\Lambda^0\) candidates were reconstructed in the channel \(\Lambda^0 \rightarrow p\pi^-\). The reconstruction of the \(V^0\) vertex and selection cuts are described in detail in Ref. [20]. The \(\Lambda^0 \rightarrow p\pi^-\) reconstruction efficiency depended strongly on the \(\Lambda^0\) momentum, and varied between 35\% and 10\%.

In the analysis presented here only \(\Lambda^0\) candidates with \(p_A > 5\) GeV/\(c\) were selected. This requirement suppresses the large background due to low energy \(\Lambda^0\)'s from fragmentation. To extract the signal of the \(\Lambda_b\) baryons, \(\Lambda^0\) candidates with an invariant mass of the \(p\pi^-\) system within two standard deviations from the nominal \(\Lambda^0\) mass were used. The \(\Lambda^0\) decay product with the higher momentum was assumed to be the proton. Its charge determined the \(\Lambda^0\) baryon number.

4.3. \(\Lambda_b\) signal selection

All events had to satisfy the selection criteria defining hadronic events from \(Z^0\) decays, requiring a charged multiplicity greater than four and a total energy of charged particles greater than \(0.12\sqrt{s}\), where \(\sqrt{s}\) was the centre-of-mass energy and all particles were assumed to be pions; charged particles were required to have a momentum greater than 0.4 GeV/\(c\) and a polar angle between 20\° and 160\°. The overall trigger and selection efficiency was over 95\%. The background, mainly from \(\tau^+\tau^-\) pairs with a smaller contribution from \(\gamma\gamma\) collisions, was below 0.7\% [20]. Additionally, events were dropped when the central tracking detectors (in particular TPC) and both electromagnetic and hadronic calorimeters were not fully operational. In total 3,498,225 events were selected for analysis.

Events were subdivided into two hemispheres by a plane perpendicular to the thrust axis and containing the interaction point. Each event was required to have the thrust axis more than 30\° from the beam directions since for the missing energy measurement it was essential to have events well contained in the detector fiducial volume where the reconstruction efficiency is high and well controlled. In order to suppress events with hard gluon radiation the calculated thrust value was required to exceed 0.75. The total visible energy in an event had to be between 30 GeV and 130 GeV.

Events with the combination of a charged lepton and a \(\Lambda^0\) in the same hemisphere were searched for. The initial sample of \(\Lambda^0l\) pairs still contained a large fraction of background events mainly due to \(\Lambda^0\) baryons from fragmentation and from non-\(b\) events. To reduce this background the following kinematic selections were applied:

- The transverse momentum of the lepton to the nearest particle jet, \(p_T\), was required to be greater than 0.8 GeV/\(c\). The LUCLUS jet finding algorithm [10] was used with \(d_{\text{join}} = 2.5\) GeV and excluding the lepton from the jet.
- The invariant mass of the \(\Lambda\) and the lepton had to lie in the range 2.1 to 4.5 GeV/\(c^2\).
- The momentum of the \(Al\) system had to exceed 11.0 GeV/\(c\).
- The angle between the lepton momentum direction and the \(\Lambda\) momentum direction could not be larger than 90\°.
- The angle between the momentum of the \(Al\) system and the thrust direction was required to be smaller than 45\°.

The first selection enriches the sample in leptons from semileptonic \(b\) decays. The next two cuts
suppress contribution from $\Lambda_c$ semileptonic decays and accidental combinations. The last two selections are fairly loose and mainly guard against accidental combinations where either the lepton or the $\Lambda^0$ belongs to a distinct hard gluon jet.

An algorithm to tag $b$ quark decays was also applied. This is based on the long $b$ hadron lifetimes and uses the large track impact parameters of the decay products [20]. The output from the $b$ tagging algorithm is expressed in terms of the probability that all charged particle tracks originate from a common primary vertex. $b$ events have their probabilities strongly peaked at zero while light quark ones have probabilities uniformly distributed from zero to one. The cut $P_{b\text{-TAG}} < 0.05$ was applied to the selected event sample. This cut suppresses about 50% of the background but only about 15% of the signal which corresponds to a drop of the background fraction from 56% to 41%. Most of the remaining background comes from $B$ events.

The overall efficiency to reconstruct the decay $\Lambda_c^0 \to l^- \nu_l \Lambda^0 X$ (where all decay modes for the $\Lambda^0$ were assumed) was found to be $0.030 \pm 0.001$ in the simulation. However, the actual knowledge of this efficiency is not needed for the polarization measurement. Only the signal purity was used and was measured using the data, as will be shown later.

The reconstructed $\Lambda^0$ mass distributions for right-sign and wrong-sign $\Lambda^0 l$ charge correlations observed in the data after the $b$-tagging cut are shown in Fig. 3. The excess of right-sign correlations over wrong-sign ones in the $\Lambda^0$ mass peak amounts to $249 \pm 19$ $\Lambda_b$ candidates. The width of the $\Lambda^0$ mass acceptance window depends on the $(p\pi^-)$ momentum and grows linearly from $\pm 9$ MeV/$c^2$ (at 5 GeV/$c$) to $\pm 38$ MeV/$c^2$ (at 30 GeV/$c$).

4.4. Background estimation and subtraction

To extract the average charged lepton and neutrino energies for the $\Lambda_b$ signal, both the background fraction and the corresponding charged lepton and neutrino average energies in the right-sign sample background have to be known. In the following it will be shown that the background contained in the right-sign sample is to a good approximation mimicked by the wrong-sign sample. The study uses the simulated hadronic $Z^0$ events described in Section 4 on which the complete $\Lambda_b$ signal selection was performed. The composition of the right-sign and the wrong-sign samples after normalizing to the luminosity of the real data is summarised in Table 1. The table also gives the total number of events in the two sign combinations reconstructed in the data. The $\Lambda_b$ production rate is overestimated in the Monte Carlo. However, the amount of background is compatible in the two samples.

All events in which the true lepton from the $\Lambda_b$ decay was reconstructed and identified were considered as the $\Lambda_b \to l^- \nu_l X$ signal. The great majority of these events contributed to the right-sign correlations. Candidates with opposite correlations originated from either fragmentation or fake $\Lambda^0$'s.

All $b$-baryon hadronic decays where the lepton candidate was either misidentified or did not come from the $b$ semileptonic decay were classified as $b$-baryon background. In this category the great majority of events contained a true $\Lambda^0$ from the baryon cascade. Here there are two physical sources of definite sign combinations. The first one, $b$-baryon $\to c$-baryon $\to l^- \nu_l \Lambda^0 X$ where the lepton from the semileptonic $c$ decay has been selected, is a source

![Fig. 3. ($p\pi$) mass distributions for $\Lambda^0$ candidates correlated with an identified high $p_T$ lepton in 1992–1995 data after the $b$-tagging selection described in the text. The excess of right-sign over wrong-sign events is attributed to the $b$ baryon signal. The curves are the result of the double-Gaussian fits.](image-url)
of wrong-sign combinations. It is highly suppressed by requiring a high lepton $p_T$ and the mass of the $\Lambda^0 l$ system to exceed 2.1 GeV/$c^2$; its contribution to the total background is smaller than 2% . The second one, $b$-baryon $\rightarrow \tau u \Lambda^0 X$ where $\tau \rightarrow \nu_l l^{-}\nu_l$, is a source of right-sign combinations. The BR($b \rightarrow \tau^{-} l^{-}$) has been experimentally estimated to be $(0.7 \pm 0.2)\%$ [22] and is not negligible. Some attenuation of this signal is obtained by requiring a high momentum lepton ($>3.0$ GeV) since the lepton from the $\tau$ cascade is less energetic. From the Monte Carlo estimation the $\tau$ background gives a small contribution (about 3%), but since it is characterised by exceptionally high missing energy (low $y$ values) it can lead to a perceptible systematic shift. The high missing energy comes from the fact that there are three escaping neutrinos in the process.

Table 1 shows that the majority of the background comes from $B$ mesons. Most of it is from accidental combinations which are not biased towards either sign combination. However, in the meson sample there are possible sources of biases between right-sign and wrong-sign samples. From a more extensive study using the JETSET [10] Monte Carlo event generator we find a systematic tendency towards 10% excess in the right-sign sample. Due to baryon number conservation, baryons are always produced in pairs in the fragmentation. The string fragmentation model used in the simulation has the effect that the more energetic baryon from fragmentation most likely contains the anti-partner of the light quark building the $B$ meson. Therefore, requiring the $\Lambda^0$ momentum to be greater than 5.0 GeV/$c$ favours pairs of the type: ($\bar{B} = b\bar{q}$) + ($qq'q'' = \text{baryon}$) which contribute to the right-sign sample. The level of induced asymmetry depends on the details of the fragmentation and will be considered as a source of systematic uncertainty. Semileptonic $B$ decays, such as $B \rightarrow \Lambda_c \bar{\Lambda} \nu X$ (where $\bar{\Lambda}$ is an antibaryon) could also give rise to an excess of right-sign combinations. The actual branching fraction for such processes is yet not measured but from the available limits the contribution of this background has been estimated to be negligible [23].

Background originating from $c$ quark jets apart from accidental combinations contains $\Lambda^0 l^+$ pairs from the process $c$-baryon $\rightarrow l^- \Lambda^0 X$ which contribute to the wrong-sign sample. Their contribution is highly suppressed by cuts on the lepton $p_T$ and mass of $\Lambda^0 l$ system and by the $b$-tagging. The contribution from this background is smaller than 2%.

Finally, the last class contains $\Lambda^0 l^+$ pairs reconstructed in the $u$, $d$, $s$ or gluon jets. These combinations are purely accidental and hence are symmetrical in the sign combination.

Average energies of the charged lepton and the neutrino as well as the resulting $y$ values in right-sign and wrong-sign background samples from simulation and in the wrong-sign data sample are given in Table 2. The values are in good agreement but possible biases arising from the particular physics processes discussed above will contribute to the systematic error.

Since the right-sign background behaviour is well reproduced by the wrong-sign sample it is possible to extract the average charged lepton energy and the average neutrino energy originating from the $\Lambda_b$.

<table>
<thead>
<tr>
<th>Event category</th>
<th>Right-Sign</th>
<th>Wrong-Sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda_b \rightarrow l\nu X$ signal</td>
<td>421.9 ± 16.4</td>
<td>25.5 ± 4.0</td>
</tr>
<tr>
<td>$b$-baryon background</td>
<td>19.7 ± 3.5</td>
<td>17.8 ± 3.4</td>
</tr>
<tr>
<td>$B$ mesons</td>
<td>134.0 ± 9.2</td>
<td>122.8 ± 8.8</td>
</tr>
<tr>
<td>$c$-jets</td>
<td>13.4 ± 2.9</td>
<td>24 ± 3.9</td>
</tr>
<tr>
<td>$u,d,s$ or $g$</td>
<td>24.8 ± 4.0</td>
<td>14.6 ± 3.1</td>
</tr>
<tr>
<td>total background</td>
<td>192.2 ± 11.1</td>
<td>179.5 ± 10.7</td>
</tr>
<tr>
<td>total events in 1992–1995 data</td>
<td>422 ± 21</td>
<td>173 ± 13</td>
</tr>
</tbody>
</table>

Table 2

Average reconstructed charged lepton and neutrino energies and their ratio $y$ in right-sign and wrong-sign simulation background and wrong-sign real data.

<table>
<thead>
<tr>
<th></th>
<th>MC background</th>
<th>MC background</th>
<th>1992–1995 data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Right-Sign (GeV)</td>
<td>Wrong-Sign (GeV)</td>
<td>Right-Sign (GeV)</td>
</tr>
<tr>
<td>$\langle E_\ell \rangle$</td>
<td>9.71 ± 0.31</td>
<td>9.51 ± 0.32</td>
<td>10.20 ± 0.41</td>
</tr>
<tr>
<td>$\langle E_\nu \rangle$</td>
<td>5.08 ± 0.38</td>
<td>5.23 ± 0.39</td>
<td>5.61 ± 0.50</td>
</tr>
<tr>
<td>$y$</td>
<td>1.91 ± 0.17</td>
<td>1.82 ± 0.15</td>
<td>1.85 ± 0.19</td>
</tr>
</tbody>
</table>
semileptonic decay using the following background subtraction:
\[ \langle E_{l,v} \rangle = \frac{1}{1 - f_{\text{bck}}} \left( \langle E_{l,v}^{\text{R.S.}} \rangle - f_{\text{bck}} \langle E_{l,v}^{\text{W.S.}} \rangle \right) \]
and
\[ f_{\text{bck}} = \frac{N^{W.S.}}{N^{R.S.}} \]
where \( \langle E_{l,v}^{\text{R.S.}} \rangle \) and \( \langle E_{l,v}^{\text{W.S.}} \rangle \) are the average charged lepton or neutrino energies measured in the right-sign and in the wrong-sign samples respectively. \( N^{R.S.} \) and \( N^{W.S.} \) are the number of selected events found in the right-sign and wrong-sign samples.

4.5. Neutrino energy reconstruction

The neutrino energy \( (E_v) \) is not directly measurable in the experiment. It was approximated by the missing energy \( (E_{\text{miss}}) \) in the hemisphere containing the \( \Lambda^0 l \) system (\( \Lambda_b \) hemisphere):
\[ E_v \approx E_{\text{miss}} = E_{\text{TOT}} - E_{\text{vis}} \]
\[ E_{\text{TOT}} = \frac{1}{2} \sqrt{s} + \frac{(M^{\Lambda_b})^2 - (M^{\text{oppo}})^2}{2\sqrt{s}} \]
where \( E_{\text{vis}} \) is the sum of all charged particle energies and neutral calorimeter energy deposits in the \( \Lambda_b \) hemisphere. \( E_{\text{TOT}} \) is the total energy available in the \( \Lambda_b \) hemisphere. The lower equation results directly from four-momentum conservation applied to the entire event. \( M^{\Lambda_b} \) and \( M^{\text{oppo}} \) are the \( \Lambda_b \) hemisphere invariant mass and the opposite hemisphere invariant mass respectively. \( \sqrt{s} \) denotes the total energy in the center-of-mass of the colliding \( e^+e^- \). Individual energy deposits in both electromagnetic calorimeters (HPC or FEMC) and hadronic calorimeters (HCAL) are clustered according to the spatial resolution of the given calorimeter to form bigger deposits which are likely to come from single particle showers. Then a matching between reconstructed charged particle tracks and the calorimeter showers is performed. The deposits not associated to any charged particle track are assumed to originate from a neutral particle cascade. Together with all reconstructed charged particle tracks they contribute to the total visible energy \( E_{\text{vis}} \) and to the computation of the hemisphere masses \( M_{\text{vis}}^i (i: \Lambda_b, \text{oppo}) \). For the reconstruction of hemisphere masses the formula \( M^i = M_{\text{vis}}^i - \frac{\sqrt{s}}{2E_{\text{vis}}} \) was found to be the best approximation. The correction accounts for both detector effects and for the missing neutrino.

The resolution of the neutrino energy reconstruction \( (E_v^\text{rec} - E_v^\text{gen}) \) obtained from the simulation is shown in Fig. 4. The two distributions correspond to contributions from purely hadronic \( \Lambda_c \) decays and semileptonic \( \Lambda_c \) decays. In the latter case there is an additional neutrino from the \( \Lambda_c \) decay escaping from the apparatus. The distributions for the hadronic \( \Lambda_c \) decays and semileptonic \( \Lambda_c \) decays were fitted with Gaussian functions yielding widths of 4.2 GeV and 4.5 GeV, respectively. Moreover, the \( E_v \) residuals for hadronic \( \Lambda_c \) decays are centered on zero while the semileptonic \( \Lambda_c \) decay subsample shows a large offset of \( \approx 3.5 \) GeV equal to the average energy of the neutrino from the \( \Lambda_c \) decay. The analysis presented here did not distinguish between hadronic and semileptonic \( \Lambda_c \) decays in the real data. The two contributions were considered together and the \( \Gamma(R(\Lambda_c \rightarrow \Lambda^0 X)) \) found in the simulation was assumed.

The uncertainty on this ratio was taken into account in the systematic error. The possibility of tagging double semileptonic decays by looking for another lepton (of the opposite sign) in the \( \Lambda_b \) hemisphere

![Fig. 4.](image)
was investigated. It was found, however, to be ineffective due to the low average energy of the charged lepton from $A_c$ decay. The decays giving most distortion of the missing energy spectrum have large neutrino energies and low lepton momenta where DELPHI has poor identification ability.

The stability of neutrino energy reconstruction versus $A_c$ polarization was checked in the simulation as well. No systematic dependence was observed.

The data/simulation agreement on the missing energy was checked using different event samples within the hadronic event selection described at the beginning of Section 4.3.

The total visible event energy comparison exhibits very good agreement between data and simulation. The average values agree to a few parts in a thousand. Such a comparison, however, is inclusive and moreover cannot reveal possible distortions from the hemisphere separation. Therefore, a final cross-check

---

**Fig. 5.** Comparison of different reconstructed energy spectra in the data (points with error bars) and in the simulation (shaded histogram). The global event selections were applied (see Section 4.3). All histograms are normalized to the unit area. (a) momentum spectrum of identified muons with $p > 3.0$ GeV/c and $p_T > 1.0$ GeV/c in $b$-tagged events ($P_{b_{tag}} < 0.01$); (b) momentum spectrum of identified electrons with $p > 3.0$ GeV/c and $p_T > 1.0$ GeV/c in $b$-tagged events ($P_{b_{tag}} < 0.01$); (c) missing energy in the muon hemisphere (same sample as a); (d) missing energy in the electron hemisphere (same sample as b); (e) visible energy in the hemisphere opposite to the muon (same sample as a) but excluding events with an identified lepton with $p > 3.0$ GeV/c in this hemisphere; (f) visible energy in the hemisphere opposite to the electron (same sample as b) but excluding events with an identified lepton with $p > 3.0$ GeV/c in this hemisphere.
Table 3
Hemisphere energy in GeV reconstructed in the inclusive semileptonic events without requiring a $A_b$. For calculation of $\langle E\rangle_{\mu\tau\nu}$ the opposite hemispheres containing identified leptons with $p > 3.0$ GeV/c have been excluded.

<table>
<thead>
<tr>
<th>$\langle E\rangle$ (GeV)</th>
<th>data</th>
<th>simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>muons, $P_{b-\text{TAG}} &lt; 0.01$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p &gt; 3.0$ GeV/c, $p_T &gt; 1.0$ GeV/c</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\langle E\rangle_{\mu}$</td>
<td>10.30 ± 0.07</td>
<td>10.38 ± 0.07</td>
</tr>
<tr>
<td>$\langle E\rangle_{\text{miss}}$</td>
<td>8.89 ± 0.11</td>
<td>8.90 ± 0.11</td>
</tr>
<tr>
<td>$y = \langle E\rangle_{\mu}/\langle E\rangle_{\text{miss}}$</td>
<td>1.159 ± 0.018</td>
<td>1.166 ± 0.018</td>
</tr>
<tr>
<td>$\langle E\rangle_{\mu\tau\nu}$</td>
<td>41.39 ± 0.12</td>
<td>41.03 ± 0.12</td>
</tr>
<tr>
<td>electrons, $P_{b-\text{TAG}} &lt; 0.01$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p &gt; 3.0$ GeV/c, $p_T &gt; 1.0$ GeV/c</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\langle E\rangle_{e}$</td>
<td>10.14 ± 0.07</td>
<td>10.07 ± 0.07</td>
</tr>
<tr>
<td>$\langle E\rangle_{\text{miss}}$</td>
<td>8.45 ± 0.11</td>
<td>8.36 ± 0.11</td>
</tr>
<tr>
<td>$y = \langle E\rangle_{e}/\langle E\rangle_{\text{miss}}$</td>
<td>1.200 ± 0.018</td>
<td>1.205 ± 0.018</td>
</tr>
<tr>
<td>$\langle E\rangle_{\mu\tau\nu}$</td>
<td>41.77 ± 0.13</td>
<td>41.64 ± 0.13</td>
</tr>
</tbody>
</table>

was done on an inclusive sample of $b$-hadron semileptonic decays.

The sample was selected requiring an identified energetic lepton ($p > 3.0$ GeV/c) with a high $p_T$ ($p_T > 1.0$ GeV) contained in a $b$-tagged event ($P_{b-\text{TAG}} < 0.01$ corresponding to $b$ purity of $\approx 85\%$). Since $90\%$ of $b$'s hadronize into mesons the inclusive sample should not retain any detectable polarization. The plots shown in Fig. 5a–d show the comparison of the charged lepton and of the hemisphere missing energy, $E_{\text{miss}}$, spectra reconstructed in data and in the simulation. Plots in the left column correspond to the muon subsample and in the right column to the electron subsample. Both charged lepton spectra and the $E_{\text{miss}}$ spectra show good agreement between data and simulation. In addition, plots 5e and 5f show spectra of visible energy in the hemisphere opposite to the reconstructed lepton when this hemisphere did not have any identified leptons with $p > 3.0$ GeV/c. The detailed numerical results of the whole cross-check are summarised in Table 3. The table contains four quantities extracted for each sample:

1. the average energy of the charged lepton ($\mu$ or $e$);
2. the average missing energy in the lepton hemisphere obtained using the same algorithm as for the $E_{\mu}$ reconstruction;
3. the ratio of the above two mean values which is the observable directly sensitive to polarization;
4. the average energy of the opposite hemisphere; events that have identified leptons with $p > 3.0$ GeV/c in the opposite hemisphere are excluded.

Data/simulation discrepancies in both lepton and neutrino mean energies and in the resulting $y$ value are within one standard deviation of their statistical uncertainty. Therefore, taking a conservative value of $2\sigma$ it can be assumed that the systematic error on $\langle E_{\mu}\rangle$ does not exceed 220 MeV.

5. The results

The results obtained for the background-free reference $A_b$ simulation, the simulated unbiased $q\bar{q}$ events and the data are summarised in Table 4. The $R_{b}$ and polarization $P$ for the background-free $A_b$ simulation sample are by definition equal to one and zero respectively. The whole analysis applied to the simulation of the unbiased $q\bar{q}$ events gives a result which is compatible with zero and within their errors the observables are in good agreement with the ones.
obtained from the background-free reference simulation. This result additionally confirms the validity of assumptions about the background behaviour and its subtraction done in Section 4.4. The last column of Table 4 gives relevant results extracted from the data. Fig. 6 shows the charged lepton and neutrino energy spectra for both right-sign and wrong-sign samples and for the $A_b$ signal obtained from the subtraction. The statistical error on $\langle E_\nu \rangle$ is not much worse than on $\langle E_\ell \rangle$ because, although the resolution on an individual measurement of $E_\nu$ is poorer, the errors on the averages are dominated by the width of the distributions. The result reads:

$$R_y = \frac{y^{\text{data}}}{y^{\text{MC}}} = 1.21^{+0.16}_{-0.15} \text{(stat.)}. \quad (7)$$

The polarization is extracted from this value of $R_y$ using the calibration curve from Fig. 2c. Since the correlation between $R_y$ and $P_{A_b}$ is not linear the error on the latter becomes asymmetric. The $A_b$ polarization is found to be:

$$P_{A_b} = -0.49^{+0.32}_{-0.30} \text{(stat.)}. \quad (8)$$

The systematic error estimation is described in the following section.

5.1. Systematic uncertainties

The individual contributions to the total systematic uncertainty, summarised in Table 5, are discussed below.

As mentioned in Section 4.5, there is a large offset in the reconstructed neutrino energy when the $A_\ell$ decays semileptonically. Therefore, the result obviously depends on the semileptonic branching fraction of $A_\ell$, $R_{sl} = \frac{\text{BR}(A_\ell \to l \nu A^0)}{\text{BR}(A_\ell \to A^0 \nu)}$. Most of the uncertainty on this number is due to the poorly measured $\text{BR}(A_\ell \to A^0 \nu)$ which is estimated to be $(35 \pm 11)\%$ [24]. Taking the PDG value for the $\text{BR}(A_\ell \to l \nu A^0)$ leads to $R_{sl} = (9 \pm 5)\%$. Assuming that the process $A_\ell \to l \nu A^0$ dominates the $A_\ell$ semileptonic decays and the CLEO result for $\text{BR}(A_\ell \to e^+ X)$ is $(3.4 \pm 0.4)\%$ [25], an estimate of the upper limit on $R_{sl} \leq (19^{+5}_{-3})\%$ is obtained. To account for this large uncertainty $R_{sl}$ was allowed to vary by $\pm 8\%$ around the $14\%$ assumed in the simulation. This variation corresponds to a systematic uncertainty on the measured missing energy of $\pm 280$ MeV leading to an error on $R_y$ of $0.010$.

The $A_\ell$ polarization affects the average missing energy measurement in $A_\ell$ semileptonic decays because the average energy of the neutrino escaping from $A_\ell$ depends on the polarization. Fortunately the dependence is not so strong in this process [11]. Therefore, the expected variation of the average reconstructed neutrino energy for unit change in $A_\ell$ polarization does not exceed 50 MeV corresponding to $\sigma(R_y) = \pm 0.010$.

Table 5

<table>
<thead>
<tr>
<th>Source</th>
<th>$\sigma(R_y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{BR}(A_\ell \to l \nu A^0 X)$</td>
<td>$0.006$</td>
</tr>
<tr>
<td>$A_\ell$ polarization</td>
<td>$0.010$</td>
</tr>
<tr>
<td>neutrino energy</td>
<td>$0.044$</td>
</tr>
<tr>
<td>background bias</td>
<td>$0.017$</td>
</tr>
<tr>
<td>$A_\ell \to \tau \nu X$ and $\tau \to \nu l_\nu$</td>
<td>$0.014$</td>
</tr>
<tr>
<td>$A_\ell$ fragmentation function</td>
<td>$0.018$</td>
</tr>
<tr>
<td>MC reference</td>
<td>$0.020$</td>
</tr>
<tr>
<td>theory</td>
<td>$0.005$</td>
</tr>
<tr>
<td>Total</td>
<td>$0.082$</td>
</tr>
</tbody>
</table>
As discussed in Section 4.5, a systematic discrepancy between data and simulation in the hemisphere energy estimation cannot fake the neutrino energy measurement by more than 220 MeV yielding \( \sigma(R_y) = 0.044 \).

Residual differences between the wrong-sign sample and the right-sign background can lead to a shift in the measured \( \Lambda_b \) polarization. The shift comes both from a different effective \( y \) reconstructed in the two samples and their unequal population faking the apparent \( \delta_{\text{bck}} \). Possible sources of such biases were discussed in Section 4.4. To extract the induced final systematic error they were added incoherently. The summed error on \( R_y \) does not exceed 0.017.

The contribution from \( \Lambda_b \rightarrow \tau \nu_X \) with the subsequent decay \( \tau \rightarrow \nu / l / \nu \) gives rise to the extra right-sign \( \Lambda^0 \) correlations. This background source might lead to an error on the observed \( R_y \) of 0.014.

In principle the measurement should not be sensitive to the \( \Lambda_b \) fragmentation function. However, selection cuts, efficiency functions, etc. could introduce a certain limited dependence. The value of \( \langle E_y \rangle \) observed in data is almost 2\( \sigma \) lower than expected under the assumption that the \( \Lambda_b \) fragmentation function is identical in data and in the simulation. The possible influence of the fragmentation on the polarization measurement was studied using the background-free \( \Lambda_b \) simulated events. In the subsequent event samples the generated \( \Lambda_b \) spectrum was varied in order to reproduce a large range of mean \( \Lambda_b \) energy. The linear fit to the \( y \) behaviour presented in Fig. 7 shows a very limited dependence of the reconstructed \( y \) on the \( \Lambda_b \) average energy. A variation of the mean \( \Lambda_b \) energy by as much as \(-25\%\) (from 34.0 GeV to 25.5 GeV) corresponds to an error on the reconstructed \( R_y \) of 0.018.

Limited statistics of the simulated \( \Lambda_b \) calibration sample led to an uncertainty on \( R_y \) of \( 0.020 \).

The theoretical error arises mainly from the uncertainty on the value of \( m_\tau / m_\tau \) and is small. This uncertainty enters the analysis implicitly via the parameters of the Monte Carlo event generator. The value \( m_\tau / m_\tau = 0.27 \) was used. Variation in the large range between 0.20 and 0.36 corresponds to a systematic error on \( R_y \) smaller than 0.005 [26]. As mentioned already in the introduction, both QCD perturbative and non-perturbative corrections were neglected being tiny relative to other sources of systematic uncertainties.

All systematic error contributions were added in quadrature resulting in \( \sigma(R_y) = 0.082 \) corresponding to the total uncertainty on \( P_{\Lambda_b} \) of \( 0.17 \).

### 5.2. Consistency check using B mesons

\( B^0 \) mesons being scalar objects do not carry any polarization. The polarization measured on the \( B^0 \) sample should be consistent with zero. Therefore, an independent measurement of the \( B \) meson polarization can serve as a test of the consistency of the analysis.

Events of \( B^0 \) semileptonic decays via the process \( B^0 \rightarrow D^+ l^- \nu \) were selected [26]. The \( D^+ \) mesons were reconstructed in the channel \( D^+ \rightarrow D^0 \nu_{\text{soft}} \).

#### Table 6

<table>
<thead>
<tr>
<th>sample</th>
<th># of ( B^0 ) candidates</th>
<th>( \delta_{\text{bck}} )</th>
<th>( \langle E_y \rangle ) (GeV)</th>
<th>( \langle E_x \rangle ) (GeV)</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC ( b\bar{B} )</td>
<td>2371 ( \pm ) 17</td>
<td>0.056 ( \pm ) 0.005</td>
<td>10.75 ( \pm ) 0.12</td>
<td>6.99 ( \pm ) 0.14</td>
<td>1.54 ( \pm ) 0.04</td>
</tr>
<tr>
<td>data</td>
<td>386 ( \pm ) 9</td>
<td>0.090 ( \pm ) 0.015</td>
<td>10.61 ( \pm ) 0.30</td>
<td>6.73 ( \pm ) 0.37</td>
<td>1.58 ( \pm ) 0.11</td>
</tr>
</tbody>
</table>

Fig. 7. Dependence of the reconstructed \( y \) on the \( \Lambda_b \) fragmentation function in the background-free simulation. The plot shows corresponding reconstructed \( y \) values as a function of mean \( x = \langle E_{\text{lab}} / E_{\text{beam}} \rangle \). The dashed line represents the best linear fit to the points.
where $D^0 \to K^+ \pi^-$. Next, the $D^+$ candidates were correlated with high $p_T$ leptons found in the same hemisphere. The lepton selection was the same as the one described in Section 4.1. A sample of 386 ± 9 $B^0 \to D^{*+} l^- \nu_l$ signal candidates was collected.

The whole procedure to extract the polarization was identical with that used in the $\Lambda_b$ analysis. The relevant polarization observables obtained from the data and the reference simulation are summarised in Table 6. These yield the polarization:

$$P_b = -0.08 \pm 0.20({\text{stat}}.) \pm 0.08({\text{MC ref}}.)$$

(9)

where the second error is the systematic uncertainty coming only from the limited statistics of the Monte Carlo reference sample. The result is compatible with zero polarization in the $B$ meson sector. Although the statistical significance of this result is limited it excludes the existence of a severe systematic discrepancy between data and MC in the missing energy estimation and proves the general correctness of the experimental procedure.

### 6. Conclusions

The $\Lambda_b$ polarization has been measured using semileptonic decays selected from $\approx 3.5 \times 10^6$ hadronic $Z^0$ decays collected with the DELPHI detector between 1992 and 1995. The $\Lambda_b$ event selection is based on charge correlations in pairs of high $p_T$ leptons and $\Lambda^0$ baryons found in the same event hemisphere. The final sample contains 249 ± 19 $\Lambda_b$ candidates observed as an excess of right-sign over wrong-sign $\Lambda b$ pairs.

The polarization is determined from the ratio of the average energies of charged leptons and neutrinos from $\Lambda_b$ decays which is an experimental observable both highly sensitive to polarization and practically free from theoretical uncertainties.

The measured value of $\Lambda_b$ polarization is:

$$P_{\Lambda_b} = -0.49 \pm 0.32({\text{stat}}.) \pm 0.17({\text{syst}}.)$$

The result is in good agreement with those obtained by ALEPH [27], $P_{\Lambda_b} = -0.23 \pm 0.24$ (stat.) $\pm 0.05$ (syst.) and OPAL [28] $P_{\Lambda_b} = -0.56 \pm 0.12$ (stat.) $\pm 0.09$ (syst.). Bearing in mind the SM prediction for $b$ polarization of $-0.94$, within the model [6] (see Section 1) all three results favour the scenario where a substantial fraction of $\Lambda_b$'s are produced in the decays of $\Sigma_b^+$ and $\Sigma_b^0$ states which live long enough to allow for a spin flip.

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Measurement of the $E_{T,\text{jet}}^2/Q^2$ dependence of forward-jet production at HERA

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Abstract

The forward-jet cross section in deep inelastic $e^+p$ scattering has been measured using the ZEUS detector at HERA with an integrated luminosity of 6.36 pb$^{-1}$. The jet cross section is presented as a function of jet transverse energy squared, $E_{T,\text{jet}}^2$, and $Q^2$ in the kinematic ranges $10^{-3} < E_{T,\text{jet}}^2/Q^2 < 10^2$ and $2.5 \cdot 10^{-4} < x < 8.0 \cdot 10^{-2}$. Since the perturbative QCD predictions for this cross section are sensitive to the treatment of the log terms, this measurement provides an important test. The measured cross section is compared to the predictions of a next-to-leading order pQCD calculation as well as to various leading-order Monte Carlo models. Whereas the predictions of all models agree with the measured cross section in the region of small $E_{T,\text{jet}}^2/Q^2$, only one model, which includes a resolved photon component, describes the data over the whole kinematic range. © 2000 Elsevier Science B.V. All rights reserved.
1. Introduction

The wide kinematic range available at the HERA ep collider at DESY has allowed QCD to be tested in regions of phase space not available to previous experiments. Both the H1 and ZEUS collaborations have studied the forward-jet cross sections [1,2] in order to search for BFKL effects [3,4]. For these analyses, the two hard scales involved in jet production in deep inelastic scattering (DIS), the negative square of the four-momentum transfer at the lepton vertex, \( Q^2 \), and the squared transverse jet energy, \( E_{T,jet}^2 \), were chosen to be of the same order of magnitude. This paper extends our previous study [2] by investigating the forward-jet cross section as a function of the ratio of these two scales, \( E_{T,jet}^2 / Q^2 \), for the entire available range.

Three different kinematic regions can be distinguished, depending on the dominant scale. In the first region, \( Q^2 \gg E_{T,jet}^2 \), \( Q^2 \) is the standard deep inelastic process hard scale. Typically, leading-order (LO) Monte Carlo models approximate pQCD contributions in this regime by parton showers. In the second region, where \( E_{T,jet}^2 \approx Q^2 \), all terms with \( \log(Q^2/E_{T,jet}^2) \) become small and the effects of DGLAP evolution [5] are suppressed. Therefore BFKL effects are expected to be observable in this region, which was selected for the analysis of forward-jet production [2], where it was discussed in detail. In the third region, where \( Q^2 \ll E_{T,jet}^2 \), the NLO pQCD prediction is sensitive to the treatment of terms proportional to \( \log(E_{T,jet}^2/Q^2) \), which ought to be resummed. Conventional Monte Carlo models do not include these terms.

In this letter, measurements of the forward-jet cross sections covering all three regions are presented and compared to the predictions of various LO Monte Carlo models in which the hard-scattering process is described by direct photon diagrams, namely boson–gluon fusion and QCD Compton diagrams. The models under consideration differ in their way of describing the higher-order contributions to the LO process. LEPTO [6] and HERWIG [7] use parton showers that evolve according to the DGLAP equations. ARIADNE [8] employs the color–dipole model, in which gluons are emitted from the color field between quark–antiquark pairs. Since color dipoles radiate independently, the gluons are not ordered in transverse momentum, \( k_T \). The linked-dipole-chain model, LDC [9], implements the structure of the CCFM equation [10], which is intended to reproduce both DGLAP and BFKL evolution in their respective ranges of validity. In all these models, \( Q^2 \) is normally used as the relevant scale. Finally, RAPGAP [11] introduces a resolved photon contribution in addition to the direct photon cross section and uses \( Q^2 + E_{T,jet}^2 \) as the factorization scale. The inclusion of the resolved photon contribution partially mimics the higher-order contributions to the direct photon component, namely the \( \log(E_{T,jet}^2/Q^2) \) terms, which are not included in the conventional DIS LO Monte Carlo models. The scattering of the partons from those contained in the resolved photon can lead to final state partons with a high transverse momentum in the forward direction. Since this process was suggested to provide an explanation for the observed excess in forward-jet production [12], the previously published forward cross section [2] as a function of the Bjorken scaling variable, \( x \), is compared to predictions of the RAPGAP model.

2. Measurement

This study is based on data taken with the ZEUS detector in 1995, corresponding to an integrated luminosity of 6.36 pb\(^{-1}\). As the analysis follows very closely that for the forward-jet cross section [2], details about the experimental setup, event selection, jet finding and systematic error are not repeated here.

The selected DIS events were required to have a scattered electron with a minimum energy of \( E_e = 10 \) GeV. The fractional energy transfer by the virtual photon had to be \( y > 0.1 \). The \( x \) range was extended

| \( Q^2 > 10 \text{ GeV}^2 \) |
| \( 2.5 \times 10^{-4} < x < 8 \times 10^{-2} \) |
| \( y > 0.1 \) |
| \( E_e > 10 \) GeV |
| \( \eta_{jet} < 2.6 \) |
| \( E_{T,jet} > 5 \) GeV |
| \( x_{jet} > 0.036 \) |
| \( p_T > 0 \) |

\( \eta_{jet} \) is the pseudorapidity of the jet.

Table 1
Selected kinematic region for the cross section measurement.
with respect to [2] from $4.5 \cdot 10^{-4} < x < 4.5 \cdot 10^{-2}$ to $2.5 \cdot 10^{-4} < x < 8.0 \cdot 10^{-2}$. An additional cut, $Q^2 > 10$ GeV$^2$, was applied in order to be well within the DIS regime.

Jets were selected with a cone algorithm in the laboratory frame. The cone radius, $R$, was chosen to be 1.0. The transverse energy of the jets in the laboratory frame, $E_{T,jet}$, was required to be larger than 5 GeV and the jet pseudorapidity $\eta_{jet}$ range was restricted to $\eta_{jet} < 2.6$. The scaled longitudinal jet momentum $x_{jet} = p_{Z,jet}/p_{beam}$, where $p_{beam} = 820$ GeV, had to be larger than 0.036 to select forward jets [4]. Furthermore, only jets with a positive Z-momentum in the Breit frame were considered, thus avoiding those jets originating from the scattered quark at large values of $x$. These cuts are given in Table 1.

The jet cross sections presented here have been corrected to the hadron level for detector acceptance and smearing effects using the ARIADNE model, since it gave the best description of the data [2]. The purity for reconstructing forward jets in the given phase space rises from 40% to 80% with increasing $E_{T,jet}$ and efficiency are around 20%, but here the statistical errors are large. The factors required to correct the data for detector effects lie between 0.8 and 1.4 and increase as $E_{T,jet}/Q^2$ increases.

3. Results

The forward-jet cross section is presented in Fig. 1 as a function of $E_{T,jet}/Q^2$. The numerical values are given in Table 2. The treatment of the systematic errors closely follows the published results [2] and leads to errors of similar size. The shaded band corresponds to the uncertainty coming from the energy scale of the calorimeter. The Monte Carlo predictions from ARIADNE thick, full line, LEPTO dashed line, HERWIG dotted line and LDC thin, full line are shown for comparison. The vertical dashed lines indicate the region used for the previous forward cross section measurement [2].

Predictions from different LO Monte Carlo models are shown in Fig. 1 and Fig. 2. Three regions are distinguished, separated by the dashed vertical lines. In the region where $Q^2 \gg E_{T,jet}$, all the models describe the data reasonably well.

In the regime $Q^2 = E_{T,jet}$, only ARIADNE 4.08 and RAPGAP 2.06 reproduce the measured distributions. In RAPGAP the resolved component of the virtual photon is modeled using the SaS-2D parametrization for the parton distribution function (pdf) of the photon [13], which in this $Q^2$ range evolves as $\sim \log(E_{T,jet}/Q^2)$. The factorization scale has been set to $\mu^2 = E_{T,jet} + Q^2$.

In Fig. 3 the $x$-dependence in this regime is compared with RAPGAP, using the cuts $0.5 < E_{T,jet}/Q^2 < 2.0$ and $4.5 \cdot 10^{-4} < x < 4.5 \cdot 10^{-2}$ [2]. RAPGAP gives a good description of the cross section. The contribution of the direct photon component is indicated separately. As expected, it matches the LEPTO prediction.

51 The ZEUS coordinate system is defined as right-handed with the Z-axis pointing in the proton beam direction, referred to as forward direction, and the X-axis horizontal, pointing towards the center of HERA. The pseudorapidity is defined as $\eta = -\ln(\tan(\theta/2))$, where the polar angle $\theta$ is taken with respect to the proton beam direction.
Table 2
Cross section values and errors for the corrected data in bins of $E_{T,\text{jet}}^2/Q^2$. The last column shows the systematic errors due to the energy scale uncertainty of the calorimeter, which is not included in the central column. The table refers to the points shown in Fig. 1. The phase space under investigation is defined by the cuts: $\eta_{\text{jet}} < 2.6$, $x_{\text{jet}} > 0.036$, $E_{T,\text{jet}} > 5$ GeV, $E_\gamma > 10$ GeV, $y > 0.1$, $Q^2 > 10$ GeV$^2$, $p_T(Jet) > 0$ and $2.5 \times 10^{-4} < x < 8.0 \times 10^{-2}$.

<table>
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<tr>
<th>$E_{T,\text{jet}}^2/Q^2$</th>
<th>$d\sigma/d(\eta/E_\gamma^2/Q^2)$</th>
<th>stat. ± syst. pb</th>
<th>syst. $E_{\text{CAL}}$-scale pb</th>
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</thead>
<tbody>
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<td>0.01 – 0.03</td>
<td>59.5 ± 32.12 ± 8.8</td>
<td>($-0.1,19.5$)</td>
<td></td>
</tr>
<tr>
<td>0.03 – 0.1</td>
<td>164 ± 33.4 ± 1.5</td>
<td>($-9.2$)</td>
<td></td>
</tr>
<tr>
<td>0.1 – 0.3</td>
<td>255 ± 22.7 ± 6.2</td>
<td>($-6.13$)</td>
<td></td>
</tr>
<tr>
<td>0.3 – 1.0</td>
<td>288 ± 12.1 ± 1.3</td>
<td>($-16.9$)</td>
<td></td>
</tr>
<tr>
<td>1.0 – 3.0</td>
<td>190 ± 6.7 ± 2.9</td>
<td>($-19.18$)</td>
<td></td>
</tr>
<tr>
<td>3.0 – 10.</td>
<td>41.2 ± 1.4 ± 0.4</td>
<td>($-4.0,3.5$)</td>
<td></td>
</tr>
<tr>
<td>10. – 30.</td>
<td>2.95 ± 0.19 ± 0.16 ± 0.07</td>
<td>($-0.33,0.27$)</td>
<td></td>
</tr>
<tr>
<td>30. – 100.</td>
<td>0.120 ± 0.020 ± 0.014</td>
<td>($-0.021,0.014$)</td>
<td></td>
</tr>
</tbody>
</table>

For $Q^2 \ll E_{T,\text{jet}}^2$, none of these models, except RAPGAP, reproduces the data. In particular ARIADNE overshoots the data by up to an order of magnitude at the upper limit of the displayed range. The other models, LEPTO 6.5, HERWIG 5.9 and LDC 1.0, lie far below the data. These comparisons using corrected cross sections are similar to those made previously [2], using the uncorrected distributions. The same data are shown in Fig. 2 together with the prediction of the RAPGAP Monte Carlo model, which describes the data well over the full range of $E_{T,\text{jet}}^2/Q^2$.

Recently, the parton level NLO calculation JetViP [14] has become available, to which our data can also be compared, with the proviso that the hadronization corrections are model-dependent and are of the order of up to 20%. JetViP sums contributions from the direct and resolved virtual photon and uses the SaS-1D photon pdf [13]. For the first three bins in Fig. 2 only the direct contribution has been taken into account, since $Q^2$ is large enough ($Q^2 > 83$ GeV$^2$) that the resolved component can be neglected. The renormalization and factorization scales have been set to $E_{T,\text{jet}}^2 + Q^2$ [15]. The agreement over the full range of $E_{T,\text{jet}}^2/Q^2$ is good. The $x$ dependence of the
The cross section in the range $0.5 < E^2_{\text{jet}}/Q^2 < 2.0$ has also been calculated with JetViP [16] and good agreement was found. The fact that only RAPGAP and JetViP describe the data implies that a resolved photon component is necessary for $E^2_{\text{jet}}/Q^2 > 1$.

The necessity of a resolved photon component in a DIS process has also been discussed by the H1 collaboration in the context of dijet production in a $Q^2$ range of 5 to 100 GeV$^2$ [17], where the measured dijet cross section could only be described with the inclusion of the resolved component.

In comparing the performance of RAPGAP and JetViP it should be noted that while they both agree with the data, their predictive power is limited. On the one hand both RAPGAP and JetViP use the SaS photon pdf, which for $Q^2 > 0$ is not very well constrained by experimental data. On the other hand there is a large variation of the results when the factorization scale is varied, as shown by the light shaded band in Fig. 3 for RAPGAP. A similar effect is seen for JetViP [16].

4. Summary

The cross sections for forward-jet production over a wide range of $E^2_{\text{jet}}/Q^2$ have been compared to different Monte Carlo models. All leading-order Monte Carlo models tested here give a good description of the region in which $E^2_{\text{jet}} \ll Q^2$. However, only those models which include non-$k_T$-ordered gluon emissions, or contributions from a resolved photon, reproduce the $E^2_{\text{jet}} \approx Q^2$ region. The full range of $E^2_{\text{jet}}/Q^2$ can be described only by the RAPGAP model and the JetViP NLO QCD calculation, both of which include a resolved photon contribution. The forward-jet differential cross section, as a function of $x$ [2], is also well reproduced by RAPGAP and JetViP. However, the large dependence of its predictions on the factorization scale diminishes the significance of this agreement.

Acknowledgements

We thank the DESY directorate for their strong support and encouragement. The remarkable achievements of the HERA machine group were essential for the successful completion of this work and are gratefully acknowledged. We also thank G. Kramer for useful discussions and B. Pöttner for providing the JetViP calculation.

References

Erratum to: ‘‘Dark matter abundance and electroweak baryogenesis in the CMSSM’’

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Fig. 1 as published was incomplete as the lettering and the legends were missing. The figure below rectifies this omission.

![Diagram](image)

Fig. 1. The light-shaded area is the cosmologically preferred region with $0.1 \leq \Omega_\chi h^2 \leq 0.3$. In the dark shaded region, the LSP is the $\tilde{\chi}_1^0$, leading to an unacceptable abundance of charged dark matter. Also shown are the isomass contours $m_{\tilde{\chi}^0} = 95, 100$ GeV (dashed) and $m_{\tilde{\chi}^0} = 95, 100$ GeV (dot-dashed), as well as an indication of the slepton bound (dotted line) from LEP [24].

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Erratum


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Received 11 January 2000

On page 15, Eqs. (17), (18) should read:

\[ L_q(Q^2) = (b^{50/81} - 1)^2 \Delta \bar{S}(\mu_b^2) \]
\[ + b^{50/81} L_q(\mu_b^2) - \frac{9}{86} (b^{50/81} - 1), \]
\[ (17) \]

\[ L_g(Q^2) = b^{50/81} \Delta g(\mu_b^2) - \Delta g(Q^2) \]
\[ + b^{50/81} L_g(\mu_b^2) - \frac{8}{25} (b^{50/81} - 1), \]
\[ (18) \]

where \( b = \alpha_s(Q^2)/\alpha_s(\mu_b^2) \).

As a consequence, Fig. 5 on page 14 should appear as below.

By looking at these figures one realizes that, for the models under scrutiny, \( L_q \) evolves at high \( Q^2 \) to a vanishingly small value, contrary to our previous result. Thus, at high \( Q^2 \) the quarks carry only spin. The results for the gluons change slightly, however the same scenario remains, a large cancellation between \( \Delta g \) and \( L_g \) takes place. Thus at large \( Q^2 \) the gluons carry over 50% of the angular momentum. On page 16, Ref. [16] should read: [16] L.M. Sehgal, Phys. Rev. D 10 (1974) 1663.

Acknowledgements

We would like to thank Drs. F. Cano and M. Wakamatsu who pointed out this error to us.

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Fig. 5. The contributions to the proton spin sum rule, Eq. (1), according to: (a) the modified IK scenario of Fig. 3; (b) the "D model" scenario of Fig. 4. The dashed curve shows $\Delta \Sigma(Q^2)$, the long-dashed one $\Delta g(Q^2)$, the dot-dashed curve is $L_g(Q^2)$, the dot-long-dashed curve gives $L_s(Q^2)$ and the full curve represents the sum of the previous four terms, giving the spin sum rule ($J = \frac{3}{2}$).
Solutions of the Wick-Cutkosky model in the light front dynamics

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Abstract

We study relativistic effects in a system of two scalar particles interacting via a scalar exchange in the Light Front Dynamics framework. The results are compared to those provided by Bethe-Salpeter and non-relativistic equations. It is found in particular that for massive exchange, the relativistic description is of crucial importance even in the limit of zero binding energy. © 2000 Published by Elsevier Science B.V. All rights reserved.

PACS: 11.10; 03.70; 03.65P
Keywords: Light-front dynamics; Relativistic equations; Quantum field theory

1. Introduction

Light Front Dynamics (LFD) is a field theoretically inspired hamiltonian approach specially well adapted for describing relativistic composite systems. First suggested by Dirac [1] it has been since widely developed (see [2–6] and references therein) and recently applied with success in its explicitly covariant version [7] to the high momentum processes measured in TJNAF [5,8]. In this approach the state vector is defined on a space-time hyperplane given by \( \omega \cdot x = \sigma \) where \( \omega = (1, \hat{n}) \) is a light-like four-vector.

We present here the first results obtained within this approach for the Wick-Cutkosky model [9]. This model describes the dynamics of two identical scalar particles of mass \( m \) interacting by the exchange of a massless scalar particle. This first step towards more realistic systems constitutes an instructive case and is presently considered by several authors [10–14]. The model has been extended to the case where the exchanged particle has non-zero mass \( \mu \) and used to build a relativistic scalar model for deuteron.

The results presented in this paper concern the S-wave bound states in the ladder approximation. They are aimed to (i) compare the LFD and Bethe-Salpeter descriptions and study their non-relativistic limits, (ii) disentangle the origin of the different relativistic effects, (iii) evaluate the contribution of higher Fock components and (iv) apply this study to a scalar model for deuteron.

2. Equation for Wick-Cutkosky model

We have considered the following lagrangian density:

\[
\mathcal{L} = \frac{i}{2} \left( \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) + \frac{i}{2} \left( \partial_\mu \chi \partial^\mu \chi - \mu^2 \chi^2 \right) - g \phi \chi^2
\]

where \( \phi \) and \( \chi \) are real fields. In the case \( \mu = 0 \) it corresponds to the Wick-Cutkosky model. The wave...
function $\Psi$, describing a bound state of two particles with momenta $k_1$ and $k_2$, satisfies in the Light-Front the dynamical equation [7,5]

$$\left[ 4(q^2 + m^2) - M^2 \right] \Psi(q, \hat{n}) = -\frac{m^2}{2\pi^2} \int dq' W(q,q',\hat{n},M^2) \Psi(q',\hat{n})$$

(1)

Variable $q$ is the momentum of one of the particles in the reference system where $k_1 + k_2 = 0$, and tends in the non-relativistic limit to the usual center of mass momentum. $M$ represents the total mass of the composite system, $B = 2m - M$ denotes its binding energy and $\epsilon_q = \sqrt{q^2 + m^2}$. In the case of S-waves the wavefunction is a scalar quantity depending only on scalars $q$ and $\hat{n} \cdot q$ [7,5].

The interaction kernel $V$ calculated in the ladder approximation is given by

$$V(q,q',\hat{n},M^2) = -\frac{4\pi\alpha}{Q^2 + \mu^2}$$

(2)

with

$$Q^2 = (q-q')^2 - (\hat{n} \cdot q)(\hat{n} \cdot q') \left( \frac{\epsilon_{q'} - \epsilon_q}{\epsilon_{q'} \epsilon_q} \right)^2$$

$$+ \left( \epsilon_{q'}^2 + \epsilon_q^2 - \frac{M^2}{2} \right) \left| \frac{\hat{n} \cdot q'}{\epsilon_{q'}} - \frac{\hat{n} \cdot q}{\epsilon_q} \right|$$

The coupling parameter $\alpha$ is defined by $\alpha = g^2/16\pi m^2$. In the limit $\epsilon_b \ll 2m$ some analytical solutions are known [15] and once removed the $\hat{n}$ dependence in the kernel – formally setting $\hat{n} = 0$ – the LFD equation turns back to the Schrödinger equation in the momentum space for the Yukawa or Coulomb potential, $\alpha$ being the usual fine structure constant.

Eq. (1) has been solved with the coordinate choice displayed in Fig. 1. We have chosen $z$ axis along $\hat{n}$ and with no loss of generality $\varphi = 0$. The $\varphi'$ dependence of the kernel (2) can be performed analytically and (1) turns into the two dimensional integral equation

$$\left[ 4(q^2 + m^2) - M^2 \right] \Psi(q,\theta) = \frac{4m^2\alpha}{\pi} \int dq' \sin \theta' dq' \sin \theta' \left( \frac{1}{\sqrt{a^2 - b^2}} \right) \Psi(q',\theta')$$

(3)

with

$$a = q^2 + q'^2 - qq' \left( 2\cos \theta \cos \theta' + \frac{(\epsilon_{q'} - \epsilon_q)^2}{\epsilon_{q'} \epsilon_q} \right)$$

$$+ \left( q^2 + q'^2 + 2m^2 - \frac{M^2}{2} \right) \left| \frac{q \cos \theta'}{\epsilon_{q'}} - \frac{q \cos \theta}{\epsilon_q} \right|$$

$$+ \mu^2, \quad b = 2qq' \sin \theta \sin \theta'$$

The kernel of (3) has an integrable singularity for $(q,\theta) = (q',\theta')$. The equation is solved by expanding the solution on a spline functions basis $S_i$, associated with coordinates $q$ and $\theta$: $\Psi(q,\theta) = \Sigma_i c_i S_i(q) S_i(\theta)$. The r.h.s. two-dimensional integral is evaluated using Gauss quadrature method adapted to treat the singularity. The unknowns of the problem are the coefficients $c_i$, which are solutions of a generalized eigenvalue problem $\lambda BC = \lambda(M^2)C$ for $M^2$ values such that $\lambda(M^2) = 1$.

3. Results

The LFD binding energy for $\mu = 0$ versus the coupling constant is displayed in Fig. 2 (solid line). It is compared with the non-relativistic values (dotted line) and a first order perturbative calculation $B_{\text{pert}}$ (dashed line), valid also for Bethe-Salpeter (BS) equation [16], given by

$$B_{\text{pert}} = \frac{m\alpha^2}{4} \left( 1 + \frac{4}{\pi} \alpha \log \alpha \right)$$

(4)
Fig. 2. Binding energy solid for $m_s^0$ compared to non-relativistic dot-dashed and perturbative dashed calculations

Corresponding numerical values – in $\hbar = c = m = 1$ units – are given in Table 1 together with the quantity $R = \langle q^2 \rangle / m^2$ usually used to evaluate the relativistic character of a system. A first sight at this figure shows a significant departure from the non-relativistic results already for $\alpha = 0.1$. This discrepancy – which keeps increasing till $B$ reaches the maximum value of $2m$ – is of 100% for $\alpha = 0.3$ whereas $R$ remains very small. When evaluated using non-relativistic solutions, $R$ is equal to $B$ (virial theorem), what gives $R \approx 2\%$ for $\alpha = 0.3$, in contrast with the 100% effect in the binding. The $R$ values obtained using the LFD solutions are even smaller (see Table 1). It is worth noticing the sizeable relativistic effects observed in a system for which both the binding energy and the average momentum are small.

A good agreement with the perturbative calculation is found up to values $\alpha \approx 0.3$ where the relative differences are 3%. The particular form of Eq. (4) ensures the existence of a non-relativistic limit, the same for LFD and BS approaches, for the weakly bound states. We will see later on, that this situation is particular to the case $\mu = 0$.

The bound-state wavefunctions presented below are normalized according to

$$\int \frac{d^3q}{e_q} = 1$$

with $e_q = m$ for the non-relativistic case. The LFD wave function $\Psi(q,\theta = 0)$ obtained for $\alpha = 0.5$ is compared in Fig. 3a (solid line) with the corresponding non-relativistic solution (dot-dashed line), that is Coulomb wave function. The sizeable difference between both functions is mainly due the differences in their binding energies: $B_{LFD} = 0.0267$ whereas $B_{NR} = 0.0625$ for the same coupling constant. In order to compare wave functions with the same energy, the value of the coupling constant for the non-relativistic solution is adjusted to $\alpha_{NR} = 0.327$. The wave function obtained (long-dashed curve) is then much closer to the relativistic one.

Furthermore, in the region of high momentum transfer, the relativistic function is smaller than the Coulomb one, as expected from the natural cut-off of high momentum components introduced by relativity. However, these differences can be accounted with the $\theta$ dependence of the LFD solutions, which exists even for $S$-waves. This angular dependence, normalized by the value of $\Psi(q,\theta = 0)$, is shown in Fig. 3b for different values of momentum $0 \leq q \leq 1.5$. As one can see, the influence of the momentum orientation compared to the light-front plane is far from being negligible. This effect increases with $q$ and, for a fixed value of the momentum, is maximum when $\hat{q} \cdot \hat{n} = 0$. For this kinematical configuration, i.e. relative momentum in the Light-Front plane, the relativistic wave function $\Psi(q,\theta = 90^\circ)$ at high momentum is even found to be bigger than non-relativistic one. To get rid of this dependence, we compared $|\Psi(q,\theta)|^2$ integrated over the $\theta$ degree of

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$10^2 B_{\mu=0}$</th>
<th>$10^2 (\varepsilon^2)$</th>
<th>$10^2 B_{\mu=0.15}$</th>
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<tr>
<td>0.3</td>
<td>1.17</td>
<td>1.1</td>
<td>–</td>
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<td>–</td>
</tr>
<tr>
<td>5.0</td>
<td>84.0</td>
<td>55</td>
<td>75.6</td>
</tr>
<tr>
<td>6.0</td>
<td>118</td>
<td>64</td>
<td>107</td>
</tr>
<tr>
<td>6.98</td>
<td>200</td>
<td>80</td>
<td>157</td>
</tr>
<tr>
<td>7.26</td>
<td>–</td>
<td>–</td>
<td>200</td>
</tr>
</tbody>
</table>
freedom, both for relativistic and non-relativistic solutions. The resulting functions, displayed in Fig. 4, measure the effective relativistic effects in the wavefunctions. At $q = 0$ they remain at the level of 5%, once the energy is readjusted. In the high momentum region the relativistic solution is — as expected — smaller than the non-relativistic one, but their differences reach a factor three at $q = 2$, and this for a moderate value of the coupling constant $\alpha = 0.5$.

In the case $\mu \neq 0$ there exists a critical $\alpha_0$ below which there is no bound solution. Fig. 5 represents the binding energy $B$ as a function of the coupling constant $\alpha$ for different values of $\mu$. They are compared with those provided by BS equation in the same ladder approximation, whose kernel incorporates higher order intermediate states. We have solved this equation using the method described in [17].

Fig. 3. (a) S-wave LFD solution for $\theta = 0$ (solid) compared with Coulomb wave function with the same coupling constant (dot-dashed) and the same energy (dashed). (b) shows the angular dependence of the wave function for different values of $q$. Curves from bottom to top correspond to increasing momenta from $q = 0$ to $q = 1.5$.

Fig. 4. Squared modulus of the wave function integrated over $\theta$. The LFD solution (solid line) and the readjusted non-relativistic one (dot-dashed line) have same binding energy. The small top right graph is a zoom of the high momentum region.

Fig. 5. Binding energy as a function of $\alpha$ for different values of $\mu$ in LFD (solid) and BS (dashed) approaches.
The first remark in comparing both approaches is that their results are seen to be close to each other. This fact is far from being obvious — specially for large values of coupling constant — due to the differences in their ladder kernel. A quantitative estimation of their spread can be given by looking into a horizontal cut of Fig. 5, i.e. calculating the relative difference in the coupling constant \( \Delta \alpha / \alpha \) for a fixed value of the binding energy. The results, displayed in Fig. 6 for \( B = 1.0, 0.1, 0.01 \), show that relative differences (i) are decreasing functions of \( \mu \) for all values of \( B \) (ii) increase with \( B \) but are limited to 10% for the strong binding case \( B = m \) which involves values of \( \alpha \geq 5 \). This indicates the relatively weak importance of including higher Fock components in the ladder kernel even for strong couplings, as already discussed in [13].

It is interesting to study the weak binding limit of both relativistic approaches and compare them with the non-relativistic calculations in the case \( \mu \neq 0 \). The results are given in Fig. 7a for \( \mu = 1 \). They show on one hand that LFD and BS (solid lines) converge towards very close, though slightly different, values of the coupling constant (\( \Delta \alpha / \alpha = 0.01 \)). On the other hand one can see, contrary to the \( \mu = 0 \) case in Fig. 2, a dramatic departure of both relativistic approaches from a non-relativistic theory (dot-dashed line), even for negligible values of binding energy. The differences increase with \( \mu \) as shown in Fig. 7b in which LFD and BS results are not distinguished. The origin of this departure lies in the fact that the integral term in Eq. (1) is dominated by the region \( q \sim \mu \), even for very small values of \( B \), and for the case \( \mu \sim m \) the \( q/m \) terms — which make the difference between the non-relativistic and relativistic kernels — are not longer negligible. We conclude from that to the non-adequacy of a non-relativistic treatment in describing systems interacting via massive fields, what is the case of all the strong...
interaction physics when not described via gluon exchange.

Some approximations of Eq. (1) have been studied in order to disentangle the different contributions to the relativistic energies \( B(\alpha) \) (see Fig. 8). Eq. (1) is formally written \( K\Psi = \frac{1}{\sqrt{s}} \mathcal{V}\Psi \). We first consider the case of a non-relativistic kernel \( \mathcal{V} \) i.e. a Yukawa potential \( E \) and \( s \) in curve \( a \) in the non-relativistic kinematics \( K = 4 q^2 + 2 mB \) and in curve \( a' \) the relativistic one \( K = 4(q^2 + m^2) - M^2 \). Curves \( b \) and \( b' \) are obtained in the same manner, but putting \( E = \sqrt{q^2 + m^2} \). The last one corresponds to the LFD equation. The results in Fig. 8 show that the kinematical term \( K \) has a very small influence on \( B \), whereas the contributions of \( E \) and \( V \) to the total binding are both essential. We conclude from this study that the kinematical corrections alone, as they are performed e.g. in minimal relativity calculations, are not representative of relativistic effects. Even by including them in the kernel through \( E \), the results obtained are wrong by a factor 2.

In case of an energy dependent kernels the normalization condition (5) is only approximate. This energy dependence denotes coupling to higher Fock components and the correct normalization condition for the model considered reads \( N(2) + N^{(3)} = 1 \) where \( N^{(3)} \) is the norm contribution from the three-body Fock component. Using (5) only the two-body part is included. One can show that the correction \( N^{(3)} \) to the two-body normalization condition is given by:

\[
N^{(3)} = -\frac{4m^2}{(2\pi)^5} \int d^3q' d^3q \int \frac{d\Psi^*}{E_q} \frac{d\Psi}{E_{q'}} \left( \Psi^* (q', \hat{n}) \frac{\partial V}{\partial M^2} \Psi (q, \hat{n}) \right)
\]

This expression can be analytically integrated over two angles \( \varphi \) and \( \varphi' \) and we are left with a four dimensional integration. The three-body correction to the norm, i.e. the ratio \( \frac{N^{(3)}}{N(2) + N^{(3)}} \), as a function of the coupling constant is shown in Fig. 9 for the case \( \mu = 0.15 \). We remark that this correction is not zero at the critical value \( \alpha = 0.35 \) corresponding to the \( B = 0 \) threshold. Its behaviour in the region of large coupling tends asymptotically towards a value non-exceeding 30%. This is in contrast with the evolution of the parameter \( R \) introduced in the same figure to estimate the norm correction for a system with a given value of \( R \). For deuteron, e.g., one has \( R \approx 1\% \) and the expected normalization corrections are of the order of 4%.

4. A scalar model for deuteron

A simple relativistic model for deuteron in the LFD is obtained by adding to the interaction kernel inclusion.
(2) a repulsive part exactly analogous except for the sign of the coupling constant. Even if this procedure is no longer based on field theory – a scalar exchange cannot produce a repulsive interaction – the potential obtained constitutes a LFD relativistic version of the Malfliet and Tjon NN potential [18] on the form:

\[ V(q, q', \hat{n}, M^2) = \frac{V_R}{Q^2 + \mu_R^2} - \frac{V_A}{Q^2 + \mu_A^2} \] (7)

The non-relativistic model corresponds to \( Q^2 = (q - q')^2 \) and for the \( ^3S_1 \) state the parameter set \( V_R = 7.29 \text{ MeV}, V_A = 3.18 \text{ MeV}, \mu_A = 0.314 \text{ GeV}, \mu_R = 2\mu_A \) inserted in Schrödinger equation ensures a deuteron binding energy \( B = 2.23 \text{ MeV} \).

By solving the LFD Eq. (1) with potential (7) we can estimate the modification in the deuteron description due to a fully relativistic treatment. The first result concerns its binding energy which becomes \( B = 0.96 \text{ MeV} \). The inclusion of relativity produces thus a dramatic repulsive effect, already drawn in [19]. We emphasize that, as mentioned before in the case of Wick-Cutkosky model, the use of relativistic kinematics alone induces a very small change in the binding energy. The sizeable energy decrease is almost entirely due to the r.h.s part of (1). To obtain a proper deuteron description in a relativistic frame it is necessary to adjust the parameters of the non-relativistic model. A binding energy of 2.23 MeV is recovered with a repulsive coupling constant \( \Lambda_R = 6.60 \text{ MeV} \) – all other parameters being unchanged – what represents an decrease of 10% with respect to its original value. Another possibility is to increase the attractive coupling constant up to \( V_A = 3.37 \). The relativistic effects in deuteron wave function depend sensibly on the way the energy is readjusted as well as on the relative angle \( \theta \) between \( \hat{n} \) and the momentum \( q \). For instance when modifying \( V_A \) and for the value \( \theta = 0 \), the zero of the relativistic wave function is shifted by \( \approx 0.1 \text{ GeV}/c \) towards smaller values of \( q \) and the differences in the momentum region of \( q = 1.5 \text{ GeV}/c \) are 50% in amplitude.

5. Conclusion

We have obtained the solutions for a scalar model in the Light Front Dynamics framework and in the ladder approximation. The results presented here concern the S-wave bound states.

We have found that the inclusion of relativity has a dramatic repulsive effect on binding energies even for systems with very small \( \langle q^2 \rangle/m \) values. The effect is specially relevant when using a scalar model for deuteron: its binding energy is shifted from 2.23 MeV down to 0.96 MeV. This can be corrected by decreasing of 10% the repulsive coupling constant, what indicates the difficulty in determining beyond this accuracy the value of strong coupling constants within a non-relativistic framework.

Light-Front wave functions strongly differ from their non-relativistic counterparts if they are calculated using the same values of the coupling constant. Once the interaction parameters are readjusted to get the same binding energy both solutions become closer but their differences are still sizeable.

The relativistic effects are shown to be induced mainly by the relativistic terms of the kernel. The relativistic kinematics has only a small influence on the binding energy; furthermore, its effect is attractive whereas the total relativistic one is strongly repulsive.

The normalization corrections due to the three-body Fock components increase rapidly for small values of the coupling constant and saturate at \( \approx 25\% \) in the ultra relativistic region. They have been estimated to \( 4\% \) in the deuteron.

The LFD results are very close to those provided by Bethe-Salpeter equation for a wide range of coupling constant despite the different physical input in their ladder kernel. However in the case of systems interacting via a massive exchanged field, they both strongly differ from the non-relativistic solutions even in the zero binding limit. This leads to the conclusion that such systems cannot be properly described by using a non-relativistic dynamics.

The case of higher angular momentum states for scalar particles requires more formal developments and is presented in a forthcoming publication [20].

Acknowledgements

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providing some results from [14] prior to its publication.

References

$0^+ \to 2^+ \, 0\nu\beta\beta$ decay triggered directly by the Majorana neutrino mass

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Abstract

We treat $0^+ \to 2^+ \, 0\nu\beta\beta$ decays taking into account recoil corrections to the nuclear currents. The decay probability can be written as a quadratic form of the effective coupling constants of the right-handed leptonic currents and the effective neutrino mass. We calculate the nuclear matrix elements for the $0^+ \to 2^+ \, 0\nu\beta\beta$ decays of $^{76}$Ge and $^{100}$Mo, and demonstrate that the relative sensitivities of $0^+ \to 2^+ \, 0\nu\beta\beta$ decays to the neutrino mass and the right-handed currents are comparable to those of $0^+ \to 0^+ \, 0\nu\beta\beta$ decays. © 2000 Published by Elsevier Science B.V. All rights reserved.

The neutrinoless double beta ($0\nu\beta\beta$) decay can take place through an exchange of neutrino between two quarks in nuclei if the electron neutrino is a Majorana particle and has a nonvanishing mass and/or right-handed couplings [1–3]. There may be other possible mechanisms such as those involving supersymmetric particles which also cause the decay of two neutrons into two protons and two electrons [4–6]. In the present work, however, we restrict ourselves to the conventional two-nucleon and $\Delta$ mechanisms of $0\nu\beta\beta$ decay through light Majorana neutrino exchange. From the analyses of experimental data on $0^+ \to 0^+ \, 0\nu\beta\beta$ decays, stringent limits on the effective neutrino mass and the effective coupling constants of the right-handed leptonic currents have been deduced (see e.g. [3,7] and the references quoted therein). On the other hand it still seems to be believed widely that $0^+ \to 2^+ \, 0\nu\beta\beta$ decays are sensitive only to the right-handed currents. In view of the theorem that the electron neutrino should have a nonvanishing Majorana mass if $0\nu\beta\beta$ decay occurs anyway [8–10], an observation of $0\nu\beta\beta$ decay due to right-handed interactions would certainly mean also a nonvanishing Majorana mass of the electron neutrino. The purpose of the present work is, however, not to investigate the role of the Majorana neutrino mass in such a sense, but to demonstrate that it causes $0^+ \to 2^+ \, 0\nu\beta\beta$ decays directly.

A direct contribution of the neutrino mass to $0^+ \to 2^+ \, 0\nu\beta\beta$ decays was considered in [3] taking...
into account the nuclear recoil currents, and the inverse half-life was given as
\[
\left[1/T_{1/2}(0^+ \rightarrow 2^+)\right]^{-1} = F_{1+} (Z_{1+})^2 + F_{1-} (Z_{1-})^2 + F_{2+} (Z_{2+})^2 + F_{2-} (Z_{2-})^2,
\]
where \(F_{j\pm} (j = 1, 2)\) are the phase space integrals and
\[
Z_{1\pm} = M_n \langle \lambda \rangle - M_q \langle \eta \rangle \pm \frac{\langle m_p \rangle}{m_e},
\]
\[
Z_{2\pm} = M_n \langle \eta \rangle \pm \frac{\langle m_p \rangle}{m_e},
\]
with the electron mass \(m_e\) and
\[
\langle m_p \rangle = \sum_j U_{ej}^2 m_j,
\]
\[
\langle \lambda \rangle = \eta \sum_j U_{ej}^2 V_{ej},
\]
\[
\langle \eta \rangle = \eta \sum_j U_{ej}^2 V_{ej}.
\]
Here \(m_j\) is the mass of the eigenstate Majorana neutrino \(N_j\), \(U_{ej}\) and \(V_{ej}\) are the amplitudes of \(N_j\) in the left- and right-handed electron neutrinos, \(\lambda\) and \(\eta\) the coupling constants of the right-handed leptonic current with the right- and left-handed hadronic currents, and the summations should be taken over light neutrinos \((m_i \ll 100\, \text{MeV})\). The nuclear matrix elements \(M_n (\alpha = \lambda, \eta, m)\) are defined by
\[
M_n = \langle 2^+ | \frac{1}{2} \sum_{n,m} \tau_n^+ \tau_m (M_{n,m}) | 0^+_\uparrow \rangle.
\]
The explicit forms of the two-body operators \(M_n, M_\eta \) and \(M'_\eta\) were given in [11] including the contribution of the \(\Delta\) mechanism, in which the \(0\nu\beta\beta\) decay proceed through an exchange of a Majorana neutrino between two quarks in the same baryon in a nucleus. On the other hand the operator \(M_n\) was derived in [3] as
\[
(M_{n,m})_{\nu m} = -\frac{i}{2} m_e \left[ r_{nm} \otimes (\sigma_n C_m - \sigma_m C_n) \right]^{(2)} + i (g_\nu / g_\lambda) \left[ r_{nm} \otimes (D_n \times \sigma_m - D_m \times \sigma_n) \right]^{(2)} + (g_\nu / g_\lambda)^2 \left[ r_{nm} \otimes (D_n - D_m) \right]^{(2)}H(r_{nm}),
\]
where \(r_{nm} = r_n - r_m\), \(H(r)\) is the neutrino propagation function, \(g_\nu\) and \(g_\lambda\) the vector and axial vector coupling constants. \(C_n\) and \(D_n\) are the recoil correction terms to the axial vector and vector nuclear currents [2, 12] given by
\[
C_n = (p_n + p'_n) \cdot \sigma_n / 2 M_n,
\]
\[
D_n = \left[ p_n + p'_n - i \mu_\beta \sigma_\alpha \times (p_n - p'_n) \right] / 2 M_n,
\]
where \(p_n\) and \(p'_n\) are the initial and final nucleon momента, \(M\) the nucleon mass, and \(\mu_\beta = 4.7\). The above expression for \(M_n\) is, however, not suitable for numerical calculations as it stands. Therefore, as was done for \(M_\lambda, M_\eta\) and \(M'_\eta\) in [11], we expand it in terms of the operators \(M_{n,m}\) with simpler spin and orbital structures,
\[
(M_{n,m})_{\nu m} = \sum_i C_{m,i} M_{n,m}^{(i)}.
\]
We define the matrix element \(M_i\) of the operator \(M_{n,m}\) analogously to Eq. (4). The coefficients \(C_{m,i}\) and the two-body operators \(M_{n,m}\) are listed in Table 1, where
\[
h = r_{nm} H(r_{nm}), \quad h' = -r_{nm} H'(r_{nm}),
\]
\[
S_{n,m} = \left[ \sigma_n \otimes \sigma_m \right]^{(2)}, \quad S_{\pm n,m} = \sigma_n \pm \sigma_m,
\]
\[
y_{k,n,m} = \left[ \hat{r}_{n,m} \otimes \hat{r}_{n,m} \right]^{(K)},
\]
\[
y'_{k,n,m} = i \left[ \hat{r}_{n,m} \otimes p_{n,m} \right]^{(K)}, \quad Y_{k,n,m} = i \left[ \hat{r}_{n,m} \otimes p_{n,m} \right]^{(K)}.
\]
\[
r_{n,m} = r_n + r_m, \quad \hat{a} = a / |a|,
\]
\[
p_{n,m} = \frac{1}{2} (p_n - p_m), \quad P_{n,m} = p_n + p_m.
\]
As was described in detail in [11], \(M_\lambda\) and \(M_\eta\) can be expanded in terms of \(M_{k,n,m}\) with \(i = 1, 5, 8, 11, 13, 16, 17\), and \(M'_\eta\) in terms of \(M_{k,n,m}\) with \(i = 6, 7\) (for the definition of \(M_{k,n,m}\) with \(6 \leq i \leq 13\), which do not appear in Table 1, see [11]). Of these operators, \(M_{k,n,m}\) with \(8 \leq i \leq 13\) are related to the \(0\nu\beta\beta\) transitions which involve virtual \(\Delta\) particles in nuclei, and they are induced by the operator \(M_{n,m}\) interpreted as acting on two quarks in a nucleon or a \(\Delta\) particle.
Table 1

The operators $M_{r_{nm}}$ and the coefficients $C_{m_i}$, the latter in units of the electron-nucleon mass ratio $m_e/M$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$M_{r_{nm}}$</th>
<th>$C_{m_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\sqrt{3}hS_zy_z$</td>
<td>$-\frac{3}{4}[\mu]\frac{(g_V/g_A)+\frac{1}{4})}{\frac{1}{4}[\mu]\frac{(g_V/g_A)-1]}$</td>
</tr>
<tr>
<td>2</td>
<td>$hS_z$</td>
<td>$-\frac{\sqrt{3}}{4\sqrt{2}}[\mu]\frac{(g_V/g_A)^2}{(g_V/g_A)^2}$</td>
</tr>
<tr>
<td>3</td>
<td>$h(S_z \otimes y_z)^{(2)}$</td>
<td>$\frac{\sqrt{3}}{4\sqrt{2}}[\mu]\frac{(g_V/g_A)^2}{(g_V/g_A)^2}$</td>
</tr>
<tr>
<td>4</td>
<td>$hy_z$</td>
<td>$\frac{1}{2\sqrt{2}}(g_V/g_A)$</td>
</tr>
<tr>
<td>5</td>
<td>$h(S_z \otimes y_z)^{(2)}$</td>
<td>$\frac{1}{2\sqrt{2}}(g_V/g_A)$</td>
</tr>
<tr>
<td>6</td>
<td>$h(S_z \otimes y_z)^{(2)}$</td>
<td>$\frac{1}{2\sqrt{2}}(g_V/g_A)$</td>
</tr>
<tr>
<td>7</td>
<td>$h(S_z \otimes y_z)^{(2)}$</td>
<td>$\frac{1}{2\sqrt{2}}(g_V/g_A)$</td>
</tr>
<tr>
<td>8</td>
<td>$h(S_z \otimes y_z)^{(2)}$</td>
<td>$\frac{1}{2\sqrt{2}}(g_V/g_A)$</td>
</tr>
</tbody>
</table>

The new operators $M_{r_{nm}}$ with $14 \leq i \leq 25$ appear only in the expansion of $M_{r_{nm}}$. In the derivation of $C_{m_i}$ listed in Table 1, we have not taken into account the $\Delta$ mechanism yet. Under the same assumption of the non-relativistic constituent quark model about the $\Delta$ mechanism as was made in [11], the operators in Table 1 except $M_{r_{nm}}$ with $i = 2, 16, 17$ do not contribute when interpreted as acting on two quarks in a nucleon or a $\Delta$ particle. Since the relation $\mu_{q}=g_{V}=g_{A}=1$ holds for the quark currents, we see $C_{m_i}=0$ for $i = 2, 16$. The only possible contribution of $M_{17n_{nm}}$ to $M_{r_{nm}}$ is estimated to be about $m_e/2M$ of the $\Delta$ mechanism contributions to $M_{r_{nm}}$ and $M_{r_{n}}$. Therefore we will neglect the $\Delta$ mechanism for the calculation of $M_{r_{nm}}$ in the present work.

We calculate the nuclear matrix elements for $0^+_1 \rightarrow 2^+_1$ $0\nu\beta\beta$ decay of $^{76}$Ge and $^{100}$Mo using the method given in [11]. We describe the initial $0^+_1$ and final $2^+_1$ nuclear states in terms of the Hartree–Fock–Bogoliubov type wave functions which are obtained by variation after particle-number and angular-momentum projection [11–13]. For the case of $^{76}$Ge decay, the calculation of the matrix elements $M_i$ with $1 \leq i \leq 13$ has been performed in [11]. In the present work we calculate only the new ones with $14 \leq i \leq 25$ using the nuclear wave functions obtained in [11]. In order to calculate all $M_i$ with $1 \leq i \leq 25$ for the $^{100}$Mo decay, the nuclear wave functions of $^{100}$Mo$(0^+_1)$ and $^{100}$Ru$(2^+_1)$ are constructed in the same manner as in the case of the $^{76}$Ge decay. Table 2 shows the calculated matrix elements $M_{r_{nm}}$ for the $^{76}$Ge and $^{100}$Mo decays as a sum of the products $C_{m_i}M_i$. It should be noted that the matrix elements of the operators with rank 0 spin
part, i.e. $M_1$, $M_4$, $M_{14}$ and $M_{15}$ have the dominant contributions to $M_{ii}$. Table 3 summarizes the calculated matrix elements $M_{ii}$, $M_{ij}$, $M'_{ij}$ and $M_{i}$ for the $^{76}$Ge and $^{100}$Mo decays.

The differential rate for $0^+ \rightarrow 2^+\ 0\nu\beta\beta$ decay with the energy of one of the emitted electrons $\epsilon_1$ and the angle between the two electrons $\theta_{12}$ can be written as

$$\frac{d^2W_{0\nu}}{d\epsilon_1 d\cos \theta_{12}} = a^{(0)}(\epsilon_1) + a^{(1)}(\epsilon_1)P_1(\cos \theta_{12}) + a^{(2)}(\epsilon_1)P_2(\cos \theta_{12}).$$

Each of the angular correlation coefficients $a^{(k)}(\epsilon_1)$ ($k = 0,1,2$) can be expressed as a sum of the products of an electron phase space factor and a second order monomial of $Z_{ij}$ defined in Eq. (2). The explicit form of $a^{(0)}(\epsilon_1)$, which yields $(\ln 2)/2$ times the right hand side of Eq. (1) upon integration over $\epsilon_1$, can be readily obtained by combining the relevant equations in [3]. Since the expressions for $a^{(1)}(\epsilon_1)$ and $a^{(2)}(\epsilon_1)$ are rather complicated, they will be given elsewhere. Numerical calculations show that $a^{(1)}(\epsilon_1)$ is dominated by a term with the factor $-(Z_{1+})^2 + Z_{2+}Z_{2-}$ times a positive function of $\epsilon_1$, whereas $a^{(2)}(\epsilon_1)$ by a term with the factor $2Z_{1+}Z_{1-} - (Z_{2+})^2 - (Z_{2-})^2$. For later reference we denote these two factors as $z^{(1)}$ and $z^{(2)}$, respectively.

Fig. 1 shows the single electron spectra $dW_{0\nu}/d\epsilon_1$ in arbitrary units and the ratios of the angular correlation coefficients $a^{(1)}/a^{(0)}$ and $a^{(2)}/a^{(0)}$ for the $0^+ \rightarrow 2^+_1\ 0\nu\beta\beta$ decay of $^{100}$Mo. They are all plotted against the kinetic energy fraction of one of the two emitted electrons, where $Q_{bb}(0^+ \rightarrow 2^+_1) = 2.494$ MeV. Only one of the three lepton number violating parameters is assumed to be nonvanishing for each of the three cases: (a) $\langle \lambda \rangle \neq 0$, (b) $\langle \eta \rangle \neq 0$ and (c) $\langle m_\nu \rangle \neq 0$.

Table 3
<table>
<thead>
<tr>
<th>$M_1$</th>
<th>$M_4$</th>
<th>$M_{14}$</th>
<th>$M_{15}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{76}$Ge</td>
<td>1.81*</td>
<td>13.37*</td>
<td>0.18*</td>
</tr>
<tr>
<td>$^{100}$Mo</td>
<td>6.33</td>
<td>3.38</td>
<td>5.17</td>
</tr>
</tbody>
</table>


Fig. 1. Single electron spectrum $dW_{0\nu}/d\epsilon_1$ in arbitrary units and the ratios of the angular correlation coefficients $a^{(1)}/a^{(0)}$ and $a^{(2)}/a^{(0)}$ for the $0^+ \rightarrow 2^+_1\ 0\nu\beta\beta$ decay of $^{100}$Mo. They are all plotted against the kinetic energy fraction of one of the two emitted electrons, where $Q_{bb}(0^+ \rightarrow 2^+_1) = 2.494$ MeV. Only one of the three lepton number violating parameters is assumed to be nonvanishing for each of the three cases: (a) $\langle \lambda \rangle \neq 0$, (b) $\langle \eta \rangle \neq 0$ and (c) $\langle m_\nu \rangle \neq 0$. 

Fig. 1 shows the single electron spectra $dW_{0\nu}/d\epsilon_1$ in arbitrary units and the ratios of the angular correlation coefficients $a^{(1)}/a^{(0)}$ and $a^{(2)}/a^{(0)}$ for the three limiting cases, (a) $\langle \lambda \rangle \neq 0$, (b) $\langle \eta \rangle \neq 0$ and (c) $\langle m_\nu \rangle \neq 0$. Since the coefficients $a^{(k)}(\epsilon_1)$ depend on the
parameters $\langle \lambda \rangle, \langle \eta \rangle$ and $\langle m_n \rangle$ through $Z_{12j}$, the results shown in Fig. 1 are independent of nuclear models for the cases (a) and (c). We can also easily understand the signs of $d^{(1)}$ and $d^{(2)}$ from the relations $z^{(1)} = -(M_n\langle \lambda \rangle)^2$ and $z^{(2)} = 2(M_n\langle \lambda \rangle)^2$ for the case (a), and $z^{(1)} = -2(M_m\langle m_n \rangle/m_0)^2$ and $z^{(2)} = -4(M_m\langle m_n \rangle/m_0)^2$ for the case (c). On the other hand for the case (b), we obtain $z^{(1)} = -(M_n\langle \eta \rangle)^2 + (M_n\langle \eta \rangle)^2$ and $z^{(2)} = 2(M_m\langle \eta \rangle)^2 - 2(M_m\langle \eta \rangle)^2$, and consequently a cancellation between the contributions of $M_n$ and $M_m$ occurs when these are of comparable magnitudes. This is just the case for the $^{100}$Mo decay, but not for the $^{76}$Ge decay where $M_n$ is much smaller than $M_m$ so that there is no significant difference between the cases (a) and (b) in the angular correlation. It should also be noted in Fig. 1 that the single electron spectra for all the three cases (a), (b) and (c) have approximately the same shape. This is in contrast with the $0^+ \rightarrow 0^+$ decays where the spectrum for $\langle \lambda \rangle \neq 0$ is very different from those for $\langle m_n \rangle \neq 0$ or $\langle \eta \rangle \neq 0$ [2,3].

Using the matrix elements in Table 3 and the phase space integrals $F_{1+}$ calculated in [3], we can deduce from the experimental data $\tau_{12j}^{0/2}(0^+ \rightarrow 2^+)$ > $8.2 \times 10^{31}$ yr (90% C.L.) [14] for the $^{76}$Ge decay the constraints on the right-handed current couplings and the effective neutrino mass listed in Table 4. As for the $^{100}$Mo decay, the Osaka group has obtained the limit $\tau_{12j}^{0/2}(0^+ \rightarrow 2^+)$ > $1.4 \times 10^{22}$ yr (68% C.L.) [15] assuming $\langle \lambda \rangle \neq 0$. Because of the differences in the angular correlation as we see from Fig. 1, an analysis of the same raw experimental data might yield a half-life limit significantly different from the above value especially for the case $\langle \eta \rangle \neq 0$. However we assume here just the same half-life limit also for the cases $\langle \eta \rangle \neq 0$ and $\langle m_n \rangle \neq 0$ in order to compare the resulting constraints with those from the $^{76}$Ge data.

The limits which can be deduced from the experimental bound $\tau_{12j}^{0/2}(0^+ \rightarrow 0^+) > 5.7 \times 10^{25}$ yr (90% C.L.) [16] on the $0^+ \rightarrow 0^+$ decay of $^{76}$Ge using the nuclear matrix elements of [17] are $|\langle \lambda \rangle| < 3.8 \times 10^{-7}$, $|\langle \eta \rangle| < 2.2 \times 10^{-9}$ and $|\langle m_n \rangle| < 0.19$ eV. Comparing these limits with those of Table 4, we notice the considerable difference in the absolute sensitivities between the $0^+ \rightarrow 0^+$ and $0^+ \rightarrow 2^+$ decays, which reflects the smaller $Q$-value as well as the higher electron partial waves associated with the latter. However, it should be stressed here that the relative sensitivities to $\langle m_n \rangle$ and $\langle \eta \rangle$ are comparable in both cases. In other words, $\langle m_n \rangle = 1$ eV would give roughly the same decay rate as $\langle \eta \rangle = 10^{-8}$ in the $0^+ \rightarrow 2^+$ as well as in the $0^+ \rightarrow 0^+$ decays. At the same time it should also be noted that the $0^+ \rightarrow 2^+$ decay is relatively more sensitive to $\langle \lambda \rangle$.

In summary, we have calculated $0^+ \rightarrow 2^+$ $0\nu\beta\beta$ decay rates taking into account the recoil corrections to the nuclear currents. As a result, the expression for the decay probability becomes a quadratic form of not only the effective coupling constants $\langle \lambda \rangle$ and $\langle \eta \rangle$ of the right-handed leptonic currents but also the effective neutrino mass $\langle m_n \rangle$ which would be totally absent without the inclusion of the recoil corrections. In other words, the recoil corrections give the lowest order contribution to the $0^+ \rightarrow 2^+$ $0\nu\beta\beta$ decay for the case where $\langle \lambda \rangle = \langle \eta \rangle = 0$ and $\langle m_n \rangle \neq 0$. Furthermore, by the numerical calculation of the relevant nuclear matrix elements, we have found that the relative sensitivities of $0^+ \rightarrow 2^+$ decays to $\langle m_n \rangle$ and $\langle \eta \rangle$ are comparable to those of $0^+ \rightarrow 0^+$ decays. Of course, the calculated matrix elements and consequently the limits listed in Table 4 depend on the assumption about the nuclear model. In order to improve their reliability, further investigation may be necessary. However, since such a model dependence is expected to be weaker for the ratios of these limits, the above result concerning the relative sensitivities would probably not change very much.

Acknowledgements

The author thanks S. Yamaji at RIKEN for his warm hospitality.

Table 4

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Assuming the same limit on $\tau_{12j}^{0/2}$ as the $\langle \lambda \rangle$ mode.
References

Drell-Hearn-Gerasimov sum rule for the nucleon in the large-$N_c$ limit

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Abstract

We show that the Drell-Hearn-Gerasimov sum rule for the nucleon is entirely saturated by the $\Delta$ resonance in the limit of a large number of colors, $N_c \to \infty$. Corrections are at relative order $1/N_c^2$. $^{22}$

The Drell-Hearn-Gerasimov (DHG) sum rule for a spin-1/2 fermion relates its anomalous magnetic moment to an integral over the spin-dependent photo-production cross section [1]. Define the total photo-production cross sections $\sigma_P(\nu)$ and $\sigma_A(\nu)$, where $P$ and $A$ label the spins of the photon being parallel and anti-parallel, respectively, to the spin of the fermion and $\nu$ is the photon energy. The DHG sum rule says,

$$\int_{\nu_m}^{\infty} \frac{d\nu}{\nu} \left[ \sigma_P(\nu) - \sigma_A(\nu) \right] = \frac{2\pi^2 \alpha_{em} \kappa^2}{M^2}. \tag{1}$$

where $\kappa$ and $M$ are the anomalous magnetic moment (dimensionless) and mass of the fermion, respectively. For the case of the nucleon it is useful to break the sum rule up into isoscalar and isovector components.

It has been known empirically for a long time that the isoscalar part of the DHG sum rule for the nucleon is almost entirely saturated by the $\Delta$-resonance contribution. The theoretical expectation for the sum (i.e. the right-hand side (RHS) of Eq. (1)) is 219 $\mu$b, whereas the $\Delta$-resonance contribution to the left-hand side (LHS) integral is 240 $\mu$b [2]. To the best of our knowledge, there hasn’t been a solid theoretical explanation for this from fundamental principles of quantum chromodynamics (QCD). In this short note, we show that such phenomenon is expected in QCD in the limit of a large number of colors ($N_c \to \infty$) [3,4].

During the past several years, there has been significant progress in understanding the baryon properties in the large-$N_c$ QCD. Large-$N_c$ consistency of the theory has been exploited to show that it has a contracted $SU(2n_f)$ spin-flavor symmetry, where $n_f$ is the number of quark flavors, as discussed in Refs. [5,6]. For example, the symmetry is required for cancellation among Feynman diagrams with different intermediate states in a hadronic description of scattering in order to preserve the unitarity bound. The spin-flavor symmetry implies that in the large-$N_c$ limit there is a tower of states with $I = J$ which are degenerate in the large-$N_c$ limit; the
lowest two states are identified with the nucleon and the \( \Delta \)-resonance. It has also been shown the splitting between the lowest states in this tower (e.g. the nucleon and \( \Delta \)) scales as \( 1/N_c \) \([7]\). The symmetry also has the feature that certain hadronic matrix elements are related to each other in the large-\( N_c \) limit. For example, the ratio of the axial-current matrix element in the nucleon to the transition matrix element between the nucleon and the \( \Delta \) is fixed in the large-\( N_c \) limit. Moreover, there are cases, such as the axial matrix elements, where the leading correction to this ratio occurs at relative order \( 1/N_c^2 \) \([6]\).

As pointed out by Broniowski \([8]\), the same underlying symmetry principle is responsible for understanding large-\( N_c \) consistency of low-energy sum rules such as those of Adler-Weisberger \([9]\) and Cabibbo-Radicati \([10]\). In this paper, we follow Broniowski’s observation and point out that the \( \Delta \)-dominance in the isoscalar part of the DHG sum rule is required by a similar large-\( N_c \) consistency and reflects the above-mentioned spin-flavor symmetry. Moreover, we will show that the \( \Delta \) dominance is valid up to relative order \( 1/N_c^2 \).

Let us do some large-\( N_c \) counting for the electromagnetic couplings in order to understand the large-\( N_c \) consistency of the DHG sum rule. Consider two quark flavors: up and down. The proton neutron is valid up to relative order 1.

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cross section turns out to be quite special: it is of order $\alpha_{em} N_c$. Therefore, the $\Delta$-resonance contribution to the DHG sum rule can be of order $\alpha_{em} N_c^2$.

Indeed, a straightforward calculation yields the following $\Delta$-photoproduction cross section,

$$\sigma(\nu) = \delta(\nu - \Delta) 2\pi^2 \alpha_{em} \mu_{\Delta N}^2 \Delta^2 \mu_{\Delta N}^2$$

where $\Delta = M_\Delta - M$ is the nucleon-$\Delta$ mass difference and $\mu_{\Delta N}$ is the MI coupling between the photon, nucleon and $\Delta$-resonance. Its contribution to the LHS of the DHG integral is

$$2\pi^2 \alpha_{em} \mu_{\Delta N}^2$$

In the large-$N_c$ limit $\mu_{\nu \Delta} = \kappa_{\nu} / M$, and so the above contribution matches exactly the leading $N_c^2$ contribution from the RHS of Eq. (1). Since the isoscalar anomalous magnetic moment $\kappa_s / M$ is of order $1 / N_c$, the relation $\mu_{\nu \Delta N} = \kappa_{\nu} / M$ is in fact correct up to $\mathcal{O}(N_c^0)$ ($\kappa_{\nu}$ is of $\mathcal{O}(N_c^2)$).

The large-$N_c$ consistency of the isovector component of the DHG sum rule is simple: The RHS of the sum rule scales like $\alpha_{em} \kappa_{\nu} \kappa_{s} / M^2 \sim \alpha_{em}$. The isovector spin-dependent cross section goes like $\alpha_{em} (\mathcal{O}(\kappa_{\nu}, \kappa_{s}) / N_c \sim \alpha_{em}$. So the $N_c$ counting is completely consistent.

The Drell-Hearn-Gerasimov sum rule is derived under the condition that $N_c = \infty$. However, if one starts with QCD with $N_c = \infty$, then the low-energy theorem for the Compton scattering amplitude changes since the $\Delta$ is now a massless excitation. The Compton amplitude, $S_i(\nu, Q^2)$, is defined according to

$$T^{\mu\nu} = i\int d^4q e^{iq\xi} \langle PS | T J^\mu(\xi) J^\nu(0) | PT \rangle$$

$$= -ie^{\nu\alpha\beta}q^\alpha S_\nu S_\xi(\nu, Q^2) + \ldots$$

where $S_\alpha$ is the polarization vector of the nucleon. Including the $\Delta$-resonance contribution, we find

$$S_i(0, 0) = -(2 \kappa_{\nu} \kappa_{\Sigma} + \kappa_{s}^2) / M^2.$$  \hspace{1cm} (8)

The leading-order nucleon-pole contribution is exactly cancelled by the $\Delta$-resonance pole. This cancellation reflects the contracted $SU(4)$ symmetry on the LHS of the DHG, the $\Delta$-resonance contribution is absent because it is forbidden by energy-momentum conservation.

It is significant to notice that the $\Delta$ dominates the DHG sum rule up to corrections of order $\alpha_{em} N_c^0$. The easiest way to see this is to compare the two sides of the sum rule. The right-hand side scales as $\alpha_{em} N_c^2$, while the generic contributions to the left-hand side are order $\alpha_{em} N_c^0$. One concludes that the $\Delta$ contribution must therefore account for both the $\alpha_{em} N_c^2$ and $\alpha_{em} N_c^0$ contribution on the left-hand side. In fact, the lack of a correction at relative order $1 / N_c$ is easily understood in another way. As noted above, there are cases, such as the axial-vector current, where the ratios of nucleon matrix elements and nucleon-$\Delta$ transition matrix elements are fixed with corrections at relative order $1 / N_c^2$ [6]. The isovector magnetic coupling of a photon has the same structure (in terms of the couplings to quarks) as the axial current. Thus the ratio of the nucleon isovector magnetic moment and the nucleon-$\Delta$ magnetic transition moment is also fixed by the group structure up to relative corrections of order $1 / N_c^2$.

Finally, we note that the analysis of the experimental data indicates that the S11 resonance contribution to the DHG sum rule ($\sim -170$ $\mu$b) is not significantly smaller than that of the $\Delta$ resonance ($\sim 240$ $\mu$b) or of the total sum (219 $\mu$b). We have no natural explanation for the large S11 contribution in terms of the large-$N_c$ analysis discussed in this paper. However, we wish to emphasize that our analysis is for an inclusive observable and its validity relies on the notion that the total subleading correction is small. The large S11 contribution does not necessarily invalidate the $N_c$ expansion for the DHG integral. This situation is somewhat analogous to deep-inelastic scattering where a perturbative parton calculation works for the total inclusive cross section and fails completely for production of a particular final state.

To summarize, we argued that the DHG sum rule is saturated by the $\Delta$ resonance in the large-$N_c$ limit up to correction terms of relative order $1 / N_c^2$. The result reflects the contracted spin-flavor $SU(4)$ symmetry in the large-$N_c$ QCD with two flavors. Turning the argument around, the contracted $SU(4)$ symmetry indicates that the spin-dependent photonucleon cross section must be of order $\alpha_{em} N_c^0$.

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References

Effects of baryon resonances on nucleon–nucleon interactions in a quark model

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Abstract

Nucleon–nucleon S-wave scattering is studied by using the resonating group method, which includes not only the nucleon and Δ but also their excited states. As quark–quark interactions, the one-gluon-exchange and one-pion-exchange potentials are employed with phenomenological confinement potential. Compared with the N and Δ, coupling effects induced by excited-state baryons are small.

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Keywords: Resonating group method

There have been many attempts to explain the nucleon–nucleon (NN) interaction by using the resonating group method (RGM) in the nonrelativistic constituent quark models [1–11]. In these works, the one-gluon-exchange potential (OGEP) and/or the one-meson-exchange potential (OMEP) have been mainly used as quark–quark (qq) interactions, supplemented with phenomenological confinement potential.

The repulsive core of the NN potential has been successfully explained by the Pauli principle for the constituent quarks and the color magnetic interaction of the OGEP [1,2]. The medium- and long-range attractive parts of the NN interaction, however, cannot be reproduced by the OGEP. In order to obtain this attraction, additional one-meson-exchange potentials (OMEP) have been introduced [1–4,7,10,11]. It recently has been shown that the OMEP also can induce the short-range NN repulsion [8,9].

The standard RGM calculations, which usually contain only the “ground-state” baryons, i.e., the N and Δ, have been in a reasonably good agreement with scattering data [1–4,7]. Particular channel-couplings, such as the 3S1 NN−3D1 NN and 1S0 NN−1D1 NΔ, have played a crucial role in reproducing the experimental NN phase shifts. These couplings are mainly induced by the tensor interaction of the one-pion-exchange potential (OPEP).

There still remain some issues to be clarified. One of these issues is the role of excited states of the nucleon and Δ, i.e., N+ and Δ+, in the NN interac-

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tion. Several authors [10,11] have partially taken account of these states and shown their influence on the $NN$ scattering. However, detailed analyses of their role in the $NN$ interaction have not been performed yet. Moreover, using the Born-Oppenheimer approximation, Stancu et al. [12] recently have claimed that the commonly used RGM wave functions are not appropriate in a study of the short-range $NN$ interaction if the OEP is employed. It has been well known, on the other hand, that in the case of the OGEp excited-state baryons do not cause any significant changes in the $NN$ interaction [1±4].

The purpose of this paper is therefore to clarify the effect of baryon resonances on the $NN$ interaction. To this end, we systematically study the $NN$ $S$-wave scattering by using the coupled-channel RGM, which includes not only the $N$ and $\Delta$ but also their excited states.

Besides the confinement potential, both OGEp and OPEp are employed as $qq$ interactions. Other mesons, such as the $\sigma$ meson, are not included, in order to make the model simple. Although the nature of the $\sigma$ meson is still an open problem, many authors have claimed that this meson is required to reproduce the intermediate attraction of the $NN$ interaction [3,4,7,10,13]. Because baryon resonances are explicitly taken into account by the coupled-channel procedure in this work, some scattering processes, such as the $2\pi$-exchange process with intermediate $NN^*$ states, can mimic a part of the one-sigma-exchange $NN$ interaction. If the contribution of the $\sigma$ meson were included in the present model, the coupling strength should be different from that used in previous works that have not included baryon resonances.

The model hamiltonian for a single-baryon is

$$H_3 = \sum_{i=1}^{3} K_i - t_c - \sum_{i<j} a(\lambda_i \cdot \lambda_j) r_{ij} + \sum_{i<j} V_{ij} + 3m,$$

where $m$ is the constituent $u$- and $d$-quark mass, and $K_i$ and $t_c$ are the nonrelativistic kinetic energies of the $i$-th quark and the center of mass of the baryon, respectively. The strength of the linear confinement potential is denoted by $a$, the SU(3) color generator for the $i$-th quark by $\lambda_i$, and the distance between the $i$-th and $j$-th quarks by $r_{ij}$.

The OGEp is necessary to reproduce the $N$–$\Delta$ mass splitting if the OPEp has the quark–pion coupling constant $g_{\pi qq}$ derived from the experimental value of the pion–nucleon coupling constant $G_{\pi NN}$; the observed $N$–$\Delta$ mass splitting is about 290 MeV, while the OPEp provides 190 MeV in the present model. The $qq$ interaction $V_{ij}$ is assumed as

$$V_{ij} = V_{ij}^{\text{OGEp}} + V_{ij}^{\text{OPEp}}, \quad (2)$$

$$V_{ij}^{\text{OGEp}} = \frac{\alpha_s}{4} (\lambda_i \cdot \lambda_j) \left[ \frac{1}{r_{ij}} - \frac{2\pi}{3m^2} (\sigma_i \cdot \sigma_j) \left( \frac{\kappa}{\pi} \right)^{3/2} \right] \times e^{-\kappa r_{ij}^2} - \frac{1}{4m^2} \frac{1}{r_{ij}^4} S_{ij}, \quad (3)$$

$$V_{ij}^{\text{OPEp}} = \frac{g_{\pi qq}^2}{4\pi} \left( \tau_i \cdot \tau_j \right) \left[ S_{\pi}(r_{ij}) (\sigma_i \cdot \sigma_j) + T_{\pi}(r_{ij}) S_{ij} \right], \quad (4)$$

$$S_{\pi}(r_{ij}) = m_{\pi}^2 e^{-m_{\pi}^2 r_{ij}} - 4\pi \left( \frac{\kappa_{\pi}}{\pi} \right)^{3/2} e^{-\kappa_{\pi} r_{ij}}, \quad (5)$$

$$T_{\pi}(r_{ij}) = \left( \frac{3}{r_{ij}^2} + \frac{3m_{\pi}^2}{r_{ij}^2} + \frac{m_{\pi}^4}{r_{ij}^4} \right) e^{-m_{\pi}^2 r_{ij}}, \quad (6)$$

where $m_{\pi}$ is the pion mass and $\alpha_s$ is the quark–gluon coupling constant. The spin and isospin generators for the $i$-th quark are denoted by $\sigma$, $\tau$, respectively. $S_{ij}$ stands for the standard tensor operator. The $\delta$-function terms stemming from the contact interactions in the OGEp and OPEp are smeared out by introducing the range parameters $\kappa_{\pi}$ and $\kappa_{\pi}$ [14]. It should be noted that there is a controversy over the use of the OPEp as the short-range part of $qq$ interactions [15–19]; this problem has not been solved yet.

Since there are many papers explaining how to construct single-baryon wave functions [20], only a brief description is given here. For a baryon with total spin $j$ and total isospin $T$, the wave function $\Phi$ is schematically written as

$$\Phi(\xi) = \mathcal{A}_3 \left[ \phi_{ij} \otimes \chi^j \right] \chi^T \chi^j,$$

where $\xi$ stands for the spatial and internal coordinates, and $\mathcal{A}_3$ is the three-quark antisymmetrizer. The isospin part is denoted by $\chi^j$, and the color part by $\chi^T$, which is always color-singlet. The spatial-spin
part is represented by \([\psi_n^l \otimes \chi^s]^l\); \(\psi_n^l\) is the spatial wave function with radial node \(n\) and orbital angular momentum \(l\), and it is combined properly with the spin wave function \(\chi^s\), which has the total intrinsic-spin \(s\) of quarks. In order to make the RGM calculation tractable, we assume the single-baryon wave function is an eigenfunction of a simple Hamiltonian that contains only the harmonic-oscillator potential (i.e., \(\frac{m}{2}\sum r_{ij}^2\)) instead of the linear confinement and the residual \(qq\) interaction \(V_{ij}\) in Eq. (1). As in the previous RGM calculations [1], the baryon mass is defined by

\[
m_b = \int d\xi \Phi^* (\xi) H \Phi (\xi).
\] (8)

The harmonic-oscillator parameter \(\beta = m \omega\) is determined by the variational condition for the nucleon mass \(m_N\):

\[
\frac{\partial m_N}{\partial \beta} = 0.
\] (9)

In addition to the \(N\) and \(\Delta\), which have been usually taken into account, ground-state baryons' excited states up to \(2\hbar \omega\) excitations are also considered in this work. These states are summarized in Table 1, together with their symmetric structure, expressed by a shorthand notation of the Young diagram. The symbols \(\hat{O}\) and \(\hat{R}\) stand for the orbital and radial excitations, respectively. Excited states that belong to the same energy-shell are distinguished by their calculated masses and labeled by subscripts \((1, 2, \ldots)\), from the lightest one. Since these excited states have various combinations of the orbital, spin and isospin symmetries, the present model has much richer symmetry patterns than the conventional RGM calculations. This is an advantage of our study.

We now fix the model parameters. The value of \(G_{\pi NN}/4\pi = 14.3\) [21] by using the spin−isospin symmetry relation of the constituent quark model. (The results change little if a newly reported value of \(G_{\pi NN}/4\pi = 13.7\) [22] is used.) The pion mass \(m_\pi = 140\text{ MeV}\) is also taken from the observed value. The oscillator parameter \(\beta\) is chosen by the variational condition (9). The other parameters are determined by fitting the \(N−\Delta\) mass splitting; other mass splittings, such as \(N−N^*_1\), \(\Delta−\Delta^*\) and so on, are also used as subsidiary conditions. All the parameters obtained are summarized in Table 2. Note that there are no adjustable parameters in the following calculations of the \(NN\) scattering.

Before discussing the \(NN\) interaction, we briefly comment on baryon mass spectra. Our calculations do not reproduce the observed spectra satisfactorily, as seen in Fig. 1.

In order to improve the result, we must diagonalize the Hamiltonian (1) in terms of the single-baryon wave functions and then readjust the parameters. Although this procedure is in principle desirable, the RGM calculation with these diagonalized single-

---

**Table 1**

Properties of baryons

<table>
<thead>
<tr>
<th>Baryon</th>
<th>Shell</th>
<th>Symmetry</th>
<th>Excitation</th>
<th>Parity</th>
<th>Total spin</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td>0</td>
<td>[3] [21]</td>
<td>[21]</td>
<td>–</td>
<td>+</td>
</tr>
<tr>
<td>(N^+_1)</td>
<td>1</td>
<td>[21]</td>
<td>[21]</td>
<td>(\hat{O})</td>
<td>–</td>
</tr>
<tr>
<td>(N^+_2)</td>
<td>1</td>
<td>[21] [3]</td>
<td>[21]</td>
<td>(\hat{O})</td>
<td>–</td>
</tr>
<tr>
<td>(N^+_3)</td>
<td>2</td>
<td>[3]</td>
<td>[21] [21]</td>
<td>(\hat{R})</td>
<td>+</td>
</tr>
<tr>
<td>(N^+_4)</td>
<td>2</td>
<td>[21] [21]</td>
<td>[21]</td>
<td>(\hat{O} + \hat{R})</td>
<td>+</td>
</tr>
<tr>
<td>(N^+_5)</td>
<td>2</td>
<td>[111]</td>
<td>[21]</td>
<td>[21]</td>
<td>(\hat{O})</td>
</tr>
<tr>
<td>(N^+_6)</td>
<td>2</td>
<td>[21]</td>
<td>[3]</td>
<td>[21]</td>
<td>(\hat{O})</td>
</tr>
<tr>
<td>(\Delta)</td>
<td>0</td>
<td>[3]</td>
<td>[3] [3]</td>
<td>–</td>
<td>+</td>
</tr>
<tr>
<td>(\Delta^+_1)</td>
<td>1</td>
<td>[21]</td>
<td>[21]</td>
<td>[3]</td>
<td>(\hat{O})</td>
</tr>
<tr>
<td>(\Delta^+_2)</td>
<td>2</td>
<td>[3]</td>
<td>[3] [3]</td>
<td>(\hat{R})</td>
<td>+</td>
</tr>
<tr>
<td>(\Delta^+_3)</td>
<td>2</td>
<td>[3]</td>
<td>[3] [3]</td>
<td>(\hat{O})</td>
<td>+</td>
</tr>
<tr>
<td>(\Delta^+_4)</td>
<td>2</td>
<td>[21]</td>
<td>[21]</td>
<td>[3]</td>
<td>(\hat{O})</td>
</tr>
</tbody>
</table>
Table 2
Model parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>m (MeV)</td>
<td>340</td>
</tr>
<tr>
<td>β (1/fm²)</td>
<td>4.34</td>
</tr>
<tr>
<td>a (MeV/fm)</td>
<td>87.0</td>
</tr>
<tr>
<td>αs</td>
<td>0.420</td>
</tr>
<tr>
<td>κ (1/fm²)</td>
<td>10.0</td>
</tr>
<tr>
<td>mω (MeV)</td>
<td>140</td>
</tr>
<tr>
<td>κω (1/fm²)</td>
<td>7.00</td>
</tr>
<tr>
<td>σπee</td>
<td>2.91</td>
</tr>
</tbody>
</table>

While the baryon wave functions becomes quite complicated and time-consuming. Fortunately, it has been found numerically that our simple wave functions are the dominant components of the diagonalized single-baryon wave functions, and that similar results are finally obtained for the coupling effects of the $N'$ and $Δ'$. Even if the model parameters are largely varied. It is therefore expected that our approximation for the single-baryon wave functions works at least for qualitative arguments.

We proceed to consider a two-baryon system, i.e., a six-quark system. The model hamiltonian is given as

$$H = \sum_{i=1}^{6} K_i - T_G - \sum_{i<j} a(\Lambda_i, \Lambda_j) r_{ij} + \sum_{i<j} V_{ij} + 6m,$$

(10)

where the two-body $qq$ interactions are the same as those for a single-baryon, and $T_G$ is the kinetic energy of the center of mass of the two-baryon system.

The RGM wave function is constructed in terms of the two color-singlet baryon wave functions, $Φ_a$ and $Φ_b$, as follows;

$$Ψ_a(ξ_a, ξ_b, R) = :φ\left(\left[Φ_a(ξ_a) \otimes Φ_b(ξ_b)\right] \otimes χ_a(R)\right):\alpha,$$

where $χ_a$ is the relative-motion wave function and $α$ stands for a set of quantum numbers that completely specifies the two-baryon scattering channel. The relative coordinate between the two baryons is denoted by $R$. The six-quark antisymmetrizer is expressed by $φ$.

The RGM equation of motion [1] can be reduced to the Lippmann-Schwinger equation for the $T$-matrix. In the present model, the calculation of the transition potential $V_{aa'}$ is rather involved because various two-baryon channels are dealt with. In order to calculate $V_{aa'}$ systematically by using the Talmi-Moshinsky technique, we expand the relative-motion wave function $χ_a$ in terms of the harmonic-oscillator eigenfunction with the same oscillator parameter $b$ used in the single-baryon wave functions $F$ [24,25].

In Table 3 are listed the two-baryon channels examined in this work; they are classified according to the sum of excitation energies of the two baryons.

When the total isospin, total spin and relative angular momentum of the two-baryon system are specified, there are 42 channels coupling to the $^3S_1NN$ channel and 34 channels coupling to the $^1S_1NN$ channel. Among these channels, we first search for several channels that may affect the $NN$ $S$-wave scattering; in the case of the $^3S_1NN$ scattering, for example, we solve the two-channel Lippmann-Schwinger equation that contains one of the 42 channels and the $^3S_1NN$ channel, and compare the result with the single-channel $^3S_1NN$ calculation.

Table 3
Two-baryon channels up to $2\hbar \omega$ excitations

<table>
<thead>
<tr>
<th>Excitation energy</th>
<th>Two-baryon channel</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \hbar \omega$</td>
<td>$NN, NΔ, ΔΔ$</td>
</tr>
<tr>
<td>$1 \hbar \omega$</td>
<td>$NN', N'Δ, ΔΔ', ΔΔ'$</td>
</tr>
<tr>
<td>$2 \hbar \omega$</td>
<td>$NN'', N''Δ, NΔ', NΔ'', ΔΔ''$</td>
</tr>
</tbody>
</table>

Fig. 1. Mass spectra of $N'$ and $Δ'$. The experimental values are taken from Ref. [23]. Note that the theoretical value of the nucleon mass is 1735 MeV; in the figure, the theoretical values are shifted by 796 MeV. $N'_p$ and $Δ'_p$, which are not shown here, correspond to the so-called missing resonances.
After choosing such effective channels, we carry out multi-channel calculations and examine channel-coupling effects on the $NN$ interaction.

Here we make a brief comment on the hidden color channels, which we do not consider explicitly in this work. The explicit inclusion of the hidden color channels generally causes the problem of a long-range color van der Waals force due to the two-body confinement interaction (1). It is also known that the RGM wave function (11) with all possible combinations of color-singlet baryons provides a complete set of channels for a totally color-singlet six-quark system, and that a lot of highly excited states are required to express a configuration of two well-separated colored-baryons [1]. Since only low-lying excited states are taken into account, the present RGM wave function is affected little by the hidden color channels. Our model is therefore expected to be almost free from this undesirable force.

We first discuss the two-channel calculations, which are shown in Fig. 2. Quite large effects are observed in the cases of the $^3S_1NN-^3D_1NN$ and $^1S_0NN-^5D_0N\Delta$ calculations; however, these results are not presented in Fig. 2, and their effects can be recognized by comparing Fig. 2 with Fig. 3. This fact was already known on the baryon level and has also been reproduced on the quark level [3,4,7].

Besides these two channels, we have selected several channels that are expected to have some effects on the $NN$ interaction. They are the $^5D_1\Delta\Delta$, $^1P_1NN^+$, $^3D_1NN^+$, and $^3S_1\Delta\Delta^+$ channels for the $^3S_1NN$ scattering, and the $^5D_0\Delta\Delta$ and $^1S_0\Delta\Delta^+$ channels for the $^1S_0NN$ scattering. The $\Delta\Delta$ channels, which have been intensively studied in Refs. [3,4], have larger effects than the other channels containing excited baryons.

The selected channels (i.e., $^5D_1NN^+$, $^3S_1\Delta\Delta^+$, and $^1S_0\Delta\Delta^+$) that are newly considered in this work contain the ‘ground-state’ baryons (i.e., the $N$ and $\Delta$) and their positive-parity excited states (i.e., the $N^+$ and $\Delta^+$). These excited states have the same combination of the orbital, intrinsic-spin and isospin symmetries of the corresponding ‘ground states’; the $N^+$ is a radial excited state and the $\Delta^+$ is an orbital excited state (see Table 1). Note that this is not the case for the $^1P_1NN^+$ channel.

The effects seen in Fig. 2 are mainly induced by the tensor interaction of the OPEP. On the other hand, the OGE does not play an essential role; for example, the OGE affects the channels containing $\Delta^+$, but these effects are much smaller than those of the OPEP. It also has been verified that channel couplings are weak in the pure OGE model [1–4], in which the parameters are readjusted to fit baryon spectra, as was stated before. Concerning channel-coupling effects, we therefore conclude that the OPEP is more important than the OGE in the present model. Due to the operator $A_1\cdot A_1$, the effects of the OGE are limited within the short distance between baryons. Contrastingly, the OPEP plays a crucial role not only in the medium- and long-range region but also in the short-range region. It should be empha-

![Fig. 2](image-url)
Fig. 3. (a): $^{3}S_{1}NN$ phase shifts as a function of $E_{\text{lab}}$. The dot-dashed line shows the result of the two-channel ($^{3}S_{1}NN-^{3}D_{1}NN$) calculation. The dotted line includes the additional contribution of $^{7}D_{1}\Delta\Delta$. The solid line represents the full calculation. The crosses show the results of the partial wave analysis [26]. (b): Same as (a) except for $^{1}S_{0}NN$ phase shifts. The dot-dashed line shows the result of the two-channel ($^{1}S_{0}NN-^{3}D_{0}N\Delta$) calculation. The calculation of the dotted line includes $^{1}D_{0}\Delta\Delta$. The solid line is the full calculation. See text for details.

Channel couplings induce an attractive interaction on the $NN$ scattering all over the energy range. Apart from the huge effects of the $^{3}D_{1}NN$ and $^{2}D_{0}N\Delta$ channels, the $^{7}D_{1}\Delta\Delta$ and $^{5}D_{0}\Delta\Delta$ channels have relatively large contributions. In contrast, the other channels containing excited baryons provide small effects. It is remarkable, however, that the channel couplings with these excited states have some influence even at low energies. To show the coupling effects in the low energy region, the deuteron binding energy and the scattering length are calculated and tabulated in Table 4.

Although our results roughly reproduce the energy dependence of the experimental $NN$ phase shifts, there still remain some discrepancies. In particular, the calculation provides less attractive interaction in the $^{1}S_{0}NN$ scattering. At this stage, however, we do not expect a perfect fit to the experimental data since we have adopted various approximations in the present calculations. We are now refining the model in several aspects: using more appropriate wave functions for single-baryons, enlarging the number of two-baryon channels, including other OMEP and so on.

In this work we have analyzed the $NN$ $S$-wave scattering by solving the coupled-channel RGM equation with the OGE, OPEP and confinement potential. The potential parameters have been completely fixed by the single-baryon mass spectra. We have confirmed the large coupling effects due to the channels containing the “ground-state” baryons, i.e., the $^{3}D_{1}NN$ and $^{7}D_{1}\Delta\Delta$ channels for the $^{3}S_{1}NN$ scattering, and the $^{5}D_{0}N\Delta$ and $^{5}D_{0}\Delta\Delta$ channels for the $^{1}S_{0}NN$ scattering. On the other hand, only small effects have been found for the channels containing excited-state baryons, such as the $^{1}P_{1}NN^{*}$.

Table 4

<table>
<thead>
<tr>
<th>State</th>
<th>Coupled channel</th>
<th>B.E. (MeV)</th>
<th>$a_{0}$ (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{3}S_{1}$</td>
<td>$^{3}D_{1}NN$, $^{3}D_{1}\Delta\Delta$</td>
<td>0.58</td>
<td>9.1</td>
</tr>
<tr>
<td></td>
<td>full</td>
<td>1.1</td>
<td>7.0</td>
</tr>
<tr>
<td>$^{1}S_{0}$</td>
<td>$^{3}D_{0}N\Delta$, $^{3}D_{0}\Delta\Delta$</td>
<td>-</td>
<td>-1.1</td>
</tr>
<tr>
<td></td>
<td>full</td>
<td>-</td>
<td>-1.5</td>
</tr>
</tbody>
</table>

sized that the short-range part of the OPEP is known to be quite important in the analysis of single-baryons [15,16].

Before closing the discussion on two-channel calculations, we will comment on channels other than those mentioned above. They show quite small effects, that is, they vary the single-channel result by only a few percent; especially, the channels containing the $N_{3}^{*-}$ or $N_{4}^{*-}$ do not provide any visible effects.

In Fig. 3 are presented the full calculations, which include all the channels selected above, i.e., the $^{3}D_{1}NN$, $^{7}D_{1}\Delta\Delta$, $^{1}P_{1}NN^{*}$, $^{3}D_{1}NN^{*}$ and $^{3}S_{1}\Delta_{2}^{*}$ channels for the $^{3}S_{1}NN$ scattering, and the $^{3}D_{0}N\Delta$, $^{3}D_{0}\Delta\Delta$ and $^{1}S_{0}\Delta_{2}^{*}$ channels for the $^{1}S_{0}NN$ scattering.
$^3D_J^J NN^+^-$ and $^1S_J^J \Delta J^+_J^+$ for the $^3S_J^J NN$ scattering, and the $^1S_J^J \Delta J^+_J^+$ for the $^1S_J^J NN$ scattering. These channel-coupling effects are mainly induced by the tensor interaction of the OPEP.

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References

Abstract

Effects of the Dirac sea on the excitation energy of the giant monopole states are investigated in an analytic way within the \( \sigma - \omega \) model. The excitation energy is determined by the relativistic Landau-Migdal parameters, \( F_0 \) and \( F_1 \). Their analytic expressions are derived in the relativistic random phase approximation (RRPA) without the Dirac sea, with the Pauli blocking terms and with the full Dirac sea. It is shown that in the RRPA based on the mean field approximation the Pauli blocking terms should be included in the configuration space, according to the relativistic Landau theory. In the renormalized RRPA, the incompressibility coefficient becomes negative, if \( NN \) excitations are neglected. © 2000 Published by Elsevier Science B.V. All rights reserved.


The relativistic mean field approximation (RMFA) neglects the Dirac sea in the description of the nuclear ground state. Recently, however, it has been numerically shown that in the relativistic random phase approximation (RRPA) built on the RMFA, the monopole states cannot be well described without the Pauli blocking terms which express transitions between the Dirac sea and the occupied Fermi sea [1]. If the blocking terms are neglected, the excitation energies of the monopole states in the RRPA are much lower than those in the time-dependent relativistic mean field approximation [2].

The purpose of the present paper is to show in an analytic way the role of the Dirac sea in the excitation energy of the monopole states. We will discuss the monopole states of nuclear matter in terms of the Landau-Migdal parameters using the \( \sigma - \omega \) model. First, we will show that in the RRPA based on the RMFA, one should take into account the Pauli blocking terms in the configuration space. Second, the full effects of the Dirac sea will be discussed in the renormalized RRPA. It will be shown that \( NN \) states yield essential effects on the excitation energy through the Landau-Migdal parameter, \( F_0 \).

The Landau-Migdal parameters, \( F_0 \) and \( F_1 \), are obtained by the second derivative of the total energy density with respect to the quasiparticle distribution. In the RMFA, they are given by [3,4]

\[
F_0 = F_\sigma - \frac{1 - \nu_s^2}{1 + a_F F_\sigma}, \quad F_1 = -\frac{\nu_s^2 F_\sigma}{1 + \frac{1}{3} \nu_s^2 F_\sigma},
\]

Where \( F_\sigma \) is the Landau-Migdal parameter for the singlet channel and \( \nu_s \) is the Pauli blocking parameter.
where we have defined

\[ F_s = N_p \left( \frac{g_s}{m_s} \right)^2, \quad F_s = N_p \left( \frac{g_s}{m_s} \right)^2, \quad (2) \]

\[ N_p = \frac{2p_p E_p}{\pi^2}, \quad v_p = \frac{p_p}{E_p}, \quad E_p = \left( p_p^2 + M^*{}^2 \right)^{1/2}. \quad (3) \]

In the above equations, \( g_s \) and \( g_v \) stand for the Yukawa coupling constants, \( m_s \) and \( m_v \) the masses of the \( \sigma \)- and \( \omega \)-meson, respectively, and \( p_p \) and \( M^* \) denote the Fermi momentum and the effective nucleon mass. \( N_p \) and \( v_p \) represent the relativistic density of states at the Fermi surface and the relativistic Fermi velocity. The factor, \( a_s \), in \( F_0 \) of Eq. (1) will play an essential role in later discussions, which is given by

\[ a_s F_s = \frac{4}{(2\pi)^3} \left( \frac{g_s}{m_s} \right)^2 \int d^3 p \frac{p^2}{E_p} \theta_p, \]

\[ E_p = \left( p^2 + M^*{}^2 \right)^{1/2}, \quad (4) \]

where \( \theta_p \) denotes the step function, \( \theta(p_p - |p|) \).

In the relativistic model, the excitation energy of the monopole states is expressed as [5],

\[ E_M = \left( \frac{K}{\epsilon_F \langle r^2 \rangle} \right)^{1/2}, \quad (5) \]

where \( \epsilon_F \) denotes the Fermi energy and \( \langle r^2 \rangle \) the root mean square radius of the nucleus. The incompressibility coefficient, \( K \), is expressed in terms of the above relativistic Landau-Migdal parameters,

\[ K = \frac{3p_p^2}{\epsilon_F} \left( 1 + F_0 \right) + \frac{1}{3} F_1. \quad (6) \]

Since \( p_p \) is determined by the nucleon density, and \( \epsilon_F \) is related to the nucleon binding energy, \( E_B \), and the free nucleon mass, \( M \),

\[ \epsilon_F = E_B + M, \quad (7) \]

the excitation energy of the monopole state is a function of \( F_0 \) and \( F_1 \).

In order to see the effects of the Pauli blocking terms on the monopole states, we derive the Landau-Migdal parameters according to the RRPA.

We calculate the longitudinal RRPA correlation functions with and without the Pauli blocking terms. By comparing them with the correlation function of the Landau theory [6], we will obtain the expressions of the Landau-Migdal parameters in each approximation.

When following our previous papers [4,7], the mean field correlation function, \( \Pi_H \) [4,7], is given by the Fourier transform of the single-particle Green function, \( G_H \),

\[ \Pi_H(A,B;k) = - \frac{1}{2\pi i} \int d^4 p \text{Tr} [ \Gamma_0 G_H(p + k) \Gamma_0 G_H(p) ] \quad (8) \]

where \( k \) denotes the four-momentum, \( (k_0,k) \), and \( A \) and \( B \) the Fourier transform of the external field expressed with the mean field, \( \psi_H(x) \),

\[ A(k) = \int d^3 x \exp(ik \cdot x) \overline{\psi}_H(x) \Gamma_A \psi_H(x) \quad (9) \]

\( \Gamma_A \) being some \( 4 \times 4 \) matrices. The sum of the ring diagrams in the RRPA for the \( \sigma - \omega \) model is described as [4,7],

\[ \delta \Pi_{RRPA}(A,B;k) = \frac{\chi_e \overline{\chi}}{\det \mathbf{U}_L} \Pi_H(A,A^e;k)(U_{L})_{ab} \Pi_H(A^b,B;k), \quad (10) \]

where the contraction should be carried out with respect to the superfix and suffix, \( a,b = -1,0 \), and \( A_{-1} \) and \( A_0 \) are given by Eq. (9) with \( \Gamma_A = 1 \) and \( \gamma_0 \), respectively. Moreover, \( \chi_e \) and \( \overline{\chi} \), represent,

\[ \chi_e = \frac{1}{(2\pi)^3} \frac{g^2_e}{m^2_e - k^2}, \quad \overline{\chi} = \frac{1}{(2\pi)^3} \frac{g^2_v}{m^2_v - k^2}. \quad (11) \]

The explicit form of the \( 2 \times 2 \) matrix, \( U_L \), in Eq. (10) depends on whether or not the Pauli blocking terms are included in the mean field correlation functions as discussed below.

The Green function, \( G_H \), is given by the sum of those for a single-particle, hole and antinucleon,

\[ G_H = G_h(1 - \theta_p) + G_h \theta_p + G_T. \quad (12) \]
It is rewritten as a sum of the density-dependent and the Feynman part [7],

\[ G_{\text{H}} = G_{\text{D}} + G_{\text{F}}, \quad G_{\text{D}} = \theta_\rho(G_h - G_p), \]

\[ G_{\text{F}} = G_p + G_{\text{G}}. \]  

(13)

Hence, \( P_{\text{H}} \) is composed of the four terms like \( G_p G_{\text{D}}, G_p G_{\text{F}}, G_p G_{\text{G}} \) and \( G_{\text{G}} G_{\text{P}} \) [7]. In the RRPA based on the RMFA, the \( G_p G_{\text{P}} \) term is neglected, which is divergent, while in the previous calculations [4], the \( G_p G_{\text{P}} \) and \( G_p G_{\text{D}} \) terms, which contain the Pauli blocking NN excitations like \( G_p \theta_\rho G_{\text{P}} \), have been kept. Then we have obtained

\[ \Pi_{\text{H}} = \left( \chi_s(1 - \bar{\chi}_s \Pi_s) \chi_s \bar{\chi}_s \Pi_{sv} \right) = \left( \chi_s(1 - \bar{\chi}_s \Pi_s) \right), \]  

\[ \Pi_{\text{sv}} = \left( \chi_s(1 - \bar{\chi}_s \Pi_s) \right), \]

(14)

where the mean field correlation functions are defined as

\[ \Pi_i = \Pi_{\text{H}}(A_{i-1}, A_i; k), \quad \Pi_{sv} = \Pi_{\text{H}}(A_0, A_i; k), \]

\[ \Pi_{sv} = \Pi_{\text{H}}(A_i, A_{i-1}; k). \]  

(15)

The Landau prescription of the correlation functions is obtained at the limit \( k \to 0 \). In this limit we have

\[ \Pi_i = (2 \pi)^3 N_p \left( 1 - v_i^2 \right) \Phi(x) - a_i \],

\[ \Pi_{sv} = (2 \pi)^3 N_p \Phi(x), \]

\[ \Pi_{sv} = (2 \pi)^3 N_p \left( 1 - v_i^2 \right)^{1/2} \Phi(x), \]  

(16)

(17)

where \( \Phi(x) \) stands for the Lindhard function with \( x = k_0 / (k |v_i^2|) \) [4]. Using these equations, we obtain the generalized dielectric function from the factor of Eq. (10) as

\[ \frac{1}{\chi_s \bar{\chi}_s} \det U_L = \left( 1 + a_i F_s \right) \left( 1 + \frac{F_s - 1 - v_i^2}{1 + a_i F_s} F_s - v_i^2 F_s x^2 \right) \]

\[ \times \Phi(x) \]  

(18)

In the Landau theory, Eq. (18) should be written as [6]

\[ \frac{1}{\chi_s \bar{\chi}_s} \det U_L = c \left( 1 + \frac{F_s + \frac{F_s}{1 + \frac{1}{3} F_s} x^2 \Phi(x)}{1 + \frac{1}{3} F_s} \right). \]  

(19)

By comparing Eq. (18) with (19), we obtain the Landau-Migdal parameters which are the same as in Eq. (1).

Next we investigate the role of the Pauli blocking terms in the Landau-Migdal parameters. We calculate the mean field correlation functions neglecting the Pauli blocking terms and taking the only particle-hole states. The calculation of the mean field correlation functions is a little different from the one in including the Pauli blocking terms, since the correlation functions are not Lorentz covariant, and the continuity equation is provided in a different way,

\[ k_\mu \Pi_{sv}(A_{i-1}, \mathbf{A}_i; k) = \left\langle \left[ A_0(k), \mathbf{A}^i(k) \right] \right\rangle, \]  

\[ \left\langle \left[ A_0(k), \mathbf{A}^i(k) \right] \right\rangle = \delta(k - k') \left\langle \left[ A_0(k), \mathbf{A}^i(k') \right] \right\rangle. \]  

(20)

(21)

In including the Pauli blocking terms, the r.h.s. of Eq. (20) is always vanished, but in neglecting them, it is not for \( \mathbf{A}_{i,2,3} \). Hence, the relationship between the correlation functions due to the time- and the longitudinal component of the \( \omega \)-meson is written in the frame, \( k = (|k|,0,0) \), as

\[ \Pi_{sv}(A_i, A_{i-1}; k) = \frac{k_0^2}{|k|^2} \Pi_{sv}(A_0, A_{i-1}; k) - a_i, \]  

\[ a_i = \left\langle \left[ A_0(k), \mathbf{A}^i(k) \right] \right\rangle / |k|. \]  

(22)

(23)
Because of this fact, the back flow effects due to the longitudinal \( \omega \)-meson exchange on \( \delta \Pi_{\text{RPA}} \) are not simply normalized as \( \tilde{\chi}_v \), and \( U_L \) in this case depends on \( \alpha_v \),

\[
U_L = \left( \chi_v(1 + \alpha_v, \chi_v - \tilde{\chi}_v, \Pi_v) + \chi_v(1 - \chi_v, \Pi_v) \right),
\]

(24)

where we have defined

\[
\chi_v = \frac{1}{(2\pi)} \left( \frac{g_s}{m_v} \right)^2, \quad \tilde{\chi}_v = \tilde{\chi}_v - \alpha_v, \chi_v^2.
\]

(25)

At the limit, \( k \to 0 \), \( \chi_v, \alpha_v \) becomes to be

\[
\chi_v, \alpha_v(k \to 0) = -4 \left( \frac{g_s}{m_v} \right)^2 \int d^3p \left( \frac{2p^2/3 + M^*}{E_p} \right)^2 \theta_p
= -\frac{1}{2} \tilde{\chi}_v^2 F_v.
\]

(26)

Moreover, \( \Pi_v \) of the present case has not the term, \( \alpha_v \), in Eq. (16), while \( \Pi_v \) and \( \Pi_w \) are the same as in Eq. (17). As a result the generalized dielectric function in this case is given by

\[
\frac{1}{\chi_v, \tilde{\chi}_v} \det U_L
= \left( 1 - \frac{1}{3} \tilde{\chi}_v \right) \left( 1 + F_v - F_v(1 - \tilde{\chi}_v) \right)
= \tilde{\chi}_v \left( 1 - \frac{1}{3} \tilde{\chi}_v \right) F_v.
\]

(27)

Finally comparison of the above equation with Eq. (19) provides us with the Landau-Migdal parameters in neglecting the Pauli blocking terms,

\[
F_0 = F_v - (1 - \tilde{\chi}_v) F_v, \quad F_1 = -\tilde{\chi}_v F_v.
\]

(28)

The difference between Eqs. (1) and (28) is very clear. \( F_0 \) and \( F_1 \) in Eq. (28) have no denominator. In order to obtain the correct expressions of \( F_0 \) and \( F_1 \) within the RMFA, thus we need to include the Pauli blocking terms in the configuration space of the RRPA.

In the Landau prescription, the denominators in Eq. (1) come from the self-consistent derivative of the effective mass and the baryon current with respect to the quasi-particle distribution, \( n \), [3,5]. As to the effective mass in the RMFA,

\[
M^* = M - \left( \frac{g_s}{m_i} \right)^2 \frac{1}{V} \sum_i \frac{M^*}{E_{p_i}},
\]

(29)

we have

\[
\frac{\partial M^*}{\partial n_j} = -\frac{1}{V} \left( \frac{g_s}{m_i} \right)^2 \frac{M^*}{E_{p_i}}
- \left( \frac{g_s}{m_i} \right)^2 \frac{1}{V} \sum_i \frac{n_i}{E_{p_i}} \frac{\partial M^*}{\partial n_j},
\]

(30)

\( V \) being the nuclear volume. The coefficient of \( \partial M^* / \partial n_j \) in the r.h.s. yields \( a_i, F_i \) in the denominator of \( F_0 \), as seen in Eq. (4). In the Green function formalism, the effective mass of the RMFA is written as

\[
M^* = M + \frac{1}{(2\pi)^3} \int \frac{d^4p}{E_p} \text{Tr} G_D(p).
\]

(31)

Hence the coefficient of \( \partial M^* / \partial n_j \) in Eq. (30) is given by

\[
a_i, F_i = -\frac{1}{(2\pi)^3} \left( \frac{g_s}{m_i} \right)^2 \text{Tr} \left( \frac{\partial G_D(p)}{\partial M^*} \right).
\]

(32)

On the other hand, we have shown that

\[
a_i, F_i = -\frac{1}{(2\pi)^3} \left( \frac{g_s}{m_i} \right)^2 \Pi_{\text{Pauli}}(k = 0),
\]

(33)

where \( \Pi_{\text{Pauli}} \) represents the Pauli blocking terms in Eq. (8) for \( \Gamma^*_\chi = \Gamma^*_B = 1 \). Thus it is seen that in the Landau prescription of the RMFA, the derivative of \( G_0 \) includes implicitly the Pauli blocking terms. Indeed, we can prove that

\[
\frac{\partial G_D(p)}{\partial M^*} f(p)
= \left( G_0 G_{\pi} + G_{\pi} G_{\pi} + G_{\pi} M^* \frac{\partial}{\partial p_0} f(p) \right),
\]

(34)
where \( f(p) \) stands for an arbitrary function of \( p \). If we insert Eq. (34) into Eq. (32) and integrate its r.h.s. over \( p_0 \), the Pauli blocking terms only remain.

The same discussion is possible for the denominator of \( F_i \). The self-consistent derivative of the current, \( J \), as to the quasi-particle distribution provides [3],

\[
\frac{\partial j}{\partial n_j} = \frac{1}{V} \frac{p_j}{E_{p_j}} - \frac{4}{3} v_F^2 F_0 \frac{\partial j}{\partial n_j}.
\] (35)

The coefficient of the second term yields the denominator of \( F_i \). Using the Green function, the current of the RMFA is written as,

\[
\Sigma = \left( \frac{g_s}{m_s} \right)^2 j = -i \left( \frac{g_s}{m_s} \right)^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr}(\gamma G_D(p')) \text{d}p.
\] (36)

Hence, the coefficient of the second term in Eq. (35) is given by

\[
-\frac{4}{3} v_F^2 F_0 \delta_{ij} = -i \left( \frac{g_s}{m_s} \right)^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr}\left( \gamma \frac{\partial G_D(p')}{\partial \Sigma}\right) \bigg|_{\Sigma=0}.
\] (37)

We can show that the above derivative of \( G_D \) required in the RMFA is expressed by using the Pauli blocking terms as

\[
\frac{\partial G_D}{\partial \Sigma} f(p) = \left( G_D \gamma G_N + G_N \gamma G_D - G_D F_F \frac{\partial}{\partial p_0} \right) f(p).
\] (38)

\( f(p) \) being an arbitrary function of \( p \).

Let us explore the effects of the Pauli blocking terms in more detail. The Pauli blocking terms reduce always the contribution of the \( \sigma \)-meson to \( F_0 \) through the factor \( a_\sigma \) in Eq. (1), since \( a_\sigma \) is positive, as seen in Eq. (4). On the other hand, the contribution of the \( \omega \)-meson to \( F_0 \) is not affected. Therefore, the value of \( F_0 \) becomes always smaller, and the incompressibility, \( K \), is reduced according to Eq. (6), when the Pauli blocking terms are neglected. On the contrary, the absolute value of \( F_i \) in Eq. (28), which has no denominator, becomes always larger, compared with the correct one in Eq. (1), so that \( K \) is enhanced in neglecting the Pauli blocking terms.

We calculate the values of \( F_0, F_1 \) and \( K \) using the following parameters [8] as an example,

\[
M = 939, \quad m_\omega = 520, \quad m_\pi = 783 \text{ (MeV)},
\]

\[
g_\omega^2 = 109.626, \quad g_\pi^2 = 190.431,
\] (39)

which reproduce the nucleon binding energy, \( E_B = -15.75 \text{ MeV} \) at \( p_F = 1.30 \text{ fm}^{-1} \). In this case, we have \( M^* = 0.541 M, \quad v_F = 0.451 \) and \( a_\pi = 9.07 \times 10^{-3} \). These values provide us with

\[
F_0 = 0.569, \quad F_1 = -1.151, \quad K = 544 \text{ MeV}
\] (40)

in taking the Pauli blocking terms, and

\[
F_0 = -0.368, \quad F_1 = -1.866, \quad K = 357 \text{ MeV}
\] (41)

in neglecting the Pauli blocking terms. Thus both Landau-Migdal parameters are strongly affected by the Pauli blocking terms, and, in particular, \( F_0 \) changes its sign. Consequently, the value of \( K \) is fairly reduced in neglecting the Pauli blocking terms. In \(^{208}\text{Pb}\), the reduction amounts to about 2.7 MeV for the present parameters. This fact may be observed in Ref. [1] by numerical calculations.

Now full effects of the Dirac sea should be explored with the renormalized RRPA, where the \( G_D G_F \) term is also calculated in the Hartree correlation functions. Then, the Pauli exclusion principle in both the Fermi sea and the Dirac sea is correctly taken into account. Such calculations based on the renormalized Hartree approximation (RHA) have been done by the present authors in Ref. [9]. For complete discussions, we quote those results here. The Landau-Migdal parameters in this case are given by

\[
F_0 = F_0 - \alpha_\sigma E_s F_s, \quad F_1 = \frac{-v_F^2 F_0}{1 + 3v_F^2 F_0},
\] (42)

where we have used the abbreviations:

\[
E_s = N_F \left( \frac{g_\pi}{m_0} \right)^2, \quad \alpha_\sigma = \frac{1 - v_F^2}{1 + a_\pi F_s + a_0}.
\] (43)
Formally the above equation is similar to Eq. (1), but the mass of the \(\omega\)-meson is replaced by the bare mass, \(m_0\), in \(F_0\) and the Dirac sea yields an additional effect, \(a_0\), in \(\alpha_{\text{ren}}\) [9].

As to \(F_1\), essentially there is no additional effects from the Dirac sea. The renormalized correlation function from the \(G^A G^B\) term due to the \(\omega\)-meson exchange disappears at the limit \(k \to 0\) and has no contribution to \(F_1\). Replacement of \(m_0\) by \(m_1\) comes from the fact that the \(\omega\)-meson propagator is written in terms of the bare mass \(m_1\). The value of \(F_1\), however, depends on those of the Yukawa coupling constants used in the RHA. In order to reproduce the nucleon binding energy and the Fermi momentum mentioned before [9], the RHA requires \(g_s = 66.117\) and \(g_\alpha = 79.927\). These values give \(M^* = 0.7306M\) and \(m_0 = 691.171\) MeV, so that we obtain \(F_1 = -0.620\).

On the other hand, \(F_0\) is strongly affected by the Dirac sea through \(a_1\) in Eq. (43). In using \(\epsilon_F = 0.3502\) of the present RHA, its value is much larger than that of \(a_1\).

\begin{equation}
\frac{a_0}{a_1} = 0.405, \quad a_0 F_1 = 0.0296.
\end{equation}

This Dirac sea effects reduce strongly the contribution of the \(\sigma\)-meson to \(F_0\) through \(\alpha_{\text{ren}}\), and we have

\begin{equation}
F_0 = 0.676.
\end{equation}

If \(a_0\) were neglected, then the value of \(F_0\) would be \(-1.56\), which means \(K < 0\). This fact reflects that N-degrees of freedom play an important role to stabilize the nucleus in the RHA.

Finally we give two comments. First, since the restoring force of the giant quadrupole states comes mainly from the distortion of the kinetic energy density, its excitation energy depends on \(F_1\) [5].

\begin{equation}
E_q = \left( \frac{6 \rho_q}{5 \epsilon_F \langle r^2 \rangle} \frac{1}{1 + \frac{1}{3} F_1} \right)^{1/2}.
\end{equation}

Thus, the Pauli blocking terms affect also the excitation energy of the quadrupole states in the RRPA based on the RMFA. Moreover, the Pauli blocking terms should be taken into account in the description of the center of mass motion, which requires the correct \(F_1\) [5]. This fact has been observed in arguments on the spurious state of RRPA by Dawson and Furnstahl [10]. In the same way the Pauli blocking terms are necessary for discussions of the nuclear current or magnetic moments [11]. The isovector dipole states depend on \(F_1\) and \(F_1'\), which also require the Pauli blocking terms. The detail will be published in a separate paper.

Second, we note that Eqs. (5), (6) and (46) are formally the same as those in nonrelativistic models, except for \(\epsilon_F\) in their denominators [5]. In nonrelativistic models, it is replaced by the nucleon mass, \(M\). However, they are related to each other as Eq. (7) through the nucleon binding energy which is negligible compared with the nucleon mass. Thus the relativistic correction to the excitation energies of the monopole and quadrupole states is less than 1% for \(E_0 = -15.75\) MeV, if the values of the Landau-Migdal parameters in relativistic models are the same as in nonrelativistic ones.

In conclusion, in the relativistic random phase approximation (RRPA) based on the relativistic mean field approximation, the Pauli blocking terms should be taken into account for consistent descriptions of the Landau-Migdal parameters. The full effects of the Dirac sea on the excitation energy of the monopole states are studied in the RRPA built on the renormalized Hartree approximation. Then the nucleon-antinucleon states affect strongly the excitation energy of the monopole states through the Landau-Migdal parameter, \(F_0\). The incompressibility coefficient becomes negative, if antinucleon-degrees of freedom are neglected. The Landau-Migdal parameter, \(F_1\), is not affected formally by the renormalization.

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References


Crystal manyfold universes in AdS space

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Abstract

We derive crystal braneworld solutions, comprising of intersecting families of parallel \( n + 2 \)-branes in a \( 4 + n \)-dimensional AdS space. Each family consists of alternating positive and negative tension branes. In the simplest case of exactly orthogonal families, there arise different crystals with unbroken 4D Poincaré invariance on the intersections, where our world can reside. A crystal can be finite along some direction, either because that direction is compact, or because it ends on a segment of AdS bulk, or infinite, where the branes continue forever. If the crystal is interlaced by connected 3-branes directed both along the intersections and orthogonal to them, it can be viewed as an example of a Manyfold universe proposed recently by Arkani-Hamed, Dimopoulos, Dvali and the author. There are new ways for generating hierarchies, since the bulk volume of the crystal and the lattice spacing affect the 4D Planck mass. The low energy physics is sensitive to the boundary conditions in the bulk, and has to satisfy the same constraints discussed in the Manyfold universe. Phenomenological considerations favor either finite crystals, or crystals which are infinite but have broken translational invariance in the bulk. The most distinctive signature of the bulk structure is that the bulk gravitons are Bloch waves, with a band spectrum, which we explicitly construct in the case of a 5-dimensional theory. © 2000 Published by Elsevier Science B.V. All rights reserved.

Recently a remarkable observation by Arkani-Hamed, Dimopoulos and Dvali that there may exist additional sub-millimeter spatial dimensions [1] has generated tremendous interest. In such theories, the usual Standard Model degrees of freedom are localized to a 3-brane which is embedded in a higher-dimensional bulk, and thus at low energies are indifferent to the extra dimensions. On the other hand, gravity and, typically, other weakly coupled fields live in the bulk, probing equally all spatial dimensions. The observed weakness of gravity in four dimensions is generated naturally, because by spreading through all spatial dimensions gravity becomes softer, as can be seen immediately from Gauss law. Hence the hierarchy between the Planck scale, \( M_{Pl} \sim 10^{19} \text{ GeV} \) and the electroweak scale, \( m_{EW} \sim \text{TeV} \) and its radiative stability, can be naturally explained if the size of the \( n \) internal dimensions \( r \) is large: \( M_{Pl}^2 = M_{Pl}^{2 + n} r^n \Rightarrow M_{Pl}^2 \sim m_{EW}^2 \) [1].

Studies of phenomenological constraints in [1] have confirmed that such models are consistent with observations, with the unification scale as low as few \( \times 10 \) TeV. There has subsequently been much interest in models with large extra dimensions [2–9]. String theory may give rise to such models, with the radius of

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compactification only a few orders of magnitude above the Planck scale, such as in Hořava-Witten [10,11], intermediate sizes $\sim (\text{TeV})^{-1}$ [12,13], and even $\sim (M_P/M)^2/M^{-1} \lesssim \text{mm}$ [2,14].

The concept of localization of matter to the branes plays a key role in masking the large extra dimensions from observation at low energies. In fact, the universe as a domain wall in a non-compact space has been considered before [15,16], but there gravity was higher-dimensional at all scales. But if gravity itself is localized to the 3-brane, then 4D gravity can be reproduced at low energies even if the extra dimensions are much larger than a mm. A very interesting recent construction by Randall and Sundrum [17] provides an elegant example of this idea for the case of one extra dimension, which is linearly infinite, but has finite proper volume. Similar ideas have been explored also in [18]. Subsequently it has been shown that gravity can be localized to intersections of $n+2$ branes in $4+n$-dimensional spaces [19]. More examples were found later [20], and various issues [21–25] were considered. Since the Randall-Sundrum proposal [17] employs AdS bulk geometry, it is possible to view the model in the framework of the AdS/CFT correspondence [26,27]. Interpreting the fifth dimension as the scale of the 4D theory [28] and the evolution along it as the holographic renormalization group [29], this model can be rephrased as a 4D CFT theory coupled with gravity [30], without invoking extra dimensions. The AdS/CFT correspondence has also been pursued in [31].

On a different track, it is possible that our 3-brane is not alone in the bulk. In fact, the bulk may be populated by many other branes, if we wish to understand the weak breaking of symmetries by shining in the bulk [5]. Hence, the universe may contain many nearby branes, which are dark because the light travels only along them, but are felt by their gravity [32]. This Manyfold universe has been considered recently by Arkani-Hamed, Dimopoulos, Dvali and the author [32], and it has been shown that it can be consistent with the observations, while giving rise to many new phenomena: dark matter candidates which are dissipationful, dark stars, hybrids, neutrino mixing, SUSY breaking et cetera. Some solutions representing multi-brane configurations in 5D spacetimes have been constructed [33], and some aspects of brane networks considered [34,35].

In this paper, we will construct explicit solutions of Einstein’s equations which represent intersecting families of parallel $n+2$-branes in $4+n$-dimensional AdS spacetimes. Each family will contain branes of alternating tension. While we are a little reluctant to treat the negative tension branes on equal footing with the positive tension ones, we will ignore this when considering the solutions because of considerable technical simplifications. If the bulk cosmological constant is replaced by a potential, there may exist solutions which involve only positive tension branes, while sharing some generic properties with our solutions. Alternatively, a first-principles construction of branes with negative tension may yet ensue. We will show that there arise several different crystal solutions, all of which support the usual 4D gravity. However phenomenological considerations strongly favor finite crystals or crystals which are infinite but have strongly broken translational symmetry, where our world arises as a lattice defect. If a crystal is interlaced by a grid of 3-branes, some along the intersections and others stretching between them, the low energy physics of such a solution is identical to that of a Manyfold universe [32]. Such a crystal would have to be subject to the same astrophysical constraints as any generic Manyfold [32]. We will further show that the spectrum of bulk gravitons has band structure, with a gap at zero energy, because the bulk gravitons are Bloch waves, like electrons in ordinary crystals. This may allow the string scale in a 5D theory to be as low as $M_5 \sim 100 \text{ TeV}$, at the cost of having $N \sim 10^{16}$ branes in the 5D AdS spacetime. However, the reward consists of reproducing the 4D Planck scale $M_4 \sim 10^{19} \text{ GeV}$, the electroweak scale $m_{EW} \sim 1 \text{ TeV}$ and suppressing the corrections to Newton’s law at distances $\gtrsim 1 \text{ mm}$. It may be possible to reduce the number of required branes by going to more than one extra dimensions.

We begin by deriving the solution describing the crystal braneworlds. Consider an array of $n$ orthogonal families of $n+2$-spatial dimensional branes in a $4+n$-dimensional spacetime, with a bulk cosmological constant $\Lambda$. For simplicity we take the branes in each family to have identical tension $\sigma_k$. The field equations can be derived from the action

$$S = \int_M d^{4+n}x \sqrt{g_{4+n}} \left( \frac{R}{2\kappa_{4+n}^2} + \Lambda \right) - \sum_{k=1}^n \sum_{j_k} \int_{J_k\lambda_k} d^{4+n}x \sqrt{g_{4+n}} \sigma_k.$$  

(1)
Here $\kappa_{s+n}^2 = 8\pi / M_s^{n+2}$, where $M_s$ is the fundamental scale of the theory. In the brane actions, $g_s$ refers to the induced metric on the brane. Note that the measure of integration differs between each brane, and between the branes and the bulk. This will be reflected in the field equations, where ratios $\sqrt{g_{3+n}} / \sqrt{g_{4+n}}$ weigh the $\delta$-function sources. The gravitational field equations are

$$G_{ab} = \kappa_{4+n}^2 \Delta \delta_{ab} - \sum_{k=1}^{n} \frac{\sqrt{g_{3+n}}}{\sqrt{g_{4+n}}} \kappa_{s+n}^2 \sigma_j \sum_{j} (-1)^j \delta(z^k - j\ell_k) \text{diag}(1,1,1,1,\ldots,0,\ldots,1).$$  \hspace{1cm} (2)

where the coordinates $z^k$ parameterize the extra dimensions. In general, these equations should be supplemented by the equations of motion for the branes, which however are solved by placing branes at fixed locations parallel to the $AdS$ boundary. The issue of stability would then have to be addressed, and is equivalent to providing the brane Goldstone bosons with mass terms. We will not address this in detail here. The ratios $\sqrt{g_{3+n}} / \sqrt{g_{4+n}}$, which appear in (2) reduce to $\sqrt{g/4}$ for simple diagonal metrics, however in general they cannot be gauged away.

Deriving the solutions is straightforward [19]. Away from the branes, the bulk geometry consists of patches of the $4+n$-dimensional $AdS$ space. On the branes, there are additional $\delta$-function singularities, which cause discontinuities in the derivatives of the metric. To find the global solution, valid on the complete manifold, we should seek the solution in the form

$$ds^2 = \Omega^2 \left[ \eta_{ab} dx^a dx^b + \sum_{k=1}^{n} (dz^k)^2 \right]$$

and adjust the boundary conditions for $\Omega^{-1}$ such that it solves Einstein equations (2), and between the branes reproduces the metric of $AdS$ bulk. It is convenient to recast the Einstein equations (2) in the conformal frame $\bar{g}_{ab} = \Omega^{-2} g_{ab}$. We use the standard relation for $4+n$ spacetime dimensions

$$G_{ab} = \bar{G}_{ab} + (n+2) \left( \bar{\nabla} \log \Omega \right) \eta_{ab} - \bar{\nabla} \log \Omega \right) \eta_{ab} + (n+2) \bar{g}_{ab} \left( \bar{\nabla} \log \Omega + \frac{n+1}{2} \left( \bar{\nabla} \log \Omega \right)^2 \right).$$

Since the conformal metric is flat, $\bar{G}_{ab} = 0$. Then substituting (4) into (2) yields two equations for $\Omega^{-1}$:

$$\nabla^2 \Omega^{-1} = \frac{\kappa_{s+n}^2 \sigma_j}{(n+2)} \sum_{k=1}^{n} \sum_{j} (-1)^j \delta(z^k - j\ell_k), \quad (\nabla \Omega^{-1})^2 = 2 \frac{\kappa_{4+n}^2 \Lambda}{(n+2)(n+3)}.$$  \hspace{1cm} (5)

Note that in deriving the first equation it was important to use $\sqrt{g_{3+n}} / \sqrt{g_{4+n}} = \Omega^{-1}$, which ensures that the brane tensions are constant.

It is now clear from the first of Eqs. (5) that $\Omega^{-1}$ is a linear combination of Green’s functions for each family of parallel branes. Since the tensions $\sigma_j$ are the strengths of sources, they control the discontinuity of the derivatives of $\Omega^{-1}$ on each brane. It is easy to see that the discontinuities must satisfy $\left( \partial_j \Omega^{-1} \right)^2 = \kappa_{s+n}^2 \sigma_j^2 / 4(n+2)^2$. Thus the discontinuity in the gradient of $\Omega^{-1}$ on each intersection of branes must satisfy $(\nabla \Omega^{-1})^2 = n \kappa_{s+n}^2 \sigma_j^2 / 4(n+2)^2$. But because the second of Eqs. (5) is defined globally, these two expressions must coincide, leading to

$$\kappa_{s+n}^2 \sigma_j^2 = \frac{8(n+2)}{n(n+3)} \kappa_{4+n}^2 \Lambda.$$  \hspace{1cm} (6)

Hence the condition for the existence of the solutions is to fine tune the tensions of the branes to the bulk cosmological constant according to (6). Then as we see from (3) the induced metric on the intersections is Minkowski, meaning that the conditions (6) amount to fine tuning the effective $4D$ cosmological constant on each intersection to zero.
We can now solve the first of (5) as follows. Suppose that in each family there is \( N \) parallel branes, which for reasons of simplicity we have taken to be equidistant, separated by \( l \), as is seen in Eqs. (2) and (5). The solution can be easily generalized when the distances between the branes vary within each family, as will become clear from the construction below. Eqs. (5) yield the solution for the conformal factor \( \Omega \)

\[
\Omega^{-1} = K \sum_{k=1}^{n} \mathcal{P}(z^k) + 1, \tag{7}
\]

where \( K = (\sqrt{n}L)^{-1} \), \( L \) is the AdS radius, and the functions \( \mathcal{P}(z^k) \) satisfy

\[
\frac{d^2 \mathcal{P}(z^k)}{d(z^k)^2} = 2 \sum_j (-1)^j \delta(z^k - j l_k), \quad \left| \frac{d \mathcal{P}(z^k)}{dz^k} \right| = 1. \tag{8}
\]

These two equations follow from (5) by using linear superposition, Eq. (6) and the definition of the AdS radius. The integration constant in (7) must be nonzero in order to excise the AdS boundary from the manifold, and is set to unity by gauge choice. As a result, the warp factor never diverges; otherwise there could not exist a normalizable 4D graviton mode.

It is easy to see that the function \( \mathcal{P}(z^k) \) which solves Eqs. (8) must be the sawtooth function. We can write it as follows:

\[
\mathcal{P}(z^k) = \begin{cases} 
\ldots \\
2pl_k - z^k, & \text{for } (2p - 1)l_k < z^k < 2pl_k; \\
2pl_k - z^k, & \text{for } 2pl_k < z^k < (2p + 1)l_k; \\
\ldots 
\end{cases} \tag{9}
\]

The solutions differ globally depending on the boundary conditions for each \( \mathcal{P}(z^k) \). In general there are four distinct types of boundary conditions:

(i) There is a discrete infinity of branes along the \( k \)th axis, and the crystal is infinite in this direction;

(ii) There is \( N_k = 2N_1 + 1 \) branes, and \( N_1 + 1 \) have positive tension and \( N_1 \) negative; the crystal ends on branes with positive tension in each direction along the \( z^k \) axis; outside of the crystal there is the near-horizon geometry of AdS space, with the horizon located at infinite proper spatial distance from each end;

(iii) The crystal is semi-infinite along the \( z^k \) axis, with infinitely many branes, and boundary conditions satisfying (i) on one end and (ii) on the other; the brane at one end must have positive tension;

(iv) The crystal is finite along \( z^k \), with an even number \( N_k = 2N_k \) of distinct branes along the \( z^k \) axis, a half of them with positive and a half with negative tension; the solution can be viewed as case (ii) but with the 0th and 2\( N_k \)th branes identified; this means, that the \( z^k \) axis is compactified on a circle.

By linear superposition, the global solution for the crystal is a combination of any of the four possibilities realized along individual axes. For the case (i), the solution for the sawtooth functions is adequately represented by an infinite array of the form (9). In the other three cases, by choosing the origin of the coordinate system in the bulk to lie at the end brane with positive tension, we can rewrite the solution in a more compact form:

\[
\mathcal{P}(z^k) = 2 \sum_{j=0}^{N_k} (-1)^j \theta(z^k - j l_k)(z^k - j l_k) - z^k, \tag{10}
\]
where $\mathcal{N}_e = 2N_e + 1$ in the case (ii), $\mathcal{N}_e \to \infty$ in the case (iii), and $\mathcal{N}_e = 2N_e$ in the case (iv). Therefore, in terms of these functions, the metric can be written down as

$$ds_{n+n}^2 = \frac{1}{\left(K \sum_{k=1}^{n} \mathcal{S}(z^k) + 1\right)^2} \left(\eta_{\mu\nu} dx^\mu dx^\nu + \sum_{j=1}^{n} (dz^j)^2\right). \tag{11}$$

This is the solution for the crystal Manyfold universe. By the conditions (6), all the vertices of the crystal lattice correspond to the 4D Minkowski spacetimes, and can accommodate 3-branes which localize the Standard Model particles. Therefore, our world could reside at any one of the vertices, leading to different low energy theory, as long as there exists a massless 4D graviton mode. The 3-branes lying on the other vertices would behave as mirror worlds, and can be connected to our own by additional 3-branes embedded in the intersecting $n+2$-branes. Hence a crystal interlaced by 3-branes would give rise to a Manyfold universe structure of [32].

It is straightforward to verify that the theory defined by the action (1) in general admits 4D massless graviton modes on the backgrounds (11). Hence at large distances these modes will give leading contributions to the gravitational interactions between particles localized to the intersections, reproducing the usual Newton’s law and masking the extra dimensions. The wave function of these modes is

$$h_{\mu\nu 0}(x^\mu, z) \sim \Omega^{(n-4)/2}(z) e^{ik_{\mu} x^\mu}, \tag{12}$$

as we will see below in more detail. Using (1) and (12), by ignoring the massive KK modes and reducing the theory (1) to four dimensions, we find that the 4D Planck mass is

$$M_{\text{Pl}}^2 = M_{\text{pl}}^{n+2} \int_{\text{crystal space}} d^n z \Omega^{n+2}. \tag{13}$$

The integral has to be performed over the whole crystal space, which is composed of a number of patches of a different kind. Depending on the type, there may be semi-infinite patches comprised of the near-horizon segments of $AdS$ bounded by branes of positive tensions. Each such patch is identical to the patches considered in the case of a single intersection of branes, contributing $\frac{1}{(n+1)^j K^j} M_{\text{pl}}^{n+2}$ to $M_{\text{pl}}^2$ [19]. However, since now there are also patches of geometry which are completely bounded by branes, there arise additional contributions to $M_{\text{pl}}^2$. Different patches, or cells, can be classified by counting how many branes they are bounded by, such that the integrations in those directions would be bounded between 0 and $l_g$. It is clear from (7) and (9) that all topologically identical segments give the same contribution to the mass integral. Each distinct integral needs to be weighed by the symmetry factor, which counts the total number of cells of a given class in the crystal. The integrals $I = \int_{\text{cell}} d^n z \Omega^{n+2}$ in general assume the form

$$I = \int_0^{l_g} dz^p \cdots \int_0^{l_{p+1}} dz^p+1 \int_0^{l_{p}} dz^p \cdots \int_0^{l_1} dz^1 \frac{1}{\left(K \sum_{j=1}^{n} z^j + 1\right)^{n+p}}, \tag{14}$$

where $p$ is the number of directions which end on the $AdS$ horizon. The integral can be evaluated straightforwardly, giving

$$I = \left(\frac{1}{(n+1)! K^j}\right)^n \frac{1}{(n+p)! \sum_{j=0}^{n-p} (-1)^j \sum_{j \in \mathcal{A}} \frac{1}{1 + K \sum_{i \in \mathcal{A}} l_i}}, \tag{15}$$

where $\mathcal{A}$ are combinations of $\{l_{p+1}, \ldots, l_1\}$ of length $j$. To compute the symmetry factors, we break up the crystal into topologically inequivalent cells. These are distinguished by how many of their sides end on $AdS$.
horizon. Clearly, the cells inside the crystal only end on branes, and correspond to taking \( p = 0 \) in (15). We can count the cells in each topological class as follows. In the case of 1D crystals, we can divide the crystal into two types, \( \mathcal{A} \) and \( \mathcal{B} \). The type \( \mathcal{A} \) are the cells inside the crystal, which end on branes on both sides, and there is \( \mathcal{A} \) of them. The type \( \mathcal{B} \) cells end on the AdS horizon, and there is at most two of them. Then the crystal (11) can be represented formally as a direct product

\[
\text{Brane Crystal} \equiv \prod_{\alpha j=1}^{n} \left[ \mathcal{A}_j \mathcal{A} + \zeta \mathcal{B} \right].
\]

where the number of end branes is \( \zeta \) = 0, 1 or 2 depending on the boundary conditions. It is then obvious that the number of elementary cells which end on the AdS horizon in \( p \) directions is given by the coefficient of \( \mathcal{B}^p \) in the expansion of (16):

\[
\nu_p = \zeta^n \sum_{\alpha \in \mathcal{A}} \prod_{\alpha j=1}^{n} \left[ \frac{\mathcal{A}_j}{\zeta} \right].
\]

where \( \mathcal{A}_p \) is any set of \( n - p \) distinct elements from the set \{1, \ldots, n\}, and \([r]\) denotes rounding off the number \( r \) to the nearest smaller integer. Specifically, the number of cells inside the crystal, with \( p = 0 \) is \( \nu_0 = \zeta^n \prod_{i=1}^{n} \left[ \mathcal{A}_i / \zeta \right] \). Note that the total number of branes along \( k \) direction is set by the boundary conditions (i)–(iv). Hence we obtain the following formula for the 4D Planck mass:

\[
M_{\text{Pl}}^2 = \frac{\zeta^n M_s^{n+2}}{(n+1)! K^n} \sum_{p=0}^{n} \sum_{\alpha \in \mathcal{A}_p} \prod_{\alpha j=1}^{n} \left[ \mathcal{A}_j \right]^{-n-p} \sum_{q=0}^{n} \frac{1}{(1 + K \sum_{i=1}^{N} l_i)^q}.
\]

where \( \mathcal{B}_j \) are combinations of length \( j \) of elements of \( \{l_{1}, \ldots, l_{n-p}\} \), and \( \epsilon(k) \in \mathcal{A}_p \). In the special case of finite crystals with cubic global symmetry and two end branes, there is the same number of branes in each direction, \( \mathcal{A}_i = \ldots = \mathcal{A}_n = \mathcal{A} \), and the lattice spacing is the same in all directions, \( l_{1} = \ldots = l_{n} = l \). Therefore all \( \mathcal{A}_p \)'s of the same length coincide, the symmetry factors \( \nu_p \) are \( \nu_p = 2^n [\mathcal{A}/2]^{n-p} n!/p!(n-p)! \), and hence

\[
M_{\text{Pl}}^2 = \frac{2^n M_s^{n+2}}{(n+1)! K^n} \sum_{p=0}^{n} \sum_{j=0}^{n-p} \frac{(-1)^j}{j! (n-p-j)!} \frac{1}{(1 + j Ki)^2} \left[ \mathcal{A} \right]^{n-p}.
\]

It is now obvious that for either class of finite crystals, (ii) or (iv), the formula (18) gives a finite value for the 4D Planck mass, meaning that there is a 4D graviton. In effect, those crystal configurations effectively ‘compactify’ the extra dimensions. In the case of infinite crystals (i) and (iii), although each individual contribution to (18) is finite, formally the sum diverges since there is an infinite number of cells to sum over. However, the sum can be renormalized at the expense of loosing a one-to-one correspondence between the 4D and (4 + n)D Planck scales. Indeed, for infinite crystals \( M_{\text{Pl}}^2 \sim \mathcal{A}^n M_s^{n+2} l^n \rightarrow \infty \), since \( \mathcal{A} \rightarrow \infty \). Since the divergence comes from the bulk volume being infinite, the renormalization must be a bulk phenomenon. To renormalize \( M_{\text{Pl}} \), we recall that since by the fine-tuning condition (6) the bulk cosmological constant is related to the brane tensions, and since the brane tensions must be unaffected by a bulk renormalization, the AdS radius must also be unaffected. Hence, the only way to remove infinity from \( M_{\text{Pl}} \) is to take \( M_\star \) to be infinite, and renormalize it by \( M_{\text{bare}} \rightarrow \mathcal{A}^{-n/(n+2)} M_\star \). But this means that the resulting mass \( M_\star \) need not be related to the scale of the brane physics, and hence this would seem to require another fine tuning to explain the hierarchy between \( M_{\text{Pl}} \) and \( m_{\text{EW}} \).

On the other hand, Eq. (19) shows that for finite crystals there are additional possibilities for generating hierarchies. This is because \( M_{\text{Pl}} \) depends on four different scales: the fundamental scale \( M_\star \), the AdS radius
$L = (\sqrt{\pi K})^{-1}$, the lattice spacing $l$ and the total volume of the crystal $\sim (\mathcal{A} \ell)^n$. Hence it is easier to generate exponential hierarchies in large crystals, even if all the other scales are of the same order of magnitude.

Before we move on to study the bulk gravitons, we should note that infinite and semi-infinite cases (i) and (iii) are also disfavored phenomenologically, even if we perform a renormalization as outlined above. In an infinite crystal there is an infinite number of intersections. Discrete translational invariance of the crystal would require that the energy density of particles on each intersection is of the same order (which would remain true even if the symmetry is weakly broken). Because at long distances the gravitational force is dominated by the zero modes, an observer on any intersection would feel the gravitational field of particles localized to all the intersections in the crystal regardless of how far in the bulk they are. Thus the localized energy density everywhere would add up to give the total energy density controlling the cosmological evolution of all intersections. Hence to sum up to a finite value, the energy density on each intersection must be infinitesimal. This implies that all but an infinitesimal amount of the energy density of the Universe we experience would have to be dark. Note that this remains true regardless of any renormalization of the scales, which may or may not be necessary (in a fashion similar to that for $M_p$). The matter on distant intersections would still be dark, there would be infinitely many of them, and renormalization would not change the ratio of dark matter to visible matter. Since the amount of visible matter in our Universe is not infinitesimal, we see that at least one intersection in the crystal must have finite $\rho$, and most of the others cannot. Hence the discrete translational invariance of the crystal must be strongly broken. Our intersection is a defect of the lattice. It cannot deform the lattice too strongly, in order not to loose the 4D graviton. But it does suggest that the graviton zero mode accumulates around the defect, which deforms the background more than most of the other vertices in the lattice. This can generate a peaked profile for the wave function of the zero mode, making other intersections weakly coupled. Such possibilities for generating asymmetric mirror worlds have been discussed in [32]. In practice, therefore, an infinite crystal would have to be physically hardly distinguishable from a finite crystal, with the exception that the strong coupling regime, discussed in [17], could be absent. A finite crystal would have to be consistent with observational constraints much like the Manyfold universes in theories with sub-millimeter internal dimensions [32]. Thus, an infinite crystal with unbroken discrete translational invariance may be only a mathematical idealization. We will retain it, however, since it is quite useful for the study of bulk gravitons, to which we now turn.

The field equation for linear perturbations around the background solution can be obtained by expanding the 4D part of the metric according to $g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is in the transverse traceless gauge with respect to the background $\nabla_\mu h^{\mu\nu} = h^{\mu\mu} = 0$, and expanding the Einstein’s equations to linear order in $h_{\mu\nu}$. The perturbations satisfy

$$\delta R_{\mu\nu} = \frac{1}{2} R^i_{\phantom{i}j\nu} h_{\mu\nu} + \frac{1}{2} R^i_{\phantom{i}j\mu} h_{\lambda\nu},$$

(20)

After evaluating $\delta R_{\mu\nu}$, substituting the background crystal metric (11), and defining the wave function $\Psi$ by

$$h_{\mu\nu} = \Omega^{(2-n)/2} \Psi,$$

(21)

where $\Psi$ in general denotes a complex function, which arises as a linear combination of $+$ and $\times$ polarizations, we find

$$\frac{1}{2} \Box_4 \Psi + \frac{1}{2} \nabla^2 \Psi - \frac{n(n + 2)(n + 4)}{8 \left( \sum_k \mathcal{C}(z^k) + \sqrt{n} L \right)} \Psi + 2 \left( \sum_k \mathcal{C}(z^k) + \sqrt{n} L \right) \sum_{k=1}^{n} \sum_j (-1)^j \delta(z^k - j l_k) \Psi = 0.$$  

(22)
In this equation, the $4D$ D'Alembertian is solved by the expansion into plane waves, $\Box \Psi = m^2 \Psi$, where $m$ is the KK mass. The resulting equation is

$$\nabla^2 \Psi + \left( m^2 - \frac{n(n+2)(n+4)}{4(\sum_{k} \mathcal{R}(z^k) + \sqrt{n} L)^2} + \frac{n}{(\sum_{k} \mathcal{R}(z^k) + \sqrt{n} L)^2} \sum_{k=1}^{n} \sum_{j} (-1)^j \delta(z^k - jl) \right) \Psi = 0.$$ (23)

This equation is a static Schrödinger equation for an electron in a periodic potential. It is now clear that the lowest energy eigenfunction of (23), which corresponds to the $4D$ graviton zero mode, is $\Psi_0 \sim \text{const} \times \Omega^{(n+2)/2}$ with $m^2 = 0$, which by (21) reproduces precisely (12), as we claimed. For $m^2 > 0$, the analogy with the electron in a crystal shows that the solutions of (23) are Bloch waves, satisfying $\Psi_q(z) = \exp(i \mathbf{q} \cdot \mathbf{z}) \psi_q(z)$, where by Floquet’s theorem $\psi_q$ are periodic functions under the action of the discrete translation group. The energy spectrum has a band structure due to resonant scattering. The bands are continuous in infinite crystals, and discrete in finite ones. In addition, there is a mass gap, separating the lowest-lying excited state from the ground state by a finite energy. In the language of our crystal Manyfold, this means that the lightest KK graviton must have a finite mass, and that the KK modes come in bands. As a result, their influence on the low energy physics on the intersection will be significantly suppressed when compared to a single brane of [17] or a single intersection of [19]. This is true not only for the production of bulk gravitons by the collisions of the Standard Model particles, but also for the corrections to Newton’s law of gravity. Due to the gap, there will be an exponential suppression of the higher-order corrections to the inverse square law. Mass gap was also discussed in [23] and in [35].

To be more concrete, we focus on an infinite crystal with only one internal dimension. Finite crystals will behave similarly, when they consist of many branes, except that the allowed bands will be discrete rather than continuous. We expect that the qualitative properties of crystals with more than one internal dimension will remain similar. In the case of one extra dimension, the Schrödinger equation (23) reduces to

$$\Psi'' + \left( m^2 - \frac{15}{4(\mathcal{R}(z) + L)^2} \right) \Psi + \frac{3}{\mathcal{R}(z) + L} \sum_{j} (-1)^j \delta(z - jl) \Psi = 0.$$ (24)

By the analogy with electrons in a crystal, e.g. with the Kronig-Penney model, we need to solve this equation inside two adjacent elementary cells in the crystal. By the periodicity of (11), we can choose the elementary cells $0 < z < 2l$ and $2l < z < 4l$. The $\delta$-function potentials in (24) can be recast as the boundary conditions on the derivatives of $\Psi$ at the vertices: $\Psi'(l_+) - \Psi'(l_-) = \frac{15}{4l^2} \Psi(l)$ and $\Psi'(2l_+) - \Psi'(2l_-) = -\frac{5}{2} \Psi(2l)$. Moreover, the wave function at each vertex must satisfy $\Psi(\text{vertex}_+) = \Psi(\text{vertex}_-)$ in order to conserve probability. The differential equation in the first elementary cell is

$$\Psi'' + m^2 \Psi = \begin{cases} \frac{15}{4(z + L)^2} \Psi, & \text{if } 0 < z < l; \\ \frac{15}{4(2l - z + L)^2} \Psi, & l < z < 2l. \end{cases}$$ (25)
These are Bessel equations. Using the boundary conditions above, the solution is

\[
\Psi = \begin{cases} 
\gamma m(z + L) \left[ A H_z^+ m(z + L) + B H_z^- (m(z + L)) \right], & 0 < z < l; \\
\gamma m(2l - z + L) \left[ \frac{h_1^+ h_2^- + h_1^+ h_2^+}{h_1^+ h_2^- - h_1^+ h_2^+} A + \frac{2 h_1^- h_2^+}{h_1^- h_2^- - h_1^- h_2^+} B \right] H_z^+ m(2l - z + L) + \frac{2 h_1^+ h_2^- + h_1^+ h_2^+}{h_1^+ h_2^- - h_1^+ h_2^+} A + \frac{2 h_1^- h_2^+}{h_1^- h_2^- - h_1^- h_2^+} B \right] H_z^- (m(2l - z + L)), & l < z < 2l.
\end{cases}
\]

(26)

Here \( H_z^\pm \) are the Hankel functions, which generalize the plane waves for the case of the Bessel equation. They should be used instead of the real Bessel functions, since they are needed to encode the momentum by phase rotation. It is rather amusing that both classical 4D graviton polarizations are needed to mimic the quantum-mechanical phase. Bloch waves physically correspond to rotation of the polarization plane of bulk gravitons in the crystal. In evaluating the transmission and reflection coefficients in (26) on the \( \delta \)-function, we have used a recursive relation \( dH_z^\pm / dz = -\frac{z}{l} H_z^\pm \). Then we can evaluate the functions and derivatives at \( l \), and using the boundary conditions eliminate \( C \) and \( D \) in favor of \( A \) and \( B \), and have the function in each elementary cell be determined by two (complex) constants, just like in the Kronig-Penney model.

The solution in the adjacent cell can be obtained similarly. It is given by

\[
\Psi = \begin{cases} 
\gamma m(z - 2l + L) \left[ A H_z^+ m(z - 2l + L) + B H_z^- (m(z - 2l + L)) \right], & 2l < z < 3l; \\
\gamma m(4l - z + L) \left[ \frac{h_1^+ h_2^- + h_1^+ h_2^+}{h_1^+ h_2^- - h_1^+ h_2^+} A + \frac{2 h_1^- h_2^+}{h_1^- h_2^- - h_1^- h_2^+} B \right] H_z^+ m(4l - z + L) + \frac{2 h_1^+ h_2^- + h_1^+ h_2^+}{h_1^+ h_2^- - h_1^+ h_2^+} A + \frac{2 h_1^- h_2^+}{h_1^- h_2^- - h_1^- h_2^+} B \right] H_z^- (m(4l - z + L)), & 3l < z < 4l.
\end{cases}
\]

(27)

Now we need to relate \( A, B, \hat{A}, \hat{B} \). The boundary conditions which they must satisfy are

\[
\Psi(2l) = \exp(2iql) \Psi(0), \quad \Psi(4l) = \exp(2iql) \Psi(2l), \quad \Psi(2l_+) = \Psi(2l_-),
\]

\[
\Psi'(2l_+) - \Psi'(2l_-) = -\frac{3}{L} \Psi(2l),
\]

(28)

where the former two come from the Floquet’s theorem for the Bloch waves and the periodicity of the braneworld crystal, and the latter two come from the continuity of \( \Psi \) and the jump of \( \Psi' \) at \( z = 2l \). The parameter \( q \) represents the wave vector of the Bloch wave, which is to be determined. The first three conditions yield

\[
B = \left( h_1^+ h_2^- + h_1^+ h_2^+ \right) \hat{h}_z^+ - \left( h_1^- h_2^+ - h_1^- h_2^- \right) \exp(2iql) \hat{h}_z^+ - 2 h_1^+ h_2^+ \hat{h}_z^- A,
\]

\[
\hat{B} = \left( h_1^+ h_2^- + h_1^+ h_2^+ \right) \hat{h}_z^- - \left( h_1^- h_2^- - h_1^- h_2^+ \right) \exp(2iql) \hat{h}_z^- - 2 h_1^- h_2^+ \hat{h}_z^+ A,
\]

\[
\hat{A} = \exp(2iql) A,
\]

(29)
where we have defined the numbers $\hat{h}_s^H = H_s^H(m(l + L))$ and $\hat{n}_s^L = H_s^L(mL)$. The last condition will give an equation relating $q$ and $m$, because the normalization constant $A$ factors out: it is determined by the overall normalization of the wave function. The equation for $q$ is the band equation. A straightforward computation gives

$$\cos(q) = \frac{(j_1n_1 + j_1n_2)(\hat{j}_1\hat{n}_1 + \hat{j}_1\hat{n}_2) - \hat{j}_1\hat{j}_2(j_1j_2 + 3n_1n_2) - \hat{n}_1\hat{n}_2(3j_1j_2 + n_1n_2)}{2(j_2n_1 - j_1n_2)(\hat{j}_2\hat{n}_1 - \hat{j}_1\hat{n}_2)}.$$

(30)

In this equation, we have decomposed the Hankel functions into the Bessel functions $J_s$ and $N_s$, with the definitions $j_s = J_s(m(l + L))$, $n_s = N_s(m(l + L))$, $\hat{j}_s = \hat{J}_s(mL)$ and $\hat{n}_s = \hat{N}_s(mL)$.

To see that there is a gap in the spectrum already at zero momentum, we can substitute $q = 0$ in (30). The resulting condition for the lowest-lying mass $m$ is the gap equation:

$$j_1j_2\hat{j}_1\hat{j}_2 + n_1n_2\hat{n}_1\hat{n}_2 + j_2j_1\hat{n}_1\hat{n}_2 + j_1\hat{j}_2n_1\hat{n}_2 + j_1\hat{j}_2n_1\hat{n}_2 - 3j_1\hat{j}_2n_1\hat{n}_2 - 3j_1\hat{j}_2n_1\hat{n}_2 - 3j_1\hat{j}_2n_1\hat{n}_2. \quad (31)$$

It is evident that this equation provides a non-trivial constraint on $m$. Finding the lowest positive value of $m$ which solves this equation is a straightforward, albeit tedious task. However, for $l > L$ we can readily obtain the correct order of magnitude of the gap. Since the Bessel functions in (31) are evaluated at $m(l + L)$ and $mL$, the only two candidates for mass scales for $m_{gap}$ are $L^{-1}$ and $(l + L)^{-1}$. On the other hand, in the limit $l \to \infty$, the crystal solution reduces to the single brane solution of [17], where the bulk gravitons do not have a gap. This excludes $L^{-1}$. Hence the mass gap must be

$$m_{gap} = \frac{\mathcal{O}(1)}{l + L}. \quad (32)$$

With this at hand, it is easy to estimate the corrections to Newton’s law in 4D induced by bulk gravitons. By the form of the wave functions (26), the 4D Newtonian potential for particles localized to a 3-brane at the intersection of 4-branes with positive tension (‘Planck 2 brane’), with the corrections from the bulk gravitons, is

$$V_N = -G_N \frac{M_1M_3}{r} \left(1 + a \int_{m_{gap}}^{\infty} \frac{dm}{K} \frac{m}{K} e^{-m/L}\right), \quad (33)$$

which upon integration gives

$$V_N = -G_N \frac{M_1M_3}{r} \left(1 + a \frac{L^2}{(L + l)^r} e^{-br/(l + 1)}\right), \quad (34)$$

where $a$ and $b$ are constants of order unity. Note that because of the gap, the corrections are $\mathcal{O}(r^{-3})$, in contrast to $\mathcal{O}(r^{-5})$ in [17]. On the other hand, using Eq. (11) when $l > L$, the potential for particles localized to a 3-brane at the intersection of branes with negative tension (‘TeV brane’) is, roughly,

$$V_N = -G_N \frac{M_1M_3}{r} \left(1 + a \int_{m_{gap}}^{\infty} \frac{dm}{K} \left(\frac{l}{L}\right)^3 e^{-m/L}\right). \quad (35)$$

$^2$The nickname ‘Planck’ refers to the 5D Planck scale $M_*$, not the 4D $M_{Pl}$.
which gives

\[ V_N = -G_N \frac{M_s M_2}{r} \left( 1 + a \frac{l^3}{L^2 r} e^{-br/l} \right), \]  

(36)

and again a and b are (different) constants of order unity. Clearly the corrections to Newton’s law for particles on the Planck brane can remain small at sub-millimeter distances even if \( L \gg l_P \sim 10^{-34} \text{ mm} \), despite the fact that there is only five spacetime dimensions. However, the corrections become more significant on the TeV brane, as is clear from (36). But if we are to solve the hierarchy problem while suppressing the corrections to 4D gravity to satisfy the observable bounds, our world should be a TeV brane. Therefore, the existing observational constraints yield \( l^3 \leq 1 \text{ mm} \cdot L^2 \); i.e. \( l \leq 1 \text{ mm} \cdot \left( \frac{1}{2} \right)^2 < 1 \text{ mm} \). Further, if \( l > L \), Eq. (19) gives \( M_P^2 \sim M_s^3 L', \) and so the 4D Planck mass depends on the size of the crystal and the lattice spacing. The fundamental scale should satisfy \( M_s \sim \frac{l}{2} \cdot \text{TeV} \). Hence \( M_P^2 \sim (\text{TeV})^3 \frac{1}{2L} \cdot \mathcal{A}' \), and using \( l \sim 1 \text{ eV}^{-1} \ll 1 \text{ mm} \), for which \( L \sim 10^{12} M_s^{-1} \) ensuring the validity of the supergravity description of the solution (11), gives \( \mathcal{A}' \sim 10^{16} \). With this choice of numbers, the Standard Model on the TeV branes has scale \( m_{\text{EW}} \sim 1 \text{ TeV} \), and its mirror on the Planck branes has scale \( M_s \sim 100 \text{ TeV} \). While the number of branes required is clearly very large, being equal to \( M_P^3/\text{TeV} \), i.e. to the hierarchy between the scales, it is smaller by sixteen orders of magnitude than the maximal number of branes which fit inside the flat extra dimensions [4]. Note also that the size of the crystal is \( \mathcal{A}' l \sim 10^{12} \text{ mm} \), with the choice \( M_s \sim 100 \text{ TeV} \), which is four orders of magnitude smaller than the size of a flat compact dimension required to generate the same ratio \( M_P^3/m_{\text{EW}} \) [1]. The required number of branes could be reduced in the case of more extra dimensions, but the details are beyond the scope of the present article.

It is straightforward to generalize this discussion when the lattice spacing varies within each family of parallel branes. In the extreme case, the configuration may be more similar to few separate crystals residing at different places in the \( AdS \) bulk. The effective scale would change from crystal to crystal, and in the infrared limit the low energy 4D physics on distant crystals would be similar to that on a 3-brane far from an intersection [19], or far from the Planck brane in 5D [22], or to a Manyfold with bulk profiles [32]. With several crystals in the bulk, it is easier to generate hierarchy with fewer branes in each crystal, as is straightforward to verify. In fact, separated ‘small crystals’ would correspond precisely to the cases of distant branes discussed in [19,22], where hierarchy is generated by gravitational shining.

We close with several concluding remarks. We have constructed general solutions describing intersecting families of parallel \( n + 2 \)-branes in 4 + \( n \)-dimensional \( AdS \) space. The families intersect at right angles, and consist of branes with both positive and negative tension. As a result, there arise several different types of crystals, both finite and infinite. All of them give rise to the usual 4D gravity, but phenomenological considerations strongly favor either finite crystals or infinite ones with broken translational invariance, profiles in the bulk and our world as a lattice defect. If such solutions are interlaced by 3-branes, at low energies they behave the same as Manyfold universes of [32]. The periodicity of the crystal lattice implies that the bulk gravitons are Bloch waves, with a band spectrum and a gap at zero energy. We construct the spectrum explicitly in the case of one extra dimension, and find that in the extreme case it may be possible to have the string scale at \( \sim 100 \text{ TeV} \), if there is \( \sim 10^{16} \) branes in the configuration. The crystal worlds raise many interesting questions: the issues of stability, early cosmology, etc. While the bulk gravitons behave as electrons in the usual crystal, one may wonder which degrees of freedom play role of the conventional phonons. It seems plausible that they should be the Goldstone modes associated with the branes in the crystal, which may get excited by sufficiently strong external perturbations. A more precise consideration would be needed to test this. Further, it would be interesting to find the properties of higher-dimensional crystals, since they may require fewer branes to resolve the hierarchy problem while ensuring Newton’s law of gravity in 4D at distances larger than a millimeter. We hope to return to some of these issues in the future.
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References

Localization of matter and cosmological constant on a brane in anti de Sitter space

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Abstract

We study two issues, the localization of various spin fields, and the problem of the cosmological constant on a brane in five-dimensional anti de Sitter space. We find that spin-zero fields are localized on a positive-tension brane. In addition to the localized zero-mode there is a continuous tower of states with no mass gap. Spin one-half and three-half states can be localized on a brane with “negative tension”. Their localization can be achieved on the positive-tension brane as well, if additional interactions are introduced. The necessary ingredient of the scenario with localized gravity is the relation between the bulk cosmological constant and the brane tension. In the absence of supersymmetry this implies fine-tuning between the parameters of the theory. To deal with this issue we introduce a four-form gauge field. This gives an additional arbitrary contribution to the bulk cosmological constant. As a result, the model gives rise to a continuous family of brane Universe solutions for generic values of the bulk cosmological constant and the brane tension. Among these solutions there is one with a zero four-dimensional cosmological constant.

1. Introduction and results

Large extra dimensions offer an opportunity for a new solution to the hierarchy problem [1]. The crucial ingredient of this scenario is a brane on which standard model particles are localized. Field-theoretic localization mechanisms for scalars and fermions [2] as well as for gauge bosons [3] were found. In string theory, fields can naturally be localized on D-branes due to the open strings ending on them [4] (for string theory realization of the scenario of Ref. [1], see, e.g., [5,6]).

Up until recently, extra dimensions had to be compactified, since the localization mechanism for gravity was not known. It was suggested in Ref. [7] that gravitational interactions between particles on a brane in uncompactified five-dimensional space could have the correct four-dimensional Newtonian behavior, provided that the bulk cosmological constant and the brane tension are related. Recently, it was found by Randall and Sundrum that gravitons can be localized on a brane which separates two patches of AdS5 space-time [8]. The necessary requirement for the four-dimensional brane Universe to be static is that the tension of the brane is fine-tuned to the bulk cosmological constant [7,8]. The generalization of this framework to higher dimensions [9–12], as well
as a number of interesting phenomenological and cosmological issues were studied in the literature [13–18]. As we mentioned above, there is a localized graviton zero-mode on the brane worldvolume [8]. In addition, there is a continuum of graviton states in the brane background [8]. All the fields in anti de Sitter space should be given boundary conditions. This can be accomplished by adopting the holographic approach to the problem [20]. In this framework, the bulk continuum is described by a certain Field Theory on the brane. This field theory should be coupled to a four-dimensional graviton which is nothing but the localized graviton zero-mode in the original AdS picture. In a sense, the introduction of the brane generates the coupling of the brane boundary four-dimensional field theory which is expected to describe the bulk gravitational physics in accordance with the Maldacena conjecture [21–23], to the four-dimensional Einstein gravity. This framework sets uniquely the boundary conditions for the fields (see detailed discussions and references in [22]).

The aim of the present paper is to study localization of various spin fields on a brane in AdS space. We consider two different cases: a positive tension brane which localizes gravity [8] (the Randall-Sundrum (RS) brane) and a ‘‘negative-tension’’ brane which gives rise to the exponentially growing warp factor [7]. We show that a spin-0 massless field is localized on the RS brane. The emerging picture is similar to that of the localized graviton. There is a scalar zero-mode on the brane. In addition, there is a continuum of states with no mass gap. This fits well into the holographic approach mentioned above. Furthermore, we show that the ‘‘negative tension brane’’ with the exponentially rising warp factor does not allow to localize normalizable scalar zero-modes.

For spin-1 fields the conclusions are less promising. Neither the positive-tension nor the ‘‘negative-tension branes’’ are capable of localizing vector fields in the minimal setup. The way out for the phenomenological model-building would be to invoke either stringy mechanism of localization on a D-brane worldvolume, or, in the field theory framework, to use the Dvali-Shifman mechanism which is based on the bulk confinement [3]. In either case the introduction of new bulk physics is necessary. Furthermore, we show that in the minimal setup neither spin-1/2 nor spin-3/2 fermions are localized on the RS brane. Nevertheless, the localization of these fields can be achieved by introducing new interactions in the theory. This allows one to have localized chiral fermions on a brane. On the other hand, spin-1/2 and spin-3/2 fermions are localized by gravitational interactions on a brane with ‘‘negative tension’’ which has the exponentially rising warp factor. A peculiar feature of this mechanism is that the localized modes are not chiral.

In Section 3 we discuss the fate of the four-dimensional cosmological constant. As we mentioned above, the scenarios of Refs. [7,8] require a special fine-tuned relation of the bulk cosmological constant to the brane tension. Once this relation is enforced, the four-dimensional brane Universe is static, i.e., the four-dimensional cosmological constant is zero. Within the supergravity framework this relation is just the BPS condition [24]. However, one should not expect that the BPS relation survives the quantum
corrections after SUSY breaking. As a result, one eventually goes back to fine-tuning. We will show in Section 3 that the necessity of the fine-tuning can be removed from the theory. This is accomplished by introducing a four-form gauge field which couples to gravity only. The four-form field does not propagate physical degrees of freedom in five-dimensional space, however, it can give rise to an arbitrary (negative or positive depending on the sign of its kinetic term) cosmological constant in the theory. Therefore, for any value of the original bulk cosmological constant, the net cosmological constant in this model is arbitrary. As a result, there are infinite number of brane Universe solutions. Each of these brane Universes is labeled by the corresponding four-dimensional cosmological constant. Among all these possible brane Universes there is one with a zero cosmological constant. This removes the fine-tuning problem from the Randall-Sundrum scenario in the sense that a static solution now exists for any value of the bulk cosmological constant and the brane tension. In addition, this framework opens an opportunity for anthropic arguments.

Note that many properties of a 3-brane is $D = 5$ are similar to those of a 2-brane in $D = 4$ which were previously studied in [27] for a review see [28]. These studies were recently generalized to $D = 2$ branes domain walls in $D$ dimensions [29].

2. Localization via warp factors

We start with the five-dimensional gravity action and a cosmological constant (the $[+,−,−,−,−,−]$ signature will be assumed below)

$$ S = - M^3 \int d^5 x (R + 2 \Lambda) .$$

In addition one includes a static 3-brane with tension $T$ which is located at $y = 0$ [8]. The five-dimensional interval will be parametrized as follows:

$$ ds^2 = A(y) dx^2_{1,1} - dy^2 .$$

The Einstein equations in this case have two different solutions for the warp factor $A(y)$. For a positive-tension brane one finds [8]

$$ A(y) = \exp(-H |y|) ,$$

while for a “negative-tension” brane the warp factor is exponentially rising $^5$ [7]

$$ A(y) = \exp(H |y|) .$$

In both cases, the static solutions (3) and (4) exist if the bulk cosmological constant is fine-tuned to the brane tension

$$ \sqrt{- \frac{2 \Lambda}{3}} = \frac{|T|}{6M^3} \equiv H .$$

We will show below, that the RS solution admits the localization of a spin-0 state in addition to the already known localization of gravitons. The solution (4), on the other hand, localizes spin-1/2 and spin-3/2 fermions.

Before we go further, let us make some comments on the literature. One can consider an orbifold compactification of the fifth dimension in the RS framework. In this case, there are two 3-branes which are located at the orbifold fixed points [8]. One of these is a positive-tension brane and the other one is a negative-tension brane. The warp factor for one of them decreases as in (3), while for the other one it increases exponentially (4). Therefore, in this framework spin-0 and spin-2 states will be localized on the positive-tension brane, while spin-1/2 and spin-3/2 states will be trapped on the negative-tension one. In what follow we will be studying a single 3-brane which separates two patches of AdS$_5$ space-time. From the point of view of the orbifold construction this corresponds to the case when one of the orbifold fixed points is removed to infinity.

2.1. Spin-0 fields

In this subsection we study the localization of a real scalar field in the background (2). We will find that the solution (3) admits a localized zero-mode.

$^5$ As before, the “negative-tension” brane is a non-fluctuating slice of space with negative energy density. The exponentially rising warp factor was originally introduced in Ref. [30] in a theory with no branes.
Let us start with a massless scalar field coupled to gravity:
\[
\frac{1}{2} \int d^5 x \sqrt{g} g^{AB} \partial_A \Phi \partial_B \Phi .
\]
Here \( A, B \) denote five-dimensional indices. The corresponding equation of motion takes the form
\[
\partial_A \left( \sqrt{g} g^{AB} \partial_B \Phi \right) = 0 .
\]
Using (2) and decomposing this equation in the four-dimensional and fifth dimensional parts one finds:
\[
\frac{1}{A} \eta^{\mu \nu} \partial_\mu \partial_\nu \Phi - \frac{1}{A^2} \partial_\mu \left( A \partial_\mu \Phi \right) = 0 .
\]
Let us decompose the field \( \Phi \) as follows:
\[
\Phi ( x, y ) = \phi ( x ) \chi ( y ) .
\]
A plane wave which propagates in a four-dimensional worldvolume satisfies the corresponding four-dimensional equation of motion:
\[
\eta^{\mu \nu} \partial_\mu \partial_\nu \phi ( x ) = - m^2 \phi ( x ) .
\]
Using this, and introducing a new variable
\[
u ( y ) = A ( y ) \chi ( y ) ,
\]
one gets the following Schrödinger equation for the \( y \)-dependent part:
\[
\left[ - \partial_\gamma^2 - m^2 e^{H |y|} + H^2 - 2 H \delta ( y ) \right] \nu ( y ) = 0 .
\]
This equation coincides with the one for a localized graviton [8]. The zero-mass solution \( m^2 = 0 \) to this equation takes the form:
\[
\nu ( y ) = c e^{-H |y|} ,
\]
where \( c \) is a constant. It is interesting that, written in terms of the original variable \( \chi \), the scalar zero-mode is just a constant
\[
\chi ( y ) = c .
\]
At a first glance, such a solution could not be localized since it is not suppressed away from the brane. Nevertheless, the solution can still be considered as a localized mode. The presence of the exponential warp factor in the metric (2) allows one to perform the following decomposition for the zero-mode
\[
\frac{1}{2} \int d^4 x dy \sqrt{g} g^{AB} \partial_A \Phi_0 \partial_B \Phi_0
\]
\[
= \frac{1}{2} \int_{-\infty}^{+\infty} d y \sqrt{g} \chi^2 ( y ) g^{\mu \nu} \int d^4 x \partial_\mu \phi ( x ) \partial_\nu \phi ( x )
\]
\[
= \frac{1}{2} \int d^4 x \eta^{\mu \nu} \partial_\mu \phi ( x ) \partial_\nu \phi ( x )
\]
where we set \( c = \sqrt{H / 2} \).

Thus, for the zero-mode \( \Phi_0 ( x, y ) = \sqrt{H / 2} \phi ( x ) \).

As in the case of gravitons [8], the Eq. (12) gives rise to a tower of continuum states with \( m^2 > 0 \). These states produce a nonzero contribution of order \( 1 / r^3 \) to the usual four-dimensional \( 1 / r \) potential mediated by a massless scalar exchange on a brane. The picture is similar to that of the localized graviton [8].

2.2. Spin-1 fields

There are no localized solutions in this case. Although there is a constant solution similar to (14), nevertheless, the Lagrangian for a vector field does not yield the suppressing warp factor which was so crucial in the case of scalars. Let us see this in some details. Using the definition of the 5D field-strength
\[
F_{AB} = \partial_A V_B - \partial_B V_A ,
\]
choosing the gauge \( V_5 = 0 \), and decomposing the remaining part of the massless vector field as
\[
V_\mu ( x, y ) = v_\mu ( x ) \alpha ( y ) ,
\]
we get \( f_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu \)
\[
- \frac{1}{2} \int d^4x dy \sqrt{g} \ g^{\Lambda\Phi} g^{CD} F_{AC} F_{BD}
= - \frac{1}{2} \int_{-\infty}^{+\infty} dy \sqrt{g} \ \sigma^2(y) \ g^{\mu\nu} g^{\alpha\beta} \int d^4x f_{\mu\alpha} f_{\nu\beta}
= - \frac{1}{2} \int d^4x \eta^{\mu\nu} \eta^{\alpha\beta} f_{\mu\alpha} f_{\nu\beta} \int_{-\infty}^{+\infty} dy \ \sigma^2(y).
\]
(18)

This diverges if \( \sigma(y) \) is a constant.

How about nontrivial solutions to the equation of motion for the vector field, could they give localized solutions? One can check that such normalizable solutions do not exist. Since this issue has already been studied in the orbifold version of the RS scenario in [31,32] we skip these derivations. The net result is that neither (3) nor (4) localizes massless vector fields.

2.3. Spin-1/2 fermions

We start with the Lagrangian for massless spin-1/2 fermions
\[
\overline{\Psi} i \gamma^\mu D_\mu \Psi.
\]
(19)
The corresponding equation of motion can be decomposed as follows:
\[
(\Gamma^\mu D_\mu + \Gamma^5 D_5) \Psi(x,y) = 0.
\]
(20)
The relations between the curved-space gamma matrices \((\{\Gamma^A, \Gamma^B\} = 2 \ g^{AB}\) and the minkowskian ones \((\{\gamma^A, \gamma^B\} = 2 \eta^{AB}\) read as follows:
\[
\Gamma^\mu = \frac{1}{\sqrt{A}} \gamma^\mu, \quad \Gamma^5 = \gamma^5.
\]
(21)
The spin-connection and covariant derivative can also be calculated for the metric (2):
\[
D_\mu = \partial_\mu + \frac{\Lambda}{4A} \Gamma^\mu, \quad D_5 = \partial_5.
\]
(22)

After these conventions are set we can decompose the five-dimensional spinor into the four-dimensional and the fifth-dimensional parts: \(\Psi(x,y) = \chi(x)\alpha(y)\). We require that the four-dimensional part satisfies the massless equation of motion \(\gamma^\mu \partial_\mu \chi(x) = 0\). As a result, we obtain the following equation for the \(y\) dependent part
\[
\left(\partial_y + \frac{\Lambda}{A}\right) \alpha(y) = 0.
\]
(23)
The solution to this equation reads:
\[
\alpha(y) = \frac{c}{A(y)}.
\]
(24)

This is not normalizable in the RS case (3), but it is normalizable if the warp factor (4) is used. To see this, one has to decompose the action (19) as follows:
\[
\int d^4x \int_{-\infty}^{+\infty} dy \sqrt{g} \ \overline{\Psi}(x,y) i \Gamma^\mu D_\mu \Psi(x,y)
= \int_{-\infty}^{+\infty} dy A^{1/2}(y) \alpha(y) \int d^4x \overline{\Psi}(x) i \gamma^5 \partial^\mu \Psi(x),
\]
which is infinite for \(A(y)\) defined in (3), but is finite for \(A(y)\) from (4) (we choose \(c = \sqrt{H}/2\) to get the correct normalization).

In order to localize spin-1/2 fermions in the RS framework one could use the method of localization due to Jackiw and Rebbi [34]. For this, one introduces the interaction of fermions with a scalar field, \(\Phi \overline{\Psi}\Psi\). The scalar should interpolate between two different vacua at different sides of the brane (one could take the kink solution, for instance). This gives an effective five-dimensional mass to fermions (constant fermion mass approximation is good enough when the scalar is heavy). However, the five-dimensional fermion mass term \(M \overline{\Psi} \Psi\) would flip the sign under the reflection with respect to the brane. Under these circumstances, the chiral left(right)-handed component of the fermion can be localized on the brane [34]. The equation of motion for the fermion takes the form:
\[
\left(\partial_y + \frac{\Lambda}{A} + i\gamma_5 M \left[\theta(y) - \theta(-y)\right]\right) \Psi(x,y) = 0,
\]
(25)
where \(\theta(y)\) is the step-function. Introducing the chiral modes as \(i\gamma_5 \Psi_{L,R} = \pm \Psi_{L,R}\), one finds that the

\[\text{After we obtained these results the work [33] appeared, in which it was shown that in the orbifold RS construction massless spin-1/2 fields are not localized on a brane.}\]
chiral solution $\Psi_L \propto \exp(H - M)$ is localized for (3) as long as:

$$M > \frac{H}{4}.$$  \hfill (26)

Likewise, in the background (4), the mode $\Psi_L$ is localized if $M < H/4$.

2.4. Spin-3/2 fermions

The consideration for gravitinos is similar to that of spin-1/2 fermions. Therefore, our discussions will be brief. The equation of motion for a massless gravitino reads as:

$$\Gamma^{(A'B'C')} D_b \Psi_C = 0.$$  \hfill (27)

Here, the square brackets denote antisymmetrization w.r.t. all indices. We choose the gauge $\Psi_s = 0$ and split the remaining fields as $\Psi_s(x,y) = \psi_s(x) u(y)$. Using now the four-dimensional gauge choice $\gamma^\mu \psi_\mu = 0$, and the four-dimensional equation of motion for a massless spin-3/2 field $\gamma^{(\mu\nu\rho)} \gamma^\sigma \psi_\sigma = 0$, one finds the following equation for the $y$ dependent part:

$$\left( \partial_\nu + \frac{A'}{2A} \right) u(y) = 0.$$  \hfill (28)

The solution to this equation is

$$u(y) = \frac{c}{\sqrt{A(y)}}.$$  \hfill (29)

In the RS case this is not a normalizable function. The action

$$\int d^4x \int_{-\infty}^{+\infty} dy \sqrt{g} \bar{\Psi}_A(x,y) i \Gamma^{(A'B'C')} D_b \Psi_C(x,y)$$

$$= \int_{-\infty}^{+\infty} dy A^{1/2}(y) u^2(y) \int d^4x \bar{\psi}_\mu(x) i \gamma^{(\mu\nu\rho)} \gamma^\sigma \psi_\sigma \times \partial_\nu \psi_\nu(x),$$  \hfill (30)

diverges because of the $y$ integration. On the other hand, for the exponentially rising warp factor (4), the $y$ integral is finite and the 4D action is canonically normalized for $c = \sqrt{H}/2$. Therefore, the solution (4) admits a localized free spin-3/2 zero-mode on the 3-brane. Notice that this mode is not chiral.

Finally, let us comment on a possibility of localization of gravitinos on the background (3). One should introduce some additional interactions for this to happen (as we did for the case of spin-1/2 fields). The simplest way would be to realize the RS solution (3) in some $N = 2$ five-dimensional supergravity. In this case, one could hope to find a BPS 3-brane which preserves half of the original supersymmetries. Since the background (3) localizes gravitons, by supersymmetry, its SUGRA counterpart would also localize gravitinos. The dynamical reason for the localization of gravitinos could be their interactions with some other fields of the corresponding SUGRA. Detailed studies of these issues could be a subject of a separate project.

3. Cosmological constant on a brane

In this section we study what happens if the bulk cosmological constant is not fine-tuned to its critical value which is defined by Eq. (5). This is expected to be the generic case in any realistic brane Universe models with broken SUSY. Indeed, after SUSY is broken, Eq. (5) can no longer be protected against corrections even though it could have been obtained as a BPS equation in some five-dimensional theory of supergravity (along the lines of [24], for instance). Therefore, in what follows we consider the case when the bulk cosmological constant differs from its critical value defined by (5). In such a case the four-dimensional brane Universe is not static anymore [35,36]. The four-dimensional cosmological constant is determined by the difference between the actual bulk cosmological constant and the critical cosmological constant satisfying (5). Generically, this difference can be big, leading to a de Sitter or anti de Sitter four-dimensional brane Universe with an unacceptably large four-dimensional cosmological constant. Thus, it seems that the fine-tuning is of vital importance. However, the fine-tuning of the parameters can be avoided by introducing into the theory a four-form gauge field $A_{BCDE}$. This field cannot propagate any physical degrees of freedom in five-dimensions. Nevertheless, it gives rise to a dynamically generated cosmological constant [25,26]. The

\footnote{We are grateful to Gia Dvali for many useful discussions of the results of this section.}
value of this cosmological constant is arbitrary, since it appears as a constant of integration of the equation of motion for the $A_{BCDE}$ field. Therefore, the net cosmological constant in five-dimensions is the sum of the original cosmological constant and the dynamically generated one. Let us for simplicity assume that the original bulk cosmological constant $\Lambda$ is positive but otherwise an arbitrary number which is not necessarily fine-tuned to the brane tension. The generalization for negative $\Lambda$ is straightforward and will be given in the next section. The action for the system can be written as follows:

$$S = -M^3 \int d^4x \sqrt{g} \left( R + 2\Lambda \right)$$

$$+ \int d^4x \sqrt{g} \left\{ -\frac{1}{2 \times 5!} F_{ABCDE} F^{ABCDE} \right\}$$

$$+ S_{\text{Brane}}.$$  

(31)

Here $F_{ABCDE}$ denotes the gauge-invariant field-strength for the field $A_{BCDE}$ (the choice of the sign of the kinetic term for this field will be discussed in the next section). The action for the brane itself in (31) is denoted by $S_{\text{Brane}}$. We will simply assume that the 3-brane is a static source which is localized at $y = 0$. It has the following energy-momentum tensor:

$$T_{AB}^{\text{Brane}} = T \delta(y) \text{diag}(1, -1, -1, -1, 0),$$  

(32)

where $T$ denotes the tension of the 3-brane.

The equations of motion of the system take the following form:

$$R_{AB} - \frac{1}{2} g_{AB} R = \frac{1}{2M^2} \left( T_{AB} + T_{AB}^{\text{Brane}} \right)$$

$$+ g_{AB} \Lambda, \quad \partial_5 \left( \sqrt{g} F_{ABCDE}(y) \right) = 0.$$  

(33)

Here $T_{AB}$ is the energy-momentum tensor for the four-form field:

$$T_{AB} = \frac{1}{4!} \left\{ -F_{ACDE} F^C_{\, DEG} \right\} + \frac{1}{\pi} g_{AB} F_{CDEGH} F^{CDEGH}.$$  

As we discussed above, the crucial property of the four-form gauge field in five-dimensional space-time is that it does not propagate dynamical degrees of freedom. The whole dynamics is eliminated by the system of equations of motion, and the gauge constraints which emerge as a result of the gauge invariance of the action. Nevertheless, this system of equations has a constant field-strength solution which gives rise to a nonzero vacuum energy density. This is what happens in general with a d-form gauge invariant field-strength in d-dimensional space-time.

To make these discussions quantitative let us look for the following solution to the equations of motion (33):

$$ds^2 = \mathcal{M}(y) \left( dt^2 - a(t) dx^i dx^i \right) - dy^2,$$  

(35)

where $a(t)$ is the scale factor of the four-dimensional brane Universe. The corresponding background for the four-form field is given by

$$F_{ABCDE} = \frac{1}{\sqrt{g}} \varepsilon_{ABCDE} k,$$  

(36)

where $k$ stands for an arbitrary integration constant (in our normalizations $\varepsilon_{12345} = 1$). With this Ansatz at hand the four-dimensional components of the energy-momentum tensor for the four-form field take the form:

$$T_{AB} = -\frac{1}{2} g_{AB} k^2.$$  

(37)

This generates an additional negative contribution to the total effective bulk cosmological constant

$$\Lambda_{\text{eff}} = \Lambda - \frac{1}{4M^2} k^2.$$  

(38)

Using these expressions the Einstein equations can be written as follows:

$$\frac{3 \dot{\mathcal{M}}}{2 \mathcal{M}} = -\Lambda_{\text{eff}} - \frac{1}{2M^2} T \delta(y),$$

$$\left( \frac{3 \dot{\mathcal{M}}}{2 \mathcal{M}} \right)^2 = \frac{H_0^2}{\mathcal{M}^2} - \frac{1}{\mathcal{M}^2} \Lambda_{\text{eff}},$$

$$\left( \frac{\ddot{a}(t)}{a(t)} \right)^2 = \frac{\dot{a}(t)^2}{a(t)} = H_0^2.$$  

(39)

Here, primes denote differentiation with respect to $y$, and dots denote differentiation w.r.t. the time vari-
The solution to these equations reads as \[ \mathcal{A}(y) = \text{ch} \left( \sqrt{\frac{2}{3} \Lambda_{\text{eff}} y} \right) - \frac{T}{2 M^3 y} \text{sh} \left( \sqrt{\frac{2}{3} \Lambda_{\text{eff}}} y \right) . \] (40)

The four-dimensional Hubble constant is defined as follows:

\[ H_0 = \sqrt{\frac{T^2}{36 M^6} + \frac{2 \Lambda_{\text{eff}}}{3}} . \] (41)

Therefore, the four-dimensional cosmological constant, \( \Lambda_4 \equiv H_0^2 \), contains an arbitrary integration constant \( k \):

\[ \Lambda_4 = \frac{T^2}{36 M^6} + \frac{2}{3} \left( \Lambda - \frac{1}{4 M^3} k^2 \right) . \] (42)

Let us summarize the results of our discussions. The system (31) without the \( F \) field does not allow for a static solution for a generic, non-fine-tuned values of the bulk cosmological constant and the brane tension. However, when the \( F \) field is switched on, there are an infinite number of brane Universe solutions. These Universes differ from each other by the value of the corresponding four-dimensional cosmological constants defined in (42). Among these solutions there is the static brane with zero four-dimensional cosmological constant. This removes the necessity of the fine-tuning in the sense that static solution now exists for any values of \( \Lambda \) and \( T \). Certainly, the cosmological constant problem is not solved, instead, the introduction of the \( F \) field gives an opportunity for the anthropic arguments. Indeed, only those branes for which the four-dimensional cosmological constant fits into the Weinberg’s window [37]:

\[ \frac{\Lambda_4}{8 \pi G_N} \leq (10^{-3} \text{ eV})^4 , \] (43)

could accommodate the Universe similar to ours. All other brane Universes, not satisfying (43), would not be able to form the large scale structures. Concluding, the action (31) allows to avoid the necessity of fine-tuning of the bulk cosmological constant \( \Lambda \) to the brane tension \( T \), for any \( \Lambda \) and \( T \) there are an infinite number of possible brane Universe solutions of (31) and one of these solutions has zero four-dimensional cosmological constant.

4. Discussions and conclusions

Let us start with some comments on the form of the action (31). The four-form field in (31) reminds the Ramond-Ramond (RR) field to which a three-brane could couple. However, one can check that the sign of the kinetic term in (31) is inverse to what should have been used for the RR field. The choice of this sign is not restricted by positivity arguments in the five-dimensional theory (31), since this field does not propagate dynamical degrees of freedom. However, the negative sign is necessary in (31) if the four-form field is real and one needs to generate a negative cosmological constant in (38). This is true as long as the original cosmological constant \( \Lambda \) in (31) is an arbitrary positive number. Thus, for \( \Lambda > 0 \) the four-form field cannot be thought of as a RR field. Let as now see what happens if \( \Lambda \) is a big arbitrary negative number (in fact, without loss of generality, we take its magnitude to be bigger than the magnitude of the critical value defined by (5)). In this case one can just flip the sign of the kinetic term of the \( F^2 \) term in (31) and obtain the desired result 10. Indeed, in this case, the kinetic term for the four-form field is precisely equivalent to that for a RR field. This produces a positive cosmological constant in

\[ \text{For simplicity we present here the solution when the brane worldvolume is dS}_4. \text{ The solution can be obtained for AdS}_4 \text{ as well [35,36].} \]

\[ \text{Alternatively, one could add to the action (31) another four-form field with opposite kinetic term.} \]
the bulk. Therefore, the expression (42) takes the form:

\[ \Lambda_k = \frac{T^2}{36 M^6} + \frac{2}{3} \left( \Lambda + \frac{1}{4 M^3} k^2 \right). \]  

(44)

Since \( \Lambda \) in this case is a big negative number (its magnitude is bigger than \( T^2/24M^6 \)), one is still able to have an infinite number of brane Universe solutions parametrized by the integration constant \( k \) and one of these solutions have \( \Lambda_k = 0 \). In this case (negative \( \Lambda \)), one could think of the four-form field as some kind of RR field of string theory. However, in our case the brane at hand should not couple to this RR field, i.e., the brane should not be a D-brane. Otherwise, the effective cosmological constant produced by the RR field would be defined by the brane RR charge. Since this is quantized, one would again need fine-tuning between the brane charge and the brane tension (as before, in the supergravity framework this is the BPS condition, however it becomes fine-tuning after SUSY is broken).

Let us now go back to the case \( \Lambda > 0 \) considered in the previous section. A way to think of the four-form field in this case is to recall that in theories of supergravity there are auxiliary scalar fields. These scalars usually enter an off-shell Lagrangian in a quadratic form without kinetic terms. One of these scalars, let us call it \( \phi \), can be dualized into the five-form field strength via \( \phi \propto \epsilon_{ABCDE} F^{ABCDE} \).

Since the original scalar was an auxiliary field and the four-form does not propagate dynamical degrees of freedom either, this dualization makes sense. As a result, the \( \phi^2 \) term in the original off-shell Lagrangian of supergravity, which can emerge as \( \sqrt{\epsilon} (R + \phi^2) \) [38], is replaced by the \( F^2 \) term in the form given in (31). This is acceptable as far as bosonic modes are concerned. The actual issue is whether it is possible to perform the dualization of the whole SUGRA supermultiplet, not only its scalar part. Such a dualization of the chiral \( N = 1 \) SUGRA supermultiplet into the tensor supermultiplet was shown to be possible in the four-dimensional case [39]. The issue whether the same can be done in five-dimensions remains open.

To conclude, we studied localization of various spin fields on a brane in AdS_5. We found that spin-0 fields are localized on a brane with positive tension which also localizes gravitons. The spectrum of spin-0 field is similar to that of the localized graviton: there is a single localized zero-mode plus a continuous tower of states. This continuum can be thought of as a bulk five-dimensional field. To fix the boundary conditions, the whole picture should be viewed in a sense of the AdS/CFT correspondence. Spin-1 fields are not localized neither on a brane with positive tension nor on a brane with “negative tension”. Within the field theory framework the Dvali-Shifman mechanism [3] should be invoked for the vector field localization. In string theory, vector fields could be localized by assuming that the brane at hand is in fact a D-brane. Spin-1/2 and spin-3/2 fields are localized due to gravitational interactions on a brane with “negative tension”. The localized modes in this case are not chiral. In order to trap spin-1/2 and spin-3/2 chiral zero-modes on a positive-tension brane additional fields should be introduced. These could be scalars of five-dimensional gauged supergravity.

Finally, we have shown that fine-tuning between the bulk cosmological constant and the brane tension can be avoided if there is a four-form gauge field in the theory. This field gives rise to an additional contribution to the bulk cosmological constant. As a result, the model with any generic \( \Lambda \) and \( T \) supports an infinite number of brane Universe solutions which are parametrized by the continuous family of four-dimensional cosmological constants. Among these solutions there is one with \( \Lambda_4 = 0 \).

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**References**

A note on T-duality, open strings in B-field background and canonical transformations

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Abstract

In this paper we study T-duality for open strings ending on branes with non-zero B-field on them from the point of view of canonical transformations. For the particular case of type II strings on the two torus we show that the \( \text{Sl}(2,\mathbb{Z})_N \) transformations can be understood as a sub-class of canonical transformations on the open strings in the B-field background.

\( \text{SL}(2,\mathbb{Z}) \) metric in the zero volume limit.

In a string theoretic discussion it was shown that quantizing the open strings with the mixed boundary conditions, the coordinates of end points of such open strings, \( x_i \), which live on a D-brane with a non-zero B-field background are really noncommuting [3,6,7] and:

\[
[x_i, x_j] = 2i\alpha'(B(1 - B^2)^{-1})_{ij}.
\] (1)

Besides the noncommutative structure, it was shown that the T-duality group of type II theories on \( T^3 \), \( SO(2,2;\mathbb{Z}) \) can be written as \( \text{Sl}(2,\mathbb{Z}) \times \text{Sl}(2,\mathbb{Z})_N \), where the first is the mapping class group of the torus and the second acts on the Kähler moduli of the torus [3,4].

On the other hand, there have been many suggestions in literature trying to explain T-duality as particular canonical transformations [8–10]. This idea was initially studied for the closed strings in general backgrounds and was extended to the open strings with Dirichlet boundaries, D-branes. However, the case of D-brane bound states, i.e., open strings ending on the branes with non-zero B-field [11] has not been elaborated on well.

In this note we will study the behaviour of open strings with different boundary conditions under the...
proper canonical transformations and show that the $SL(2, \mathbb{Z})_N$ part of T-duality is really a sub-class of these canonical transformations.

2. Open string in B-field backgrounds

We start with the usual $\sigma$-model action describing the open strings in B-field backgrounds:

$$S = \frac{1}{4\pi \alpha'} \int d^2\sigma \left[ \eta_{\mu\nu} \partial_\mu X^\nu \partial_\nu X^\mu - \frac{1}{2\pi \alpha'} \Phi d\sigma A_\mu \partial_\mu \xi^i \right] + g^{ab} \partial_\mu X^\mu \partial_\nu X^\nu,$$

where $A_\mu$ is the U(1) gauge field living on the D-brane and $\xi^i$ its internal coordinates. The action is invariant under the combined gauge transformation [2]

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu A_\nu - \partial_\nu A_\mu, \quad A_\mu \rightarrow A_\mu - \partial_\mu B_{\mu\nu}.$$  \hspace{8cm} (3)

The gauge invariant field strength is then

$$F_{\mu\nu} = B_{\mu\nu} - F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  \hspace{8cm} (4)

So we can always choose $A_\mu$ so that $F_{\mu\nu} = B_{\mu\nu}$, and hereafter we will work in this gauge.

Variation of the action $S$, leads to the following mixed boundary conditions

$$\partial_\sigma X^\mu + \partial_\mu X^\sigma = 0,$$  \hspace{8cm} (5)

or in terms of the conjugate canonical momenta, $P_\mu$:

$$\partial_\sigma X^\mu + \alpha f^{\mu\nu} P_\nu = 0,$$  \hspace{8cm} (6)

with

$$f_{\mu\nu} = \left( F \right)^{-1}. \hspace{8cm} (7)$$

Since we will mostly consider the compactifications of $X^1$ and $X^2$ directions on two torus, for simplicity we assume that all the $F$ components except $F_{12}$ to be zero and in this case [2]

$$f_{\mu\nu} = \epsilon_{\mu\nu} f, \quad f = \frac{F}{1 + F^2}.$$  \hspace{8cm} (8)

$f$ is the deformation parameter of the noncommutative tours.

Let us return to the question of canonical transformations. It is well known that, although the canonical transformations do not change the equations of motion, they alter the boundary conditions and hence in our case we expect the boundary conditions, or equivalently the parameter $f$, to be changed with canonical transformations. But, we will momentarily show that for any linear canonical transformation one can find a transformed $f$, so that the form of the boundary conditions remain invariant. This argument can be justified as follows: Boundary conditions can be treated as the constraints which are of second class [12] and hence the canonical transformations are those under which the Dirac brackets are invariant. The above requirement is satisfied if we also transform $f$ in a proper way.

Since we are interested in the two torus case, we consider the canonical transformations which reproduce the $SL(2, \mathbb{Z})_N$:

a) $$\begin{cases} X_i \rightarrow \tilde{X}_i = \alpha' \int^\sigma P_i d\sigma \\ P_i \rightarrow \tilde{P}_i = \partial_\sigma X_i, \quad i = 1, 2 \end{cases}$$ together with \footnote{For the case of the torus we should take $\alpha'$ to $R_1^2$ and the metric is changed as the usual T-dualities.}

$$f \rightarrow \tilde{f} = \frac{-1}{f}.$$  \hspace{8cm} (9)

One can easily show that the above transformations do not change the form of equations of motion and the boundary conditions,

$$\partial_\sigma X^\mu + \alpha' \tilde{f}^{\mu\nu} P_\nu = 0 \rightarrow \partial_\sigma \tilde{X}^\mu + \alpha' \tilde{f}^{\mu\nu} \tilde{P}_\nu = 0.$$  \hspace{8cm} (10)

In other words under the above transformations the Dirac bracket introduced in [7,12] are invariant. It is worth noting that the above transformations are exactly the same as the transformations introduced in [8] for the closed strings.

b) $$\begin{cases} X_i \rightarrow \tilde{X}_i = X_i + \epsilon_{ij} \alpha' \int^\sigma P_j d\sigma \\ P_i \rightarrow \tilde{P}_i = P_i, \quad i = 1, 2 \end{cases}$$ together with the proper metric transformations (the usual T-duality transformations), and the $f \rightarrow \tilde{f} = f + \frac{F}{1 + F^2}$.  \hspace{8cm} (11)
1. are also the symmetry of the open string action in the same sense as the case $a$.

As we see the $Sl(2,\mathbb{Z})_\rho$ transformations which act on the Kähler structure of the torus, $\rho = iV + f$, can be generated by the transformations on canonical variables and also the $f$, background field. All the above arguments can easily be generalized to the case of $T^\alpha$.

3. Concluding remarks

In this note we showed that the idea of treating T-duality group of the perturbative closed string theories (type II strings) as canonical transformations of the $\sigma$-model can be extended to the strong couplings, using the fact that at strong coupling type II theories are perturbatively governed by the open strings ending on the D-branes and their bound states. In this case although the open string boundary conditions or equivalently the brane structure we have change under canonical transformations, we argued that by a proper change of parameters we can make our theory to be invariant under such transformations, which exactly lead to the $Sl(2,\mathbb{Z})_\rho$ part of the expected T-duality group. On the other hand it has been argued that this duality can be realized as the Morita equivalence of the noncommutative gauge bundles governing the low energy effective theory of the open strings we considered here. So it is an interesting question to study Morita equivalence in the light of canonical transformations.

One can address the same procedure in the path integral formulation too. In the path integral, canonical transformations are translated to changing the operators by similarity transformations. This point of view has recently been used for quantizing the discretized M2-branes in a three from field background [13]. In this picture T (or U)-duality is given by a special class of the unitary transformations which are characterized by C-field. This question from the string theory point of view can be understood through the DVV [14] string matrix theory.

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Noncommutative string theory, the R-matrix, and Hopf algebras

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Abstract

Motivated by the form of the noncommutative $*$-product in a system of open strings and $D_p$-branes with constant nonzero Neveu–Schwarz 2-form, we define a deformed multiplication operation on a quasitriangular Hopf algebra in terms of its $R$-matrix, and comment on some of its properties. We show that the noncommutative string theory $*$-product is a particular example of this multiplication, and comment on other possible Hopf algebraic properties which may underlie the theory.

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Keywords: Noncommutative geometry; R-matrix; Hopf algebras

1. Introduction

Although the subject of gauge theories in noncommutative geometry is not a new one [1], recently it has enjoyed something of a resurgence. It reappeared in the context of matrix models compactified on tori [2], where it was shown that such models may be reformulated as super Yang–Mills (SYM) theories on noncommutative spacetimes when the Neveu–Schwarz (NS) 2-form $B_{ij}$ is constant and nonzero. This is accomplished by replacing the usual commutative multiplication of functions on the space by a noncommutative one, denoted by $*$ (which had appeared previously in the context of the Moyal bracket [3–6]). Subsequent studies dealt with various aspects of noncommutative spaces [7–17]. It has also been realised that one gets a noncommutative SYM theory for the case of a system of open strings and $D_p$-branes in a flat spacetime (with the same condition on $B_{ij}$), provided that one not only changes all ordinary products to $*$-products, but also deforms the gauge fields and their transformations [18]. This seems to suggest that noncommutative geometry may be an underlying aspect of a large class of theories.

If this is so, then a mathematically consistent way of dealing with this noncommutativity is needed, and one possibility might be to use the language of Hopf algebras (HAs). A HA extends the notion of an algebra by including information about its representations (through the coproduct), and when it is quasitriangular also gives the algebraic structure of the modules it is represented on through its associated $R$-matrix, i.e. commutation relations between module elements. Perhaps the best-known cases where nontrivial quasitriangular HAs play key roles are quantum groups [19,20], which may be thought of consisting of matrices whose entries are noncommuting.
It is the intent of this paper to show (in Section 3) that the ∗-product mentioned above is in fact a particular case of a more general multiplication which may be defined in terms of the R-matrix of a particular quasitriangular HA. This may be a clue that there is indeed a quasitriangular HA structure to these noncommutative theories, and might serve as a tentative first step toward finding that structure. This could in turn lead toward a way of formulating gauge theories on a large class of spaces, not just commutative ones. Some of the implications of such a structure are commented upon in Section 4.

Parts of this work are somewhat pedagogical, but this is because it is intended for an audience for whom the language of HAs may not be too familiar; for those with some knowledge of the subject, Section 2 can be skimmed just to determine the notation we use herein. Others who may be curious about HAs may find the short review useful.

2. Hopf algebras

This section is meant to be a review of both the formal aspects of what constitutes a Hopf algebra and the explicit example where we consider the formal aspects of what constitutes a Hopf algebra.

2.1. Formal definitions

A Hopf algebra \( HH \) is an associative algebra with unit 1 over a field \( k \) which is also equipped with a counit \( e: HH \to k \), a coproduct (or comultiplication) \( \Delta: HH \to HH \otimes HH \) and an antipode \( S: HH \to HH \); the first two of these are defined to be homomorphisms, the third an antihomomorphism, and all three satisfy the relations

\[
(\Delta \otimes \text{id}) \Delta(f) = (\text{id} \otimes \Delta) \Delta(f),
\]

\[
(\varepsilon \otimes \text{id}) \Delta(f) = (\text{id} \otimes \varepsilon) \Delta(f) = f,
\]

\[
m((S \otimes \text{id}) \Delta(f)) = m((\text{id} \otimes S) \Delta(f)) = \varepsilon(f)1,
\]

where \( f \in HH \), \( \text{id} \) is the identity map \( f \to f \) and \( m \) is the multiplication operation on \( HH \) (which will usually be suppressed). For future reference, the first of (2.1) is often called ‘coassociativity’.

For clarity, we adopt Sweedler’s notation [21], in which we write the coproduct as \( \Delta(f) = f_{(1)} \otimes f_{(2)} \), where there is an implied summation. For example, using this convention, the third of (2.1) may be written as \( S(f_{(1)})f_{(2)} = f_{(1)}S(f_{(2)}) = \varepsilon(f)1 \).

We can define the dually paired HA to \( HH \), denoted \( HH^* \), in the following way: As a vector space over \( k \), \( HH^* \) is just the dual space to \( HH \), so there is an inner product taking \( HH^* \otimes HH \) (which we assume to be nondegenerate) to \( k \), written as \( \langle x, f \rangle \) for \( x \in HH^* \) and \( f \in HH \). \( HH^* \) may be given a Hopf algebra structure by defining the multiplication, unit, coproduct, counit and antipode on \( HH^* \) via

\[
\langle xy, f \rangle = \langle x \otimes y, \Delta(f) \rangle, \quad \langle x, 1 \rangle = \varepsilon(x).
\]

It is straightforward to show that these operations satisfy all the HA conditions.

\( HH^* \) may be thought of as an algebra of operators on \( HH \) when we define the (left) action of \( x \in HH^* \) on \( f \in HH \), denoted \( x \cdot f \), as

\[
x \cdot f := f_{(1)}(x, f_{(2)}).
\]

Since it is easy to show that \( (xy) \cdot f = x \cdot (y \cdot f) \), this is indeed an action of elements of \( HH^* \) on those of \( HH \), and thus gives a representation of \( HH^* \) with \( HH \) as the module. Furthermore, there is a sort of Leibniz rule: \( x \cdot (fg) = (x_{(1)} \cdot f) (x_{(2)} \cdot g) \). To give \( HH^* \) an interpretation, notice that \( P_0(x) := x - \varepsilon(x)1 \) and \( P_1(x) := \varepsilon(x)1 \) are projections on \( HH^* \), so \( HH^* = \ker \varepsilon \oplus k1 \). Note that for \( x \in \ker \varepsilon \), \( x \cdot 1 = 0 \), so these may be thought of as derivatives on \( HH \); elements of \( k1 \) just multiply elements of \( HH \) by elements of \( k \). Finally, if we have a concept of derivatives on \( HH \), we can define an integral on \( HH \) as a linear map \( f: HH \to k \) such that \( f \cdot f = 0 \) for any \( x \in \ker \varepsilon \).

A quasitriangular Hopf algebra is an HA \( HH \) for which there is a special invertible element \( R \in HH \otimes HH \), called the R-matrix, which has the properties

\[
(\tau \otimes \Delta)(f) = R \Delta(f) R^{-1},
\]

\[
(\Delta \otimes \text{id})(R) = R_{13} R_{23},
\]

\[
(\text{id} \otimes \Delta)(R) = R_{13} R_{12},
\]
where $\tau: f \otimes g \rightarrow g \otimes f$ and the subscripts on $R$ in the latter two above tell in which pieces of $\mathcal{F} \otimes R$ lives, i.e. if $R = r_0 \otimes r^n$ (sum implied), then $R_{13} := r_0 \otimes 1 \otimes r^n$. One consequence of these properties on the $R$-matrix is the fact that it must satisfy the Yang–Baxter equation (YBE)

$$R_{12}R_{23} = R_{23}R_{12}.$$  

(2.5)

(Nota that if $\Delta = \tau \circ \Delta$, $R = 1 \otimes 1$ satisfies all the above, so all such HAs are trivially quasitriangular.)

2.2. Hopf algebra of functions and derivatives

Let $\mathcal{F}$ be the space of functions mapping $\mathbb{R}^{p+1}$ into $\mathbb{C}$. $\mathcal{F}$ is made into a commutative associative algebra in the usual way, e.g. $(fg)(x) := f(x)g(x)$ for $f, g \in \mathcal{F}$ and $x \in \mathbb{R}^{p+1}$. Furthermore, $\mathcal{F}$ can be extended to an HA if we define the following on the coordinate maps $x^i$:

$$\Delta(x^i) := x^i \otimes 1 + 1 \otimes x^i,$$

$$\epsilon(x^i) := 0, \quad S(x^i) := -x^i. \quad (2.6)$$

Once we have the above (as well as the relations $\Delta(1) = 1 \otimes 1$, $\epsilon(1) = 1$ and $S(1) = 1$, of course), we can use the fact that $\Delta$ and $\epsilon$ are homomorphisms and $S$ is an antihomomorphism to extend them to all monomials of the coordinate functions and (ignoring questions of completeness) thus to all of $\mathcal{F}$.

We define $\mathcal{F}^*$, the dually paired HA to $\mathcal{F}$, to be spanned by elements $\{\partial_i| i = 0, \ldots, p\}$, where the inner product between $\partial_i \in \mathcal{F}^*$ and a monomial in $\mathcal{F}$ is

$$\left\langle \partial_i, \prod_{p=1}^N (x^i)^{n_p} \right\rangle = \sum_{p=1}^N \delta_{ij} \delta_{n_j, 1} \prod_{q \neq p} \delta_{n_q, 0}, \quad (2.7)$$

as well as $\left\langle \partial_i, 1 \right\rangle = 0$. Once we have this, the HA structure of $\mathcal{F}^*$ is immediate: The form of the coproduct on $\mathcal{F}$ tells us that $\mathcal{F}^*$ is commutative, the commutativity of $\mathcal{F}$ gives $\Delta(\partial_i) = \partial_i \otimes 1 + 1 \otimes \partial_i$, and $\epsilon(\partial_i) = 0$ and $S(\partial_i) = -\partial_i$.

The action of $\partial_i$ on a monomial in $\mathcal{F}$ can be computed as well; it will perhaps be no surprise to the reader that the result is

$$\partial_i \cdot \prod_{p=1}^N (x^i)^{n_p} = \sum_{p=1}^N n_p (x^i)^{n_p-1} \delta_{ij} \prod_{q \neq p} (x^i)^{n_q}. \quad (2.8)$$

Furthermore, for two arbitrary functions $f(x)$ and $g(x)$,

$$\partial_i \cdot (f(x)g(x)) = (\partial_i \cdot f(x))g(x) + f(x)(\partial_i \cdot g(x)), \quad (2.9)$$

so $\mathcal{F}^*$ is indeed the space of partial derivatives on functions over $\mathbb{R}^{p+1}$. (Note: For brevity’s sake, from now on we will omit the · when speaking of the action of a partial derivative on a function of $x$.)

3. The $*$-product

Motivated by the previously-mentioned literature on noncommuting strings, we now use the formalism just presented to introduce a new $*$-product on the Hopf algebra $\mathcal{H}$. We examine its properties, and then go on to show that the noncommutative product in string theory is a particular case of this multiplication.

3.1. Formal definition of the $*$-product

Suppose we have two dually paired HAs $\mathcal{H}$ and $\mathcal{H}^*$, and further suppose that there is an $R$-matrix on $\mathcal{H}^*$. We can thus define a new operation $*: \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$ in terms of $R$ and the usual multiplication on $\mathcal{H}$ via

$$f * g := F_{(1)}g_{(1)} \left( f_{(2)} \otimes g_{(2)} \right) \left( R_{21} \begin{array}{c} f_{(2)} \\ \otimes \end{array} g_{(2)} \right). \quad (3.1)$$

(If $\mathcal{H}^*$ is trivially quasitriangular, then $f * g = fg$.) This operation is actually associative as a consequence of the YBE. To see this explicitly, we pick $f, g, h \in \mathcal{H}$, and first compute $(f * g) * h$:

$$(f * g) * h = \left( f_{(1)}g_{(1)} \left( f_{(2)} \otimes g_{(2)} \right) \right) * h \quad \left( R_{21} \begin{array}{c} f_{(2)} \\ \otimes \end{array} g_{(2)} \right). \quad (3.2)$$
In going from the first line to the second, we used the coassociativity of \( \Delta \); next, the third of (2.2), the second of (2.4) and a relabelling of the indices so as to stick everything into one inner product; and the last step used the first of (2.2). If we now calculate \( f \ast (g \ast h) \), an exactly analogous computation gives the result above with the left argument of the inner product replaced by \( R_{21}R_{13}R_{32} \), and by using the YBE (with the first and third tensor spaces swapped), we find the two quantities are exactly the same. Hence, \( (f \ast g) \ast h = f \ast (g \ast h) \) and \( \hat{\mathcal{A}} \), the algebra constructed from (the vector space) \( \mathcal{A} \) and the \( \ast \)-product is associative. Since it turns out that \( 1 \ast f = f = 1 \ast f \), \( \hat{\mathcal{A}} \) is unital as well.

This is in general a noncommutative multiplication: Even if \( \hat{\mathcal{A}} \) itself is commutative, \( \hat{\mathcal{A}} \) may not be if \( R_{21} \neq R \). To examine this a bit more, it can be shown that the defining relations for \( R \) imply that \( (\epsilon \otimes \text{id}) R = (\text{id} \otimes \epsilon) R = 1 \otimes 1 \), so that we may define a new quantity \( \Theta \in \ker \epsilon \otimes \ker \epsilon \) as \( \Theta = R - 1 \otimes 1 \). When written in terms of \( \Theta \), the \( \ast \)-product becomes \( f \ast g = fg + f(1)g(1)\Theta_{21}f(2)g(2) \). Suppose we now define \( \Theta = \theta_0 \otimes \theta_0 \) and compute \( \theta_0 \ast (f \theta_0 \ast g) \):

\[
\theta_0 \ast (f \theta_0 \ast g) = \theta_0 \ast (fg + f(1)g(1)\Theta_{21}f(2)g(2))
\]

\[
= f(1)g(1)(\Theta_{21}f(2)g(2))\theta_0, g(3)
\]

\[
= f(1)g(1)(\Delta(\theta_0) \otimes \theta_0)
\]

\[
f(2) \otimes \Delta(g(2)),
\]

(3.3)

where in the last step we have split up \( f(2)g(2) \) to get the coproduct of \( \theta_0 \). The third of (2.4) in terms of \( \Theta \) is \( (\text{id} \otimes \Delta) \Theta = \Theta_{21} + \Theta_{12} + \Theta_{13} \Theta_{12} \); if we use this, and then get rid of \( \Delta(g(2)) \) by exchanging it for a multiplication in the \( \mathcal{A} \) argument of the inner product, we find

\[
\theta_0 \ast (f \theta_0 \ast g)
\]

\[
= f(1)g(1)(\Theta_{21} + (1 \otimes \kappa) R_{21} f(2) \otimes g(2)),
\]

(3.4)

where \( \kappa = \theta_0^2 \). Let us now suppose that there is a kind of ‘tracelessness’ condition on \( \Theta \), and \( \kappa \) vanishes. We must stress that this is purely an assumption, but if it does in fact hold, then we conclude that

\[
f \ast g = fg + \theta_0 \ast (f \theta_0 \ast g).
\]

(3.5)

in other words, \( f \ast g \) and \( fg \) differ by a ‘total derivative’, since \( \theta_0^2 \in \ker \epsilon \). It follows that if there is an integral \( f \) defined over \( \mathcal{A} \), \( f \ast g = fg \).

A comment on the above: The above can be easily extended to the case where we have a \( N \times N \) matrix-valued Hopf algebra, i.e. \( \mathcal{A} \otimes \mathcal{M}_N(k) \), in which case the \( \ast \)-product includes matrix multiplication as well: \( (f \ast g)_{ij} = (fg)_{ij} \). Note that this is not the same as a quantum group, since \( \mathcal{A} \) is not being thought of as the set of functions over the group manifold. Thus, when we take coproducts et al., the ‘matrix part’ is unaffected.

Also, this \( \ast \)-product may not be unique: If we assume that \( \ast \) is defined by (3.1) without assuming that \( R \) satisfies (2.4), then the requirement that \( \ast \) is associative means \( R \) must satisfy

\[
(\Delta \otimes \text{id})(R_{21})R_{21} = (\text{id} \otimes \Delta)(R_{21})R_{32}.
\]

As shown above, \( R_{21} \) satisfies this, but so does \( R^{-1} \), and unless the HA is triangular, i.e. \( R_{21} = R^{-1} \), there may be more than one associative \( \ast \)-product.

3.2. Noncommutative String Theory

Recently [18], it has been shown that when one considers open strings and Dp-branes in a space with metric \( g_{ij} \) and constant nonvanishing Neveu–Schwarz 2-form \( B_{ij} \) \((i,j = 0, \ldots, p)\), the theory can be reformulated as a SYM theory on a space where the coordinates no longer commute, but instead satisfy the deformed relation

\[
x^i x^j - x^j x^i = i \theta^{ij},
\]

(3.7)

where

\[
\theta^{ij} := -(2 \pi \alpha')^2 \left( \frac{1}{g + 2 \pi \alpha' B} \frac{1}{g - 2 \pi \alpha' B} \right)^{ij}.
\]

(3.8)

More generally, it is shown that if \( f(x) \) and \( g(x) \) are (matrix-valued) functions, the noncommutative product between two functions \( f(x) \) and \( g(x) \) is

\[
f(x) \ast g(x)
\]

\[
:= e^{i \theta^{ij} \frac{\partial}{\partial^j} \frac{\partial}{\partial^i} f(x + \xi) g(x + \xi)} \bigg|_{\xi = 0}.
\]

(3.9)

The action then can be expressed in SYM form provided that all ordinary multiplications are re-
placed by this $\ast$-product, the ‘closed string metric’ $g_{ij}$ is replaced by the ‘open string metric’ $G_{ij} := g_{ij} - \langle (2 \pi a^\prime)^2 (B^{-1} B) \rangle$, and the gauge field $A_i$ is replaced by $A_i$, which depends on both $A_i$ (and its derivatives) and $\theta^{ij}$.

If we compare (3.9) to (3.1), it suggests that a candidate for the $R$-matrix is

$$R = e^{\sum_{k=0}^\infty \frac{(-i)^k}{2^k(k-k)!} \sum_{p \leq k} \theta^{i,p} \partial_{\sigma(p)} \otimes \theta^{i,p} \partial_{\sigma(p)}},$$

but we must make sure it satisfies all the appropriate relations. Since $\mathcal{F}$ is commutative, and $\Delta(\partial_j)$ is symmetric, it is evident that the first of (2.4) is satisfied. To check the second of (2.4), note that

$$\Delta \left( \prod_{\ell=1}^k \partial_{\ell} \right) = \sum_{\sigma} \sum_{\ell \leq k} \frac{1}{(k-\ell)!} \prod_{\ell \leq k} \partial_{\sigma(\ell)} \otimes \prod_{\ell \leq k} \partial_{\sigma(\ell)}.$$

where $\sigma$ is a permutation of $(1, \ldots, k)$. Therefore,

$$(\Delta \otimes \text{id})(\mathbb{1}) = \sum_{k=0}^\infty \frac{(-i)^k}{2^k(k-k)!} \left( \sum_{p \leq k} \theta^{i,p} \partial_{\sigma(p)} \otimes \theta^{i,p} \partial_{\sigma(p)} \right) = \sum_{k=0}^\infty \frac{(-i)^k}{2^k(k-k)!} \left( \sum_{p \leq k} \theta^{i,p} \partial_{\sigma(p)} \otimes \theta^{i,p} \partial_{\sigma(p)} \right) \otimes \left( \prod_{q \leq k} \partial_{\sigma(q)} \right),$$

where in the last step we have switched the sums over $k$ and $\ell$, and also used the commutativity of the partial derivatives in the third tensor space to get rid of $\sigma$ (picking up a $k!$ in the process). If we now split the product in the third space into two, one from $1$ to $\ell$ and the other from $\ell+1$ to $k$, and let $k-\ell=r$, then we get

$$(\Delta \otimes \text{id})(\mathbb{1}) = \sum_{\ell,r=0}^\infty \frac{(-i)^r}{2^r(r-r)!} \sum_{\ell \leq k} \theta^{i,r} \partial_{\sigma(r)} \otimes \theta^{i,r} \partial_{\sigma(r)} \otimes \left( \prod_{q \leq r} \partial_{\sigma(q)} \right).$$

A very similar calculation confirms that the last of (2.4) holds as well, so (3.10) is in fact an $R$-matrix, and $\mathcal{F}$ a quasitriangular HA. The YBE is therefore satisfied by this $R$, and thus $\ast$ is associative, as proven above. Actually, any element of $\mathcal{F} \ast \otimes \mathcal{F} \ast$ satisfying (3.6) must have the form (3.10), which means for this particular HA, the $\ast$-product is unique.

To check that this $R$-matrix gives us the correct commutation relation (3.7), we simply compute $x^i \ast x^j$:

$$x^i \ast x^j = \langle x^i \rangle_{(1)} \langle x^j \rangle_{(2)} \otimes \langle x^i \rangle_{(2)} \langle x^j \rangle_{(1)},$$

and

$$= x^i \langle R_{21} \otimes 1 \rangle + x^j \langle R_{21} \otimes x^j \rangle + \langle R_{21} \otimes x^i \rangle + \langle R_{21} \otimes x^i \otimes x^j \rangle,$$

$$= x^i + \frac{i}{2} \theta^{ij}.$$  (3.14)

Therefore, by switching $i$ and $j$ and subtracting, we recover (3.7). (3.9) holds almost by definition, since, from (3.1), $f \ast g$ is simply the product between the action of the first tensor space of $R_{21}$ on $f$ and the action of the second on $g$.

This $R$ also satisfies the ‘tracelessness’ condition: If we subtract $1 \otimes 1$ from $R_{21}$ and multiply the two tensor product spaces together, we immediately get $\kappa$, which is evidently zero due to the antisymmetry (and constancy) of $\theta^{ij}$ and commutativity of the partial derivatives, so $f \ast g$ and $fg$ differ by a total
derivative. And since the original multiplication was commutative to begin with, \( f\tau(f \ast g) = f\tau(g \ast f) \).

4. Noncommutative gauge theories

We now make some comments on noncommutative gauge theories and how they may or may not relate to HAs.

4.1. Algebraic structure

In Section 2.2, we showed that for the commutative case, there is a HA structure to both functions and derivatives on \( \mathbb{R}^{n+1} \). However, even though the noncommutative \( \mathcal{F} \) is an associative unital algebra, it is not a HA, as can be seen from the following: As we proved, the unit in \( \mathcal{F} \) is the unit in \( \mathcal{F} \), so that if \( \mathcal{F} \) is a HA and has a counit \( \varepsilon \), then the fact that this counit is a homomorphism from \( \mathcal{F} \) to \( \mathbb{C} \) means \( \varepsilon(1) = 1 \). Now, consider \( x' = x^l - x^l \times x^i \); the counit will map this to zero. But this counit is \( i\theta \), so we have a contradiction. Therefore, \( \mathcal{F} \) cannot be a HA.

This is no real surprise. In the first place, the requirements necessary for a space to be a HA are very restrictive, and in general there is no reason to expect an arbitrary algebra to also be a HA. Furthermore, although \( \mathcal{F}^* \) has a coalgebra structure (a counit and a coproduct) due to the fact that \( \mathcal{F} \) is unital and associative, there is no ‘deformed derivative’. We can certainly define an inner product between the two spaces, but due to the lack of a coproduct on \( \mathcal{F} \), there is no action of \( \mathcal{F}^* \) on it, and thus no concept of derivative. This is borne out by the fact that we must use the ordinary partial derivative to define the noncommutative SYM field strength, via \( \tilde{F}_{ij} = \partial_{[i} \tilde{A}_{j]} - iA_{[i} A_{j]} \) [18]; if a deformed derivative were available, it would be the more natural choice, but this is not the case.

However, although there is no interpretation of elements of \( \mathcal{F}^* \) as objects with a local action on \( \mathcal{F} \), it might still be possible to interpret them as nonlocal operators. As an illustration of this, consider the case of the 2-dimensional quantum hyperplane: The coordinates \( x, y \) generate an algebra (the functions on the plane) modulo the commutation relation \( xy = qyx \), and the ‘derivatives’ \( \partial_x, \partial_y \) act on a function \( f(x,y) \) (ordered so that all \( x \)s appear to the left of all \( y \)s) as

\[
\partial_x f(x,y) = \frac{f(q^{-2}x,y) - f(x,y)}{(q^{-2} - 1)x},
\]

\[
\partial_y f(x,y) = \frac{f(q^{-2}x,y^{-2}y) - f(q^{-2}x,y)}{(q^{-2} - 1)y}.
\] (4.1)

where \( q \in \mathbb{R} \) [22]. Note that as \( q \to 1 \), these become ordinary derivatives, but otherwise they are nonlocal difference operators. The string case could be similar, with \( \theta \) playing the role of \( \ln q \). This granularisation of the spacetime might explain the absence of small instanton singularities [23,18] in the noncommuting theory, by smearing out such objects over more than one point.

4.2. Gauge fields and Hopf algebras

The fact that the \( \ast \)-product can be defined in terms of an abstract HA and includes the noncommutative string theory case hints at the possibility of describing the entire theory using a quasitriangular HA, where the R-matrix depends on \( \theta \). However, this is certainly not sufficient, since we have not considered the gauge field \( A_i \). We have also not addressed the matter of the map \( A_i \mapsto \tilde{A}_i \), which allow us to cast the action in SYM form. We now address both of these issues.

\( \theta \) is inherently a HA parameter; it appears in the R-matrix and therefore describes the HA structure of \( \mathcal{F} \) and \( \mathcal{F}^* \). This can be seen either explicitly, as in the relation \( (\tau \circ \Delta)(x) = R \Delta(x) R^{-1} \) on any \( \mathcal{H}^* \), or via the commutation relations of elements of \( \mathcal{H} \), which may be expressed as

\[
gf = \left( R f_{(1)} \otimes g_{(1)} \right) f_{(2)} g_{(2)} \left( R^{-1} f_{(3)} \otimes g_{(3)} \right).
\] (4.2)

So, motivated by these facts, it therefore seems reasonable to conjecture that all \( \theta \)-dependence in the theory is in the HA structure of \( \mathcal{F} \) and \( \mathcal{F}^* \).

If this is true, then the \( \theta \)-dependence in the noncommuting theory must arise from the underlying HA describing the commutative theory. This gives us a bit of information about the gauge fields: We know that the change of variables between the gauge fields of the two theories involves \( \theta \) [18], which means some sort of HA-derived operation is involved in
going from one to the other. Since, as we proved in the previous section, the noncommutative algebra $\mathcal{F}$ is not a HA, there must be some element $W \in \mathcal{F}$ with given HA properties (coproduct, etc.) which is related to both $A$ and $A_j$. The assumption that all $\theta$-dependence is in the HA structure and not in $\mathcal{F}$ (as a vector space) itself leads to the conclusion that $W$ is independent of $\theta$, and thus must be related to $A_j$, since this is the gauge field for the $\theta^{ij} = 0$ case. If we also assume that the only dependence on the open string metric $G_{ij}$ is from constructing Lorentz-invariant quantities in the integral, e.g. $\int d^{p+1}x \sqrt{G} G^{ij} \alpha_i \beta_j$, then $W$ should also be $G$-independent.

At this writing, we do not know precisely what this element might be, but a natural candidate would be the Wilson line (which explains why we call it $W$): Recall that the Wilson line $W(x)$ is independent of $\theta$, since this is the gauge field for the $\theta^{ij} = 0$ case. If we also assume that the only dependence on the gauge transformation, the same idea applies: A finite transformation on $W_C$ given by a unitary matrix $U(x) = e^{iA(x)}$ gives $U(x) W_C(x_0) U^{-1}(x_0)$. This should be the same as if we started with $W_C$ in terms of $*$ and $A_j$, and then transformed by $U$ expressed in terms of $*$ and a new matrix $\lambda$ via $U = e^{i\lambda}$, then by solving

$$\frac{\partial}{\partial \theta^{ij}} \left[ \hat{\theta}^{ij}(x) + \hat{\theta}^{ij}(C(x_0, x)) \right] = 0,$$

we would obtain the second of (3.5) in Ref. [18].

So the sought-after $W \in \mathcal{F}$ could be related to the Wilson loop $W_C$, even though we do not claim to have offered anything more than a vague justification for this feeling. We have not said anything about the HA properties of $W$ (although the HA enters in going from $e$ to $\hat{e}$, via $*$), nor have we said how the curve $C(x_0, x)$ would be chosen, since we have taken it to be completely arbitrary. And perhaps most importantly, we have not considered what the action might be as a function of $W$ (and $R$) such that we ultimately end up with a SYM form when we go to the noncommuting spacetime. Regardless, there is enough evidence to consider our Wilson line guess as reasonable.

Everything we have done above has been for the specific case of a noncommutative string, where we started with a commutative space ($\mathbb{R}^{p+1}$); formulating it in the HA language as we propose would also be a way of coming up with gauge theories where the original space might be noncommutative. Steps in this direction have been made when the gauge group (and possibly also the spacetime) is deformed [24], and this may mean there is some hope of success for the present problem.

5. Conclusions

We have shown that the product appearing in noncommutative string theory is simply a specific case of one which may be defined in terms of a quasitriangular HA. This fact has lead us to speculate that it may be possible to relate an arbitrary noncom-
mutative gauge theory to a quasitriangular HA in this way, and we have commented on some ways in which this may be done. We hope to address this possibility in future work.

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References

Aspects of duality and confining strings

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Abstract

We inspect the energy spectrum of a confining string, at the classical level in an effective field theory model. The spectrum can be characterized by a spectral function, and twisting and bending of the string is manifested by the invariance of this function under a duality transformation. Both general considerations and numerical simulations reveal that the spectral function can be approximated by a simple rational form.

The ground state energy of a long confining string is proportional to its length $L$. When $L$ decreases there will be corrections, and in QCD we expect for the ground state energy of the confining flux with fixed ends at distance $L$ [1,2]

$$E_0(L) = \sigma L + \varepsilon + \frac{c}{L} + \Theta \left( \frac{1}{L^2} \right)$$

(1)

Here $\sigma$ is the string tension, $\varepsilon$ the intercept, and $c = -\frac{\pi^2}{\varepsilon}$ is a universal constant, the Casimir energy of zero-point fluctuations that can be computed from the Nambu–Goto action in the Gaussian approximation [1,2].

Besides the contributions that can be described by the Nambu–Goto action, there are also additional contributions that can lead to large deformations even when the underlying displacements remain tiny [2]. For example, a small torsional rotation around the axis of a long string can cause distant cross-sections to rotate through large angles. Similarly, if a long string is slightly bent at some point, remote parts of the string can move over large distances. In the present Letter we inspect an effective field theory approach to describe such contributions to the energy spectrum of a confining string. The string appears as a classical soliton in an effective field theory, which provides an appropriate background to eventual systematic investigations of quantum corrections. Quantitatively, our arguments are based on the classical action [3]

$$\int d^4x \left[ (\partial_\mu n)^2 + \frac{1}{4\epsilon^2} (n \partial_\mu n \times \partial_\nu n)^2 \right]$$

(2)

The order parameter $n$ is a three-component unit vector $nn = 1$ and $\epsilon^2$ is a coupling constant that

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determines the length scale. This action is known to support closed knotted strings as classical solitonic solutions to its equations of motion [4,5]. In [6] it has been argued, that these solitons represent aspects of the SU(2) Yang–Mills theory in the long wavelength limit. Indeed, (2) does emerge from the Yang–Mills theory in a derivative expansion [7]. Thus it becomes natural to relate its string solutions to properties of the confining flux in the Yang–Mills theory. However, ultimately we wish to employ universality to relate its string solutions to properties of the SU(2) Yang–Mills theory in the long wavelength limit. For this we combine duality arguments with a familiar problem in the classical theory of elasticity, we expect their validity to beyond the specific details of 2.

Classically there are exactly two intrinsically small deformations of an elastic rod that can be accompanied by large global displacements, twisting about its axis and bending [8]. When a straight elastic rod is twisted around its axis, each transverse section becomes rotated by some relative angle. We are interested in the ensuing bulk contribution to the free energy, over a length \( L \). For this we align the coordinates so that the axis of the rod coincides with the \( z \)-axis and denote by \( \tau \) the (local) twist angle, defined as the angle of rotation per unit length of the rod around its axis. For a constant \( \tau \) a rod acquires a total twist \( \psi = \tau L \) over a length \( L \). From the results in [8] we then expect that asymptotically the ensuing bulk contribution to the classical free energy of a straight, twisted confining string has the form

\[
E_0 + E_{\text{twist}} = \sigma L + \varepsilon + 2 \pi^2 \varrho \frac{\mathcal{F}^2}{L} + \ldots
\]

\[
= \sigma \mathcal{F} \left( \lambda + \frac{a}{\lambda} \right) + \ldots
\]

(3) exhibits a \( \lambda \rightarrow a/\lambda \) duality, reminiscent of the T-duality in (super)string theories.) When the number of twists \( \mathcal{F} \) remains fixed the self-dual point \( \lambda = \sqrt{a} \) minimizes the ensuing contribution which yields

\[
E_{\text{min}}(Q) = 2 \sqrt{a} \sigma |\mathcal{F}| + \varepsilon + \ldots
\]

In the field theory model (2) a straight twisted string (away from the end points) can be described by an axially symmetric configuration, in cylindrical coordinates

\[
n(r, \varphi, z) = \begin{pmatrix} \sin(k \varphi + \tau z) \sin \theta(r) \\ \cos(k \varphi + \tau z) \sin \theta(r) \\ \cos \theta(r) \end{pmatrix}
\]

(5) where \( k \) is an integer that counts the degeneracy. We substitute (5) into (2), and by defining \( \rho = er \) a dimensionless variable, we find for the bulk contribution to the free energy of a string over a length \( L \)

\[
\frac{E}{2\pi} = \left( \int_0^\infty \rho d\rho \left( \theta^2 + k^2 \left[ 1 + \frac{\theta^2}{\rho^2} \sin^2 \theta \right] \right) \right) L 
\]

\[
+ \frac{4\pi^2 \mathcal{H}^2}{k} \left( \int_0^\infty \rho d\rho \left( 1 + \frac{\theta^2}{\rho^2} \sin^2 \theta \right) \right) \frac{e^2}{L}
\]

(6)

We draw attention to the similarity in the functional forms of (3) and (6) which can be used to relate the parameters \( \sigma \) and \( \varrho \) to integrals over \( \theta(\rho) \). Here \( \mathcal{H} \) is the Hopf invariant over the length \( L \), explicitly with \( F = (n, dn \wedge dn) = dA \)

\[
\mathcal{H} = \frac{1}{2\pi} \int_L F \wedge A = \frac{1}{2\pi} k \tau L = \frac{k}{2\pi} \psi
\]

so that for a straight string the Hopf invariant over the length \( L \) reduces to the corresponding number of twists \( \mathcal{H} = \mathcal{F} \), that a configuration over a length \( L \) makes around the \( z \)-axis (including the multiplicity \( k \)).

The profile \( \theta(\rho) \) that minimizes the energy (6) solves the pertinent Euler–Lagrange equation. Since this equation depends nontrivially on the various parameters in (6), we expect a priori that the solution also attains a nontrivial, nonlinear dependence on the parameters. In particular we expect that for an actual
minimum energy configuration the functional form of (6) in $L$ does not stand.

We have performed an exhaustive numerical investigation of the parameter dependence for the configuration $\theta(\rho)$ that minimizes the free energy contribution (6). The result is described by a spectral function $f(k^2, \lambda)$. It depends on the integer $k^2$ and the dimensionless combination $\lambda = (Le)/(2\pi|\mathcal{H}|)$ which is a measure of length per Hopf invariant. Quite surprisingly, we find that the functional form (3), (6) persists: We find that to a very high degree of accuracy the bulk free energy of an actual straight string solution is interpolated by a simple rational spectral function

$$E_{\text{min}}(k, \lambda) = \left| \mathcal{H} \right| \left( a\lambda + b + \frac{c}{\lambda} \right)$$

(7)

where $a, b, c$ are $k^2$ dependent numerical coefficients. In Fig. 1 we plot (7) for $k = 1$, together with several numerically computed values of the energy. The high degree of accuracy of the rational spectral function (7) over a wide range of values in $\lambda$ and $k^2$ leads us to propose that this simple rational form might actually become exact for the straight string solutions of (2), or at least to a slight perturbation of (2) within its universality class.

We note that (7) has a definite, manifest modular structure as it should. Namely, suppose we split a straight string into two so that the total energy remains intact. Obviously, the spectral form of the energy for both of the resulting strings should also coincide with (7). Since the Hopf invariant $|\mathcal{H}|$ is additive and $\lambda$ denotes length per Hopf invariant, we conclude that (7) indeed does have the requisite property under splitting and joining. The definite rational form of (7) then leads to additional observations. For this we first note that if $\lambda = \lambda_c$ minimizes (7), in parallel with (3) the rational function in (7) exhibits a $\lambda \to \lambda_c/\lambda$ duality. In order to interpret this duality we consider a long straight string with boundary conditions that clamp the ends, to keep the total Hopf invariant $|\mathcal{H}|$ intact. According to (7) the string has a tendency to adjust its effective length $\lambda$ towards a configuration with $\lambda = \lambda_c$, i.e. towards the self-dual point of the spectral function that minimizes the free energy in the bulk.

When $\lambda < \lambda_c$, the value of $\lambda$ has a tendency to increase. But when $\lambda > \lambda_c$ there is a tendency for $\lambda$ to decrease. To interpret this, we recall the relation between Hopf invariant $\mathcal{H}$, twist $\mathcal{F}$ and writhe $\mathcal{W}$ [9]

$$\mathcal{H} = \mathcal{F} + \mathcal{W}$$

(8)

We conclude that with our boundary conditions for $\lambda \neq \lambda_c$ the string has a tendency to supercoil: When $\lambda < \lambda_c$ the string is excessively twisted, and it becomes energetically favourable to decrease the twist $\mathcal{F}$. But when $\mathcal{H}$ is fixed, twist can decrease only when the writhe $\mathcal{W}$ increases. On the other hand, when $\lambda > \lambda_c$ it becomes energetically favourable to increase twist. The $\lambda \to \lambda_c/\lambda$ duality of the rational spectral function then suggest that we have a relation between two different supercoiled configuration, one with an excess amount of twist and the other with an excess amount of writhe but with an equal energy.

We observe from (8) that the Hopf invariant $\mathcal{H}$ does not change when we exchange $\mathcal{F} \leftrightarrow \mathcal{W}$. Furthermore, we have concluded that the $\lambda \to \lambda_c^2/\lambda$ duality appears to relate two configurations with different $\mathcal{F}$ and $\mathcal{W}$, in such a manner that the Hopf invariant $\mathcal{H}$ and the rational spectral function both remain intact. The previous considerations then suggests us to define a twist-writhe duality, that ex-

![Fig. 1. A comparison of the rational function (7) with the energy values of the line solitons in (2), (6). The interpolation is in the sense of least square, which yields $a \approx 4.15 \ldots$, $b \approx 5.26 \ldots$ and $c \approx 1.42 \ldots$. The agreement is consistent with finite lattice size errors.](image)
changes $\mathcal{F} \leftrightarrow \mathcal{W}$ and sends $\lambda \to \lambda^2 \lambda$ in the spectral function. The energy (7) is invariant under this twist-writhe duality transformation, it determines a symmetry in our effective field theory.

The previous arguments are based on the inspection of a straight string. But a configuration with a nontrivial writhe can not be straight, it must be bent, and in order to proceed it becomes imperative to inspect the effects of bending. For this we recall from classical theory of elasticity that the bulk free energy for bending of a thin elastic rod, when uniformly bent over a length $L$, has the form $E = \kappa L / R^2$ with $R$ the (mean) radius of curvature, and $\kappa$ a form factor that characterizes the rod [8].

We consider a confining string which has been bent uniformly, with a constant radius of curvature $R$. We assume that there is a constant rate of twist along the string. The writhe is similarly supposed to be uniformly distributed along the string, with a constant rate of increment $\omega$. The length $L(\omega)$ of the string then scales in proportion to $R$ according to $L(\omega) = L_0 + c \omega R$. Here $L_0$ is the (planar) component of the length that does not contribute to the writhe, and $c$ is a numerical constant; for simplicity we set $c = 1$. When we combine the free energy for bending with the ensuing free energy (3) for stretching and twisting, we conclude that the bulk free energy of our string has the following $\omega$ dependence,

$$E(\omega) = \sigma L(\omega) + \alpha^2 \left( \mathcal{W} - \omega \right)^2 \frac{L(\omega)}{L_0} + \beta^2 \frac{L(\omega)}{R^2}$$

with $\alpha$ and $\beta$ some parameters.

We first consider the minimum value of the free energy (9) when we minimize it w.r.t. the rate of increment $\omega$, by keeping the Hopf invariant $\mathcal{W}$ fixed. We get

$$E_{\text{min}} = 2(\rho - \alpha) \frac{\alpha}{R} \left( |\mathcal{W}| + \frac{L_0}{R} \right)$$

where we have defined $\rho^2 = \alpha^2 + \beta^2 + \sigma R^2$. We observe that (10) has the same functional form as (4). This suggests in particular, that the underlying dual structure persists. For this we subject the free energy to a minimization of $\omega$, with the condition that both the Hopf invariant $\mathcal{W}$ and the length $L = L_0 + \omega R$ remain fixed. We find

$$E_{\text{min}} = \frac{L}{S} \left( \frac{S}{|\mathcal{W}|} + (\alpha \beta)^2 \frac{|\mathcal{W}|}{S} \right)$$

where $S^2 = \alpha^2 (L - L_0)^2 + \beta^2 L^2$. The dual structure is now manifest. We also note that both (10) and (11) are consistent with the expected modular property of a uniformly writhed and twisted string under splitting and joining.

In the field theory model (2) the bulk properties of a stable, bent and twisted confining string can be (locally) described by a closed soliton in the form of a torus knot [4,5]. The radius of curvature of an actual soliton in general fails to be constant, as there are deviations due to interactions between the strands. But at least for small values of $\mathcal{W}$ the deviations from an average $R$ appear to remain small [5]. Consequently the main features of (9) should persist. Since an arbitrary space curve is described at each point by its torsion and radius of curvature, this suggests that the energy of a closed knotted soliton should be qualitatively described in terms of modular invariant quantities only. This leads us to expect that the manifestly twist-writhe dual functional form of the spectral representation (7), (1) continues to hold, with the various parameters now integrals of $n$ evaluated for the soliton. A priori such parameters depend on the details of the soliton profile, but according to (7) this dependence appears to be quite weak. Since the soliton minimizes the energy it corresponds to the self-dual minimum energy point of the relevant spectral function, and according to (7), (10) this means that the energy spectrum of a closed knotted soliton admits the now-familiar functional form

$$E_{\text{soliton}} = \gamma (|\mathcal{W}| + \sigma)$$

with some parameter $\gamma$ [10]. In particular, since $|\mathcal{W}| = 1,2,3,\ldots$ is an integer the energy spectrum will be equipartitioned, $E_n = \gamma(n + \sigma)$.

We note that (12) is consistent with the lower bound estimate in [11], with a parameter $\gamma$ that scales in the volume of the knot. This lower bound estimate is established for an energy functional that generalizes the second term in (2). But a finite energy critical point of (2) obeys a virial theorem
Fig. 2. Comparison of the linear trajectory (12) to numerics in [5]. Line/data with + denotes the toroidal $\mathcal{R} = 1, \ldots, 8$ solutions with slope/intercept 364/116 and line/data with $\circ$ the minimal energy $\mathcal{R} = 1, \ldots, 8$ solutions with slope/intercept 277/306. For the toroidal solutions a linear fit is consistent with our general arguments. But for the minimal energy solutions this is somewhat unexpected, due to variations in individual geometries. (In [5] only interpolation with a vanishing intercept was considered. See also [10].)

which states that the integrals for the two terms in (2) coincide [4]. Consequently the lower bound estimate in [11] should also extend to the present case.

Due to the high complexity of the Euler–Lagrange equations of (2) it becomes difficult to actually check the accuracy of (12) with a presumably volume dependent $\gamma$, or to inspect the underlying dual structure. Numerical simulations are also very demanding [5], and at the moment we lack computational resources to perform exhaustive simulations. Thus we rely on the numerical results that have been recently published in [5]. There, the energy of various knotted solitons is evaluated for $Q_\mu = 1, \ldots, 8$. In Fig. 2 we compare the data in [5] to the predictions of (12). When we account for the numerical uncertainties in the data as described in [5], we conclude that the agreement appears quite satisfactory. In particular, this suggest that the dual twist-writhe structure of (2) is indeed inherent.

In conclusion, we have inspected properties of a confining string by describing it as a soliton in an effective field theory. In particular, we have numerically studied the excitation energy spectrum of a straight string. We find that it can be described by a spectral function that admits a simple rational form, manifestly invariant under a duality transformation that relates twist and writhe. We have verified that the presently available 3D data [5] is also consistent with the simple rational form of the spectral function. This suggests that (2) is at least in the same universality class with a Lagrangian for which a manifestly dual, simple rational form of the spectral function is exact. It becomes natural to expect that our rational realization of the twist-writhe duality is distinctive to an effective Lagrangian that accurately describes the properties of the confining flux in the Yang–Mills theory.

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[10] According to (11) $\gamma$ and $\epsilon$ can depend on $R$ etc. properties of the soliton. This may introduce implicit $N$ dependence, restricted by the lower bound in A.F. Vakulenko, L.V. Kapitanski, Dokl. Akad. Nauk USSR 248 (1979) 810.
Abstract

Duality symmetric electromagnetic action à la Schwarz–Sen is shown to appear naturally in a chain of equivalent actions which interchange equations of motion with Bianchi identities. Full symmetry of the electromagnetic stress tensor is exploited by generalizing this duality symmetric action to allow for a space-time dependent mixing angle between electric and magnetic fields. The rotated fields are shown to satisfy Maxwell-like equations which involve the mixing angle as a parameter, and a generalized gauge invariance of the new action is established.

1. Introduction

Current interest in duality in string theory has brought about a wealth of studies on similar symmetries present in other contexts, such as abelian p-forms theories. The simplest among the latter, namely free electromagnetism, has long been known to remain invariant under the interchange of equations of motion with Bianchi identities. The first attempt to implement this symmetry at the level of the basic fields of the action [1] involved non local transformation among the $A$ and $E$ fields, in the Hamiltonian (first order) version of the electromagnetic action. Later on, a non covariant action was proposed by Schwarz and Sen [2] where the transformation is made local at the expense of doubling the number of gauge fields. When the equations of motion for some of the fields are used, the usual Maxwell action is recovered.

As it is well known, duality symmetry is actually more general than the discrete interchange of electric with magnetic fields. It is a continuous symmetry, which gets reduced to a $U(1)$ group when invariance of the symmetric stress tensor $^2$ is also imposed [3]. The conserved momentum associated with this continuous symmetry is found to be the integral of Chern–Simons terms, and full equivalence between Maxwell and Schwarz–Sen actions has been shown to remain valid at the quantum level [4]. Moreover, covariant generalizations of the Schwarz–Sen action have been found by introducing either an infinite number of auxiliary gauge fields [5] or finite additional fields in a non polynomial way [6].

$^1$ Becario CONICET.

$^2$ All future references to the stress tensor will be to the symmetric gauge-invariant one.
On the other hand, when considered as a symmetry of the stress tensor, this duality symmetry is not the most general invariance, because the stress tensor is also invariant under rotations with a different angle at each spacetime point. We call this symmetry of the stress tensor fully-local duality, to distinguish it from that of the Schwarz–Sen action, which is usually called local duality in opposition to the non-local transformations in [1]. In fact, from our perspective, the duality symmetry of the Schwarz–Sen action could be called global. Our work is reminiscent of the passage from global to local gauge invariance of complex matter fields, wherein gauge fields are introduced in the covariant derivative.

The purpose of this work is twofold. We first show how the Schwarz–Sen action emerges from a chain of equivalent actions which interchange equations of motion with Bianchi identities. Then we rewrite the Schwarz–Sen action in terms of a complex gauge field. We proceed afterwards to probe the effects of such space-time dependent rotations on Maxwell equations and their implementation in a generalized version of the Schwarz–Sen action. We find that gauge invariance remains valid when suitable generalized. At the end we present the conclusions.

2. The Schwarz–Sen action

Free electromagnetism equations of motion 3,
\[ \partial_\mu F^{\mu\nu} [A_\mu] = 0, \]  
(1)
with
\[ F_{\mu\nu}[A_\mu] = \partial_\mu A_\nu - \partial_\nu A_\mu, \]  
(2)
can be obtained from the action
\[ S[A_\mu] = -\frac{1}{4} \int d^4 x F_{\mu\nu} F^{\mu\nu}, \]  
(3)
and Bianchi identities
\[ \partial_\mu F^{\mu\nu} [A_\mu] = 0, \]  
(4)
hold automatically, with
\[ \tilde{F}^{\mu\nu} [A_\mu] = \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}[A_\mu]. \]  
(5)

Alternatively, we can vary
\[ S[Z_\mu] = -\frac{1}{4} \int d^4 x \tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu}, \]  
(6)
with
\[ \tilde{F}_{\mu\nu}[Z_\mu] = \partial_\mu Z_\nu - \partial_\nu Z_\mu \]  
(7)
obtaining
\[ \partial_\mu \tilde{F}^{\mu\nu} [Z_\mu] = 0, \]  
(8)
as equations of motion and
\[ \partial_\mu F^{\mu\nu} [Z_\mu] = 0 \]  
(9)
as Bianchi identities, with \( F^{\mu\nu}[Z_\mu] \) defined by a relation analogous to Eq. (5).

If we define \( E^i = F^{0i} \) and \( B^{ij} = -\frac{1}{2} \epsilon^{ijkl} F_{kl} \), regardless of whether \( F_{\mu\nu} \) depends on \( A_\mu \) or \( Z_\mu \), we see that the effect of passing from \( S[A_\mu] \) to \( S[Z_\mu] \) is to interchange equations of motion with Bianchi identities.

Now, \( S[A_\mu] \) is equivalent to the first order action
\[ S[A_\mu, F_{\mu\nu}] \]
\[ = \int d^4 x \left[ \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} F_{\mu\nu} \left( \partial^\nu A^\mu - \partial^\mu A^\nu \right) \right] \]
\[ = \int d^4 x \left[ \frac{1}{2} (B^2 - E^2) - E \cdot \dot{A} - E \cdot \nabla A_0 \right. \]
\[ - B \cdot \nabla \times A \right]. \]  
(10)
where \( A^\mu = (A^0, A) \). As it is well known, \( A_\mu \) and \( F_{\mu\nu} \) are independent fields in this approach, but varying with respect to \( F_{\mu\nu} \), definition (2) is recovered, and replacing it in \( S[A_\mu, F_{\mu\nu}] \), we get back \( S[A_\mu] \). Now, varying \( S[A_\mu, F_{\mu\nu}] \) with respect to \( B^i \) and \( A_0 \), yields \( B = \nabla \times A \) and \( \nabla \cdot E = 0 \), respectively. Replacing these last equations in (10) we get
\[ S[A_\mu, E^\mu_0] = \int d^4 x \left( E^\mu_0 A^\mu - \frac{1}{4} [E^\mu_0]^2 + \left( \nabla \times A \right)^2 \right) \]  
(11)

\[ ^{3} \text{We use the metric } g_{\mu\nu} = \text{diag}(+1,-1,-1,-1). \]
where $E_T$ indicates that only the transversal part of $E$ survives. Hence, another vector potential can be introduced through $E_T = \nabla \times Z$. As we shall see, $Z$ will be later identified with the spatial components of the tetravector $Z_{\mu}$, already introduced. Then $S[A_{\mu}, E_T]$ can be written as

$$S[A_{\mu}, Z_{\mu}] = -\frac{1}{2} \int d^4x \left[ \nabla \times Z \cdot \dot{A} - \nabla \cdot A \cdot \dot{Z} + (\nabla \times A)^2 + (\nabla \times Z)^2 \right], \quad (12)$$

where an integration by parts has been performed, thus exhibiting the symmetry between $Z$ and $A$. The equations of motion are now

$$\dot{E} = \nabla \times \dot{Z} = \nabla \times \nabla \times A = \nabla \times B, \quad B = \nabla \times A = -\nabla \times \nabla \times Z = -\nabla \times E \quad (13)$$

Some comments are in order. First note that the assignment in $S[A_{\mu}, Z_{\mu}]$ of (13) as equations of motion and $\nabla \cdot E = \nabla \cdot B = 0$ as “Bianchi identities”, corresponds neither to $S[A_{\mu}]$ nor to $S[Z_{\mu}]$. It is a mixture between the assignments in both actions. Secondly, we could go through steps similar to those that led us from $S[A_{\mu}]$ to $S[A_{\mu}, Z_{\mu}]$, but now in reverse order and with $Z_{\mu}$ taking the place of $A_{\mu}$. The antisymmetric disposition of $Z$ and $A$ in the first terms of $S[A_{\mu}, Z_{\mu}]$ necessitates the definition $Z_{\mu} = (A_{\nu} Z_{\nu})$, which should be compared to $A_{\mu} = (A^0_{\nu} A_{\nu})$. We would end up with an action which can be identified with $S[Z_{\mu}]$ as defined in (6).

At this point, Schwarz–Sen action is obtained from $S[A_{\mu}, Z_{\mu}]$ noting that for any functions $A_0$ and $Z_0$,

$$\int d^4x A_0 \cdot \nabla \times Z = \int d^4x Z_0 \cdot \nabla \times A = 0,$$

so that

$$S[A_{\mu}, Z_{\mu}] = S[A_{\mu}, Z_{\mu}] = -\frac{1}{2} \int d^4x \left[ \nabla \times Z \cdot (A + \nabla A_0) - \nabla \cdot (Z + \nabla Z_0) + (\nabla \times A)^2 + (\nabla \times Z)^2 \right], \quad (14)$$

which is easily recognized as the Schwarz–Sen action. Eq. (14) summarizes one of the main results of this work: we see that $S[A_{\mu}, Z_{\mu}]$ emerges in the middle point of a chain of equivalent actions that lead from $S[A_{\mu}]$ to $S[Z_{\mu}]$ and backwards, interchanging equations of motion with Bianchi identities.

It should be stressed that the $A_0$ and $Z_0$ fields of the Schwarz–Sen action bear no relation to those which appear in $S[A_{\mu}]$ and $S[Z_{\mu}]$. In the former case, advantage is taken of relation (14) to make the integral nicely dependent on two four-potentials, while in the latter case they are used to impose the constraints $\nabla \cdot E = 0$ and $\nabla \cdot B = 0$.

3. Complex field formulation

If we define $\Phi = A + iZ$, $\Phi_0 = A^0 + iZ_0$, $\Phi_{\mu} = (\Phi_0, \Phi)$, Schwarz–Sen action can be written as

$$S[\Phi_{\mu}, \Phi_{\mu}^*] = -\frac{i}{2} \int d^4x \left[ \left( \Phi + \nabla \Phi_0 \right) \cdot \nabla \times \Phi^* - \left( \Phi^* + \nabla \Phi_0^* \right) \cdot \nabla \times \Phi + \nabla \times \Phi \cdot \nabla \times \Phi^* \right], \quad (15)$$

Varying with respect to $\Phi_{\mu}^*$ yields

$$\nabla \times \Phi = i\nabla \times \nabla \times \Phi \quad (16)$$

which is the same as Eq. (13), while variation with respect to $\Phi_{\mu}$ yields the complex conjugate equation.

$S[\Phi_{\mu}, \Phi_{\mu}^*]$ is separately invariant under the local gauge transformations

$$\Phi \rightarrow \Phi + \nabla \Psi_1, \quad (17)$$

$$\Phi^* \rightarrow \Phi^* + \nabla \Psi_2, \quad (18)$$

$$\Phi_0 \rightarrow \Phi_0 + \Xi_1, \quad (19)$$

$$\Phi_{\mu} \rightarrow \Phi_{\mu} + \Xi_2, \quad (20)$$

where $\Psi_1$, $\Psi_2$, $\Xi_1$ and $\Xi_2$ are arbitrary gauge functions satisfying appropriate boundary conditions. In case we want the surface term picked by the Lagrangian to be real, conditions $\Psi_1 = \Psi_2^*$ and $\Xi_1 = \Xi_2^*$ should be further imposed. The La-

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4 With our choice $\Phi_{\mu}$ is not a tetravector since in its definition covariant and contravariant components are summed. Other choices would render it a tetravector.
transformation for $\alpha$ is not a scalar.

If we rotate them through

$$F = i D \times F,$$

which satisfies Maxwell equations

$$\nabla \cdot F = 0, \quad F' = i \nabla \times F',$$

if we rotate them through

$$F = e^{i\alpha(x)} F',$$

we have a total of eight equations. Were one to solve the four compatibility conditions turn out to be

$$\frac{i}{2} \partial_\mu \partial_\nu \alpha = \partial_\mu \partial_\nu \alpha.$$

which are just the conservation equations of the stress tensor, but now for the rotated fields. This is actually not a piece of news, since the stress tensor is invariant under (29). Of course, not every configuration satisfying the four Eq. (33) comes from a U(1)-local rotation of another configuration which satisfies Maxwell equations, since the $\partial_\mu \alpha$ field solved from Eqs. (30) and (31) must satisfy the integrability conditions $\nabla \times \Phi = 0$, where

$$\Phi = e^{i\alpha(x)} \Phi'. $$

the rotated fields can be derived from a rotated potential using a covariant rotor

$$F = e^{i\alpha(x)} F' = D \times \Phi, \quad \Phi = e^{i\alpha(x)} \Phi'$$

this equation can be obtained from the following generalization of the Schwarz–Sen action,

$$S[\Phi, \Phi', \alpha] = - \frac{i}{2} \int d^4 x \left[ \delta_\mu \Phi + D \Phi_0 \right] \cdot D \times \Phi^* + \nabla \times \left[ \Phi \cdot D \times \Phi^* \right] + \nabla \times \left[ \Phi \cdot D \times \Phi^* \right].$$

Variation with respect to $\Phi_0$ yields Eq. (36), and with respect to $\Phi^*$ yields its complex conjugate. Varying with respect to $\alpha$ yields no new equations.
$S[\Phi^\mu, \Phi^{\mu*}, \alpha]$ is now separately invariant under the local transformations

\[ \Phi \to \Phi + D\Psi_1 \]
\[ \Phi^* \to \Phi^* + D^*\Psi_2 \]
\[ \Phi_0 \to \Phi_0 + \Xi_1 \]
\[ \Phi_0^* \to \Phi_0^* + \Xi_2 \]

where the conditions $\Psi_1 = \Psi_2^*$ and $\Xi_1 = \Xi_2^*$ should again be imposed in case we want the surface term picked by the Lagrangian to be real. The Lagrangian is also invariant under the simultaneous local $U(1)$ rotations

\[ \Phi_\mu \to e^{i\beta(x)}\Phi_\mu , \]
\[ \Phi_\mu^* \to e^{-i\beta(x)}\Phi_\mu^* , \]

together with

\[ \alpha \to \alpha + \beta . \]

Under the transformations (42)-(44), the fields

\[ F = D \times \Phi = B + iE \]

are rotated into

\[ e^{i\beta(x)}F = (\cos \beta(x)B - \sin \beta(x)E) \]
\[ + i(\cos \beta(x)E + \sin \beta(x)B) . \]

We see that the action $S[\Phi_\mu, \Phi_\mu^*, \alpha]$ is a particular case of $S[\Phi_\mu, \Phi_\mu^*, \alpha]$, which is obtained from the latter through the transformations (42-44) with $\beta = -\alpha$.

The conserved current associated to the $U(1)$ symmetry in the generalized action is now

\[ j^0 = \frac{i}{4} (D \times \Phi \times \Phi - D \times \Phi \times \Phi^* ) , \]
\[ j = \frac{i}{2} \nabla \times ( \Phi \times \Phi^* + \Phi_0 \times \Phi ) \]
\[ + \frac{1}{4} (D \times \Phi \times \Phi - D \times \Phi \times \Phi^* ) \]
\[ + \frac{i}{2} \left( (D \times \Phi^*) \times \Phi - (D \times \Phi) \times \Phi^* \right) . \]

5. Conclusions

We have shown that the duality symmetric Schwarz–Sen action is the middle point of a chain of equivalent actions, which interchange equations of motion with Bianchi identities. Space-time dependent duality rotations were studied and the equations obeyed by the rotated fields were obtained, along with a generalized action from which these equations are derived. Remarkably, the gauge symmetries of this action are a natural extension of those of the Schwarz–Sen action.

An important property of the new Eqs. (30) and (31) is that for any $\alpha(x)$, every solution of them is mapped one-to-one to a solution of Maxwell equations. Among further developments of this work would be to show how this equivalence holds at the quantum level. This involves an analysis of the constrain structure of $S[\Phi_\mu, \Phi_\mu^*, \alpha]$. Moreover, coupling to external currents in a fully-local-duality-preserving way is also worth studying. We hope to deal with these topics in a next work.

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References

Supersymmetric dark matter and the energy of a linear electron-positron collider

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Abstract

We suggest that supersymmetric dark matter be used to set the energy scale of a linear $e^+e^-$ collider. Assuming that the lightest supersymmetric particle (LSP) is a stable neutralino $\chi$, as in many incarnations of the MSSM with conserved $R$ parity, previous calculations that include coannihilation effects have delineated the region of the $(m_{1/2}, m_{\chi})$ plane where the LSP cosmological relic density lies in the preferred range $0.1 < \Omega h^2 < 0.3$. We evaluate here the total cross section for $e^+e^- \rightarrow \chi \chi$ visible pairs of supersymmetric particles, for different values of $m_{1/2}$ and $m_\chi$, and investigate how much of the dark matter region can be explored by $e^+e^-$ colliders with different centre-of-mass energies $E_{CM}$. We find that a collider with $E_{CM} = 500$ GeV or 1 TeV can only explore part of the cosmological region, and that a collider with $E_{CM} = 1.5$ TeV with sufficient luminosity can explore all of the supersymmetric dark matter region. © 2000 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

One of the most promising ideas for a high-energy accelerator to complement the LHC is a linear $e^+e^-$ collider (LC) with a centre-of-mass energy $E_{CM}$ in the TeV range [1]. The crucial parameters of such a LC are $E_{CM}$ and the luminosity. The optimal choice of $E_{CM}$ is constrained by technology and cost, but should be driven by physics arguments based on the accessibility of physics thresholds. One established threshold in the energy range of interest is that for $e^+e^- \rightarrow \bar{t}t$ at about 350 GeV [2]. A second threshold likely to be in this energy range is that for Higgs boson $(H)$ production via the reaction $e^+e^- \rightarrow H + Z$ [3]. For some years [4], the precision electroweak data have favoured a relatively light Higgs boson, as suggested independently by supersymmetry. The most recent indication is that $M_H \lesssim 200$ GeV at the 95% confidence level [5], corresponding to a $H + Z$ threshold below about 300 GeV.

Since supersymmetry is widely considered to be one of the most promising possible low-energy extensions of the Standard Model, it is desirable that any new collider offer good prospects of detecting at least some supersymmetric particles, as is the case of the LHC [6]. The physics argument that has usually been employed to estimate the sparticle mass scale $\tilde{m}$ has been that of the naturalness of the gauge hierarchy, which suggests that $\tilde{m} \lesssim 1$ TeV [7]. A supporting argument has been the concordance of the gauge couplings measured at LEP and elsewhere with the predictions of supersymmetric Grand Unified Theo-
ties (GUTs) [8]. However, this argument is sensitive only logarithmically to $\tilde{m}$, and is also vulnerable to GUT threshold effects due to particles beyond the minimal supersymmetric extension of the Standard Model (MSSM). The agreement of the Higgs mass range favoured by the precision electroweak data with that calculated in the MSSM [9] is also encouraging, but is again only logarithmically sensitive to $\tilde{m}$, and hence unable to specify it with any accuracy.

An independent argument for new physics around the TeV scale is provided by calculations of cold dark matter, which yield naturally a freeze-out density in the cosmologically allowed range: $\Omega_{\text{CDM}} h^2 \lesssim 0.3$ (where $\Omega \equiv \rho/\rho_c$, the critical density, and $h$ is the Hubble expansion rate in units of 100 km/s/Mpc), and that preferred by theories of structure formation: $0.1 \lesssim \Omega_{\text{CDM}} h^2$, if the mass of the cold dark matter particle is $\lesssim 10$ TeV [10]. The upper limit on $\Omega_{\text{CDM}} h^2$ is fixed by the age of the Universe. For $\Omega_m h^2 < 1$, a lower limit on the age of the Universe of 12 Gyr implies an upper limit $\Omega_{\text{CDM}} h^2 < 0.3$ on the total matter density, and hence $\Omega_{\text{CDM}} < 0.1/h^2$. This argument does not rely on the high-redshift supernova observations [11], but they do support it.

A serendipitous prediction of $\Omega_{\text{CDM}}$ is provided by the MSSM with $R$ parity conservation [12], if the lightest supersymmetric particle (LSP) is the lightest neutralino $\chi$, as in many versions of the MSSM. Indeed, it has been shown [13] that the most ‘natural’ choices of MSSM parameters, from the point of view of the gauge hierarchy, yield a relic LSP density in the astrophysical and cosmological region $0.1 \lesssim \Omega_{\chi} h^2 \lesssim 0.3$. In this case, detailed calculations of the relic LSP abundance yield $\Omega_{\text{CDM}} \lesssim 0.3$ only for $m_\chi \leq 600$ GeV [14]. An essential role in this relic density calculation is played by $\chi - \tilde{\chi}$ coannihilation effects when the LSP is mainly a gaugino, which increase significantly the upper limit on the LSP mass quoted previously [15].

The idea we propose in this paper is that the relic density calculation be used to specify the likelihood that a LC with given $E_{\text{CM}}$ will be above the sparticle pair-production threshold, and able to detect at least some supersymmetric cross section. The answer is necessarily higher than $E_{\text{CM}} = 2 \times m_{\chi}^{\text{max}}$, since the process $e^+ e^- \rightarrow \chi \bar{\chi}$ is not directly observable in models with a stable neutralino LSP $\chi$. On the other hand, as we discuss in more detail below, $m_{\chi}^{\text{max}} \sim 600$ GeV is attained when $m_\chi \sim m_\gamma$, with $m_{\mu}, m_{\tau}$ not much heavier, so one might expect that a LC with $E_{\text{CM}}$ not far above 1200 GeV should be sufficient. As we show in more detail below, a LC with $E_{\text{CM}} = 500$ GeV or 1 TeV would only be able to detect supersymmetry in a fraction of the preferred dark matter region of MSSM parameter space. A LC with $E_{\text{CM}} = 1.5$ TeV would probably cover the preferred region, but might miss some part of the $\chi - \tilde{\chi}$ coannihilation ‘tail’ at large $m_{1/2}$, depending on the luminosity it attains. A LC with $E_{\text{CM}} = 2$ TeV would, on the other hand, be able to cover all the cosmological region with a comfortable safety margin in terms of cross section, kinematic acceptance and astrophysical uncertainties.

2. Summary of LSP density calculations

We assume $R$ parity is conserved, otherwise there would be no stable supersymmetric dark matter to interest us. We work within the constrained MSSM, in which all the supersymmetry-breaking soft scalar masses are assumed to be universal at the GUT scale with a common value $m_0$, and the gaugino masses are likewise assumed to be universal with common value $m_{1/2}$ at the GUT scale [16]. The constrained MSSM parameters are chosen so as to yield a consistent electroweak vacuum with a value of tan$\beta$ that is left free. The LEP lower limits on MSSM particles, including the lightest Higgs boson, suggest that tan$\beta \gtrsim 3$, so we consider this and the higher value tan$\beta = 10$. We consider two possible values of the trilinear soft supersymmetry-breaking parameter: $A = 0, -m_{1/2}$, the latter being the value for which the constraint that the lowest-energy state not break charge and colour (CCB) is weakest [17], consistent with parameter choices out to the point at the tip of the cosmological region.

When calculating the relic density of LSPs $\chi$, it is assumed that they were in thermal equilibrium prior to freeze-out at some temperature $T_f$. The relic density after freeze-out is then determined by the competition between the expansion rate of the Universe and the neutralino annihilation rate. Ultimately, the relic density is inversely related to the effective annihilation cross section $\sigma_{\text{eff}}$, which falls off as the square of the supersymmetry breaking scale. Thus,
as the supersymmetry breaking scale is increased, the annihilation cross section decreases and the relic density increases. This is why an upper limit to the relic density puts an upper limit on the sparticle mass scale, and on the mass of the neutralino LSP, in particular. In regions where the neutralino is mainly a gaugino (usually a bino), as in many models of interest, such as those with GUT-scale universality relations among the sparticle masses, the annihilation rate is dominated by sfermion exchange. As one approaches the upper limit on the neutralino mass, the cross section is maximized by taking sfermion masses as small as possible: in this case, the sleptons \( \tilde{\ell} \) are nearly degenerate with the neutralino LSP \( \chi \).

When the LSP is nearly degenerate with the next-to-lightest supersymmetric particle (NLSP), it is known [18] that new important coannihilation channels must be included to determine the relic neutralino density. Thus, in addition to the self-annihilation process \( \chi \chi \rightarrow \text{anything} \), the effective annihilation cross section includes important contributions from coannihilation processes involving slightly heavier particles \( \tilde{X}, \tilde{Y} \): \( \chi \tilde{X} \rightarrow \text{anything}, \tilde{X}\tilde{Y} \rightarrow \text{anything} \), weighted by the corresponding Boltzmann density suppression factors:

\[
\sigma_{\text{eff}} \sim \sigma(\chi\chi) + \sum_{\tilde{e}, \tilde{\mu}} e^{-(m_{\tilde{e}}-m_\chi)/T} \sigma(\tilde{e}\tilde{X}) + \sum_{\tilde{e}, \tilde{\mu}} e^{-(m_{\tilde{e}}+m_{\tilde{\mu}}-2m_\chi)/T} \sigma(\tilde{\mu}\tilde{X})
\]

In the parameter region of interest after taking into account the LEP exclusions of light sparticles, the most important coannihilation processes are those involving the NLSP \( \tilde{\tau} \) and other sleptons: \( \tilde{\ell}, \tilde{\mu} \), which are all taken into account in the following analysis [14]. Several of these coannihilation cross sections are much larger than that for \( \chi \chi \) annihilation close to threshold, because they do not exhibit \( P \)-wave suppressions. Therefore, coannihilation is an essential complication.

As noted above, since the resulting LSP relic density \( \Omega_\chi h^2 \) increases as \( \sigma_{\text{eff}} \) decreases, and since \( \sigma_{\text{eff}} \) decreases as \( m_\chi, m_{1/2} \) increase, one expects generically that \( \Omega_\chi h^2 \) should increase with increasing \( m_\chi, m_{1/2} \). This simple correlation is complicated in the presence of nearby \( s \)-channel \( Z^0 \) and Higgs poles in the annihilation cross sections, but the LEP exclusions now essentially rule out this possibility [19]. As mentioned earlier in the paper, the preferred range of cold dark matter density is \( 0.1 \leq \Omega_{\text{CDM}} h^2 \leq 0.3 \). It is possible that all the cold dark matter may not consist of LSPs \( \chi \), so we can at best assume that \( \Omega_\chi h^2 \leq \Omega_{\text{CDM}} h^2 \leq 0.3 \). However, this upper limit on \( \Omega_\chi h^2 \) is sufficient to infer an upper limit on \( m_\chi, m_{1/2} \). In [14], the values of the two key supersymmetry-breaking inputs \( m_\chi, m_{1/2} \) were constrained so that neutralino relic density should fall within the desired range. Roughly speaking, when \( m_{1/2} \leq 400 \) GeV, there is a relatively broad allowed range for \( m_\chi \) between about 50 and 150 GeV, depending on \( \tan \beta \) and the sign of \( \mu \). For values of \( m_{1/2} \geq 400 \) GeV, coannihilation becomes important, and \( m_\chi \) is restricted to a relatively narrow range of typical thickness \( \delta m_\chi \sim 20 \) GeV. The maximum value of \( m_{1/2} \) is determined by the point where there is no longer any value of \( m_\chi \), such that the neutralino mass is less than the \( \tilde{\tau}_R \) mass and \( \Omega_{\text{CDM}} h^2 < 0.3 \). This occurs when \( m_{1/2} \approx 1400 \) GeV, corresponding to the neutralino mass of about 600 GeV mentioned previously.

This is the essence of our argument that the relic density calculation can be used to specify the \( e^+ e^- \)-collider energy required to produce sparticles.

The upper limit to the neutralino mass including coannihilation effects of \( m_\chi \leq 600 \) GeV is relatively insensitive to such MSSM parameters as \( \tan \beta \) and \( A \). As in [14], we consider here the two cases \( \tan \beta = 3.10 \), and initially set \( A \) close to the weak-CCB value \( A = -m_{1/2} \). As mentioned earlier, the upper limit on \( m_\chi \) implies that the threshold for pair-producing sparticles must be at least \( E_{\text{CM}} = 1200 \) GeV. In fact, when the limit \( m_\chi \sim 600 \) GeV is reached, one also has \( m_\chi = m_{1/2} \), where the NLSP \( \tilde{\tau}_1 \) is the

---

The GUT universality conditions then imply that the squarks are considerably heavier.

---

\( ^2 \) On the other hand, the lower bound on \( \Omega_{\text{CDM}} h^2 \geq 0.1 \) cannot be transferred to a lower bound on \( \Omega_\chi \), and hence there are no corresponding lower bounds on \( m_\chi, m_{1/2} \), except for those imposed by slepton searches and/or the requirement that the \( \tilde{\tau} \) not be the LSP.
lighter stau mass eigenstate, so the threshold for the reaction $e^+ e^- \rightarrow \tilde{\tau}^+ \tilde{\tau}^-$ is also $\sim 1200$ GeV. Moreover, the mass of the $\tilde{\tau}$ is also not far above 600 GeV, so the threshold for $e^+ e^- \rightarrow \tilde{\tau}_R \tilde{\tau}^-_R$ is also not far beyond 1200 GeV. In addition, it is easy to check that even if one allows $m_{\chi} < m_{\tilde{\tau}}$, which is possible if $m_{\tilde{\tau}} < 600$ GeV, the threshold for $e^+ e^- \rightarrow \tilde{\tau}^+ \tilde{\tau}^-$ is never above 1200 GeV. These arguments are all suggestive that $E_{\rm CM} = 1200$ GeV may be sufficient for an $e^+ e^-$ linear collider to observe supersymmetry, but any such conclusion must hinge upon the analysis of the observability of the sparticle pair-production cross section that we undertake next.

3. Analysis of sparticle pair-production cross sections

In order to determine the region of the $(m_{\tilde{\chi}}, m_{1/2})$ plane that can be explored with a linear $e^+ e^-$ collider of given $E_{\rm CM}$, we have calculated the total observable production cross section for the pair production of sparticles $e^+ e^- \rightarrow X \tilde{Y}$, where $X$ and $Y$ are not necessarily particle and antiparticle [20]. In this context, ‘observable’ means that we do not include pair production of the LSP: $e^+ e^- \rightarrow \chi \tilde{\chi}$. Nor do we include sneutrino pair production: $e^+ e^- \rightarrow \tilde{\nu} \tilde{\nu}$, although some $\tilde{\nu}$ decays might be visible. Also, the production cross sections for heavier neutralinos $\chi'$, e.g., $e^+ e^- \rightarrow \chi \chi'$, are corrected for invisible $\chi'$ decay branching ratios. Finally, we assume that the ordinary particles emitted in a sparticle decay chain are observable only if the mass difference $\Delta M > 3$ GeV.

We assume an integrated luminosity $L = 100$ fb$^{-1}$ [1]. In order to estimate the corresponding sensitivity to the new-physics cross section $\sigma$, the relevant quantity is $B \equiv \sigma_{\tilde{\chi} \bar{\chi}} / \epsilon$, where $\sigma_{\tilde{\chi} \bar{\chi}}$ is the residual cross section for background processes, and $\epsilon$ is the signal-detection efficiency. As usual, a five-standard-deviation discovery is likely if $\sigma > 5 \times B / \sqrt{L}$, whereas, in the absence of any observation, new-physics processes with $\sigma > 2 \times B / \sqrt{L}$ will be excluded at about the 95% confidence level. At LEP 2, for mass differences between the produced sparticle and the LSP that are not too small, the background to searches for charginos $\chi^\pm$ and sleptons is mainly due to $W^{\pm}$ production, and typical values for $B$ were in the range 3–6 (fb$^{+1/2}$). At the LC we expect cleaner background conditions for both slepton and chargino searches because the $W^{\pm}$ should be more easily distinguishable, and also $\sigma(e^+ e^- \rightarrow W^+ W^-)$ is smaller. It is therefore likely that $B$ is smaller than at LEP 2. We adopt a conservative approach and scale $B$ roughly by $\sigma(e^+ e^- \rightarrow W^+ W^-)$, taking $B = 2$ (fb$^{+1/2}$), which gives a lower limit on the discoverable cross section of 1 fb, and an exclusion upper limit of 0.4 fb.

Fig. 1 shows the physics discovery reach in the $(m_{\tilde{\chi}}, m_{1/2})$ plane for $\tan \beta = 3$, 10 provided by the processes $e^+ e^- \rightarrow t\bar{t}$, neutralinos and charginos $\chi^+ \chi^-$ for collisions at $e^+ e^-$ collisions at $E_{\rm CM} = 500, 1000, 1250, 1500$ GeV, compared with the allowed cold dark matter region (shaded).

The solid lines in Fig. 1 correspond to the estimated discovery cross section of 1 fb for $e^+ e^- \rightarrow t\bar{t}$, and the broken lines to the kinematic limit $m_{\tilde{\chi}} = E_{\rm CM}/2$. We see no big differences between the plots for the different signs of $\mu$, nor indeed for the different values of $\tan \beta$. We note that $e^+ e^- \rightarrow t\bar{t}$ (solid lines) provides the greatest reach for each of the values $E_{\rm CM} = 500, 1000, 1250, 1500$ GeV studied, and that chargino pair production $e^+ e^- \rightarrow \chi^+ \chi^-$ (broken lines) becomes progressively less important as $E_{\rm CM}$ increases.

We see in Fig. 1 the extent to which the region favoured by the cosmological requirement that $0.1 \leq \Omega h^2 \leq 0.3$ may be covered by LC searches at different energies. In particular, about a half of this region is covered by sparticle searches at $E_{\rm CM} = 500$ GeV, a somewhat larger fraction (but not all) is covered at $E_{\rm CM} = 1000$ GeV, and full coverage of the favoured region is approached only when $E_{\rm CM} = 1500$ GeV $^3$. The reason why more than 1200 GeV is required is the $P$-wave threshold suppression for the observable processes with the lowest thresholds near the point of the cosmological region, namely the reactions $e^+ e^- \rightarrow t\bar{t}$, $s\bar{s}$.
Cross section limit $\sigma_{\text{lim}} = 1 \text{ fb}$

Fig. 1. Discovery sensitivity in the $(m_0,m_{1/2})$ plane for $\tan \beta = 3$ (left panels) and 10 (right panels) provided by searches for $e^+ e^- \rightarrow f^+ f^-$ and neutralinos (solid lines) and $\chi^+ \chi^-$ (broken lines) for collisions at $E_{\text{CM}} = 500, 1000, 1250, 1500$ GeV. The allowed cold dark matter regions are shaded. The top (bottom) panels are for $\mu < (>) 0$, and the value $A = -m_{1/2}$ is used.

Fig. 2 shows as three-dimensional 'mountains' the full observable sparticle cross section for $\tan \beta = 10$ and $\mu > 0$ for $E_{\text{CM}} = 500, 1000, 1250$ and 1500 GeV, including also other pair-production processes. The irregularities in the outline of the three-dimensional 'mountain' plot correspond to the opening up of different sparticle pair-production thresholds. We see again that $E_{\text{CM}} = 500$ GeV is not adequate to cover much of the cosmological region, that $E_{\text{CM}} = 1000$ GeV does not cover a significant fraction of the high-$m_{1/2}$ tail opened up by coannihilation, and that $E_{\text{CM}} \geq 1500$ GeV covers the cosmological region. We find similar features for $\tan \beta = 10$ and $\mu < 0$, and also for $\tan \beta = 3$ and both signs of $\mu$ (not shown).

We now return to the tip of the cosmological tail, which occurs when $m_0 \sim 630(610)$ GeV for $\tan \beta = 3(10)$ for our default option $\Omega_h^2 \leq 0.3$, and explore
in more detail how much $E_{\text{CM}}$ beyond 1200 GeV is required to be sure of detecting supersymmetry. Fig. 3 shows the contributions to the effective observable cross section from the dominant reactions $e^+e^- \rightarrow \tilde{\chi}^0 \tilde{\chi}^0$. Close to threshold, only pair production of the $\tilde{\chi}^0$ states is accessible, which exhibits a $P$-wave suppression. The associated-production process $e^+e^- \rightarrow \tilde{e}_L \tilde{\ell}_R$ kicks in at somewhat higher energies, and rapidly dominates, because of its $S$-wave threshold. This is the origin of the kink seen in the rise of the total cross section in each of the panels of Fig. 3, where the discovery and exclusion sensitivities are also shown as horizontal broken lines. We see that $E_{\text{CM}}$ only just above $2m_{\chi} \sim 1200$ GeV is not sufficient for sparticle discovery, because of the small observable cross section. We recall that, for our assumed integrated luminosity of 100 fb$^{-1}$ and de-
Fig. 3. The total observable cross section for $e^+e^-\rightarrow f\bar{f}$ processes, as a function of $E_{\text{CM}}$, for the points at the tips of the region of $m_{1/2}, m_0$ allowed for $\Omega_\chi h^2 \leq 0.2, 0.3$ (our preferred choice) and 0.4, with the usual choices $\tan\beta = 3, 10$ and both signs of $\mu$. The solid lines are for $A = -m_{1/2}$, and the dashed lines for $A = 0$. The horizontal broken lines are our estimates of the possible discovery and exclusion cross sections.

Detector performances, the discovery cross-section limit would be 1 fb, as indicated by the upper horizontal broken line in Fig. 3. Of course, this may be altered by different assumptions on the integrated luminosity and/or detection efficiency $^4$.

Each of the panels in Fig. 3 exhibits alternative curves to be compared with our default choices $\Omega_\chi h^2 = 0.3$ and $A = 0$. The curves for $\Omega_\chi h^2 = 0.4$ are for instruction only. In this case, one finds $m_\chi \leq 740(710)$ GeV for $\tan\beta = 3(10)$, but it is very difficult to reconcile such a large value of $\Omega_\chi h^2$ with the emerging measurements of cosmological parameters $^5$. In fact, we actually believe that allowing $\Omega_\chi h^2 \leq 0.3$ is already quite conservative. For $^4$ We note, in particular, that higher luminosities may be achievable at higher $E_{\text{CM}}$.

$^5$ For the record, for $\Omega_\chi h^2 < 0.5$, the upper limit on the neutralino mass increases to $m_\chi \leq 830(800)$ GeV for $\tan\beta = 3(10)$. 


the preferred observational value $h \sim 1/\sqrt{2}$, this would correspond to $\Omega_\gamma \leq 0.6$, which extends far beyond the currently favoured range $\Omega_\gamma \leq 0.4$. If instead one enforces $\Omega_\gamma h^2 \leq 0.2$, one finds that the maximum value of the LSP mass becomes $m_\chi \sim 520(500)$ GeV, for $\tan \beta = 3(10)$ and $E_{\text{CM}} = 1500$ TeV would be adequate, as seen in Fig. 3. Indeed, in this case, $E_{\text{CM}} = 1200$ GeV would be sufficient to cover all the region of the $(m_\chi, m_{1/2})$ favoured by cosmology. We also show in Fig. 3 comparisons between the cross sections at the extreme points for $A = 0$ and $-m_{1/2}$. Our conclusions are clearly insensitive to the ambiguity in the choice of $A$.

4. Conclusions

Finally, we show in Fig. 4 the fraction of the cosmologically-allowed region of the $(m_{1/2}, m_0)$ plane that can be explored by a LC as a function of the accessible limiting cross section $\sigma_{\text{lim}}$, for different values of $E_{\text{CM}}$. When the detector performances are specified, the values of $\sigma_{\text{lim}}$ correspond to different values of $E_{\text{CM}}$. We also show in Fig. 3 comparisons between the cross sections at the extreme points for $A = 0$ and $-m_{1/2}$. Our conclusions are clearly insensitive to the ambiguity in the choice of $A$.

![Graph showing discoverable fractions](image)  

Fig. 4. The discoverable fractions of the region of the $(m_{1/2}, m_0)$ plane allowed by cosmology for $0.1 \leq \Omega_\gamma h^2 \leq 0.3$ that is accessible to a LC as a function of accessible cross section $\sigma_{\text{lim}}$, for different values of $E_{\text{CM}}$. For each of our scenarios $\tan \beta = 3, 10$ and both signs of $\mu$, assuming $A = -m_{1/2}$. Also indicated is the correspondence between luminosity and $\sigma_{\text{lim}}$ for the detector performances assumed in this paper.
ferent values of the available luminosity, as indicated.

We see in Fig. 4 that a LC with \( E_{\text{CM}} = 1.5 \) TeV would cover all the cosmological region if \( \sigma_{\text{lim}} \leq 5 \) fb \(^6\) and one with \( E_{\text{CM}} = 1.25 \) TeV if \( \sigma_{\text{lim}} \leq 0.5 \) fb. On the other hand, a LC with \( E_{\text{CM}} = 1 \) TeV could never cover all the cosmological region, and a LC with \( E_{\text{CM}} = 0.5 \) TeV covers \( \sim 60\% \) of it \(^7\).

The conclusions to be drawn from this analysis are somewhat subjective, since they depend how much you are prepared to bet at what odds. It could well be that new cosmological data might inform better your choice. For example, you could become more sanguine about the prospects for a lower-energy LC if the upper limit on \( V_h^2 \) could be decreased to 0.2. Our point in this paper has been to establish that there is a phenomenological connection between the LC energy and supersymmetric dark matter, and we believe that Fig. 4 summarizes the best advice we can offer at the beginning of the third millennium.

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**References**


\(^6\) A LC with \( E_{\text{CM}} = 2 \) TeV would always cover all the cosmological region, even for a very pessimistic assumption on \( \sigma_{\text{lim}} \).

\(^7\) Fig. 4 is plotted using a linear measure for the cosmological region. The prospects for lower-energy machines would seem brighter if one used a logarithmic measure of the parameter space, e.g., using this measure, a LC with \( E_{\text{CM}} = 0.5 \) TeV would cover over 80\% of the cosmological region.
Superfield BRST charge and the master action

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Abstract

Using a superfield formulation of extended phase space, we propose a new form of the Hamiltonian action functional. A remarkable feature of this construction is that it directly leads to the BV master action on phase space. Conversely, superspace can be used to construct nilpotent BRST charges directly from solutions to the classical Lagrangian Master Equation. We comment on the relation between these constructions and the specific master action proposal of Alexandrov, Kontsevich, Schwarz and Zaboronsky.

1. Introduction

In two recent papers [1,2], it has been shown that quantization, in both the Hamiltonian operator language and a phase space path integral, has an equivalent superfield formulation. The superspace consists of ordinary time $t$ and a new Grassmann-odd direction $\theta$. All original phase space coordinates $z^A(t)$ are just the zero-components of super phase-space coordinates $z^A(t, \theta)$. It follows that $z^A(t, \theta)$ has the same statistics as $z^A_0(t)$, denoted by $\epsilon(z^A)$. The superspace derivative replaces the ordinary time derivative. It satisfies

$$D^2 = \frac{d}{dt},$$

and one can indeed show that the usual Heisenberg equations of motion on the original phase space variables are obtained by applying $D$ twice on the extended variables.

We shall here consider another extension of the usual time derivative,

$$D \equiv \frac{d}{dt},$$

which obviously satisfies $D^2 = 0$. This derivative will turn out to play a central role in understanding BV-quantization [3] in phase space [4]. We note that $D$ can be assigned a definite ghost number according to the ghost number we assign to $\theta$, and we choose $\text{gh}(\theta) = 1$. 

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We now make some general observations concerning a relation between the even Poisson bracket and an associated (Grassmann-odd) antibracket on the super path space that can be induced by it [1,7]:

\[
(\tilde{z}^A_0(t),\tilde{z}^B_0(t)) = 0,
\]

\[
(\tilde{z}^A_0(t),\tilde{z}^B_0(t)) = \omega^{AB}(\tilde{z}_0) \delta(t-t'),
\]

\[
(\tilde{z}^A(t),\tilde{z}^B(t)) = z^{\dagger}_0 \partial_\theta \omega^{AB}(\tilde{z}_0) \delta(t-t'),
\]

where \(\omega^{AB}(z) = (z^A, z^B)\). This antibracket obviously carries one unit of Grassmann parity and one unit of ghost number. Another important property of (5) is the following: let \(f(z)\) be a function on phase space and let the functional \(F\) be defined by

\[
F[z] = \int dt d\theta f(z(t,\theta)).
\]

so that the Grassmann parity of \(F\) is opposite that of \(f\). Then for any functions \(f, g\) on the phase space we have

\[
(F,G) = \int dt d\theta \{f, g\},
\]

where \(F,G\) are the corresponding functionals obtained by (6). In particular, if a Grassmann-odd function \(f\) satisfies the “Hamiltonian Master Equation” \((f,G) = 0\) then the corresponding Grassmann-even functional \(F = \int dtd\theta f\) satisfies the BV Master Equation with respect to the antibracket:

\[
(F,F) = 0.
\]

This has an obvious generalization, because even if \(f = f(z,t,\theta)\) is a function on the original phase space that also explicitly depends on time \(t\) and its superpartner \(\theta\), and \(f\) still satisfies the Hamiltonian Master Equation \((f,G) = 0\) for any \(t\) and \(\theta\), then the corresponding functional \(F = \int dtd\theta f(z(t,\theta),t,\theta)\) also satisfies the Master Equation with respect to the antibracket (7).

\[\text{1 See Section 4 for details.}\]

2. Superfield realization of the phase space antifield formalism

Let us now consider a system with first class constraints. We thus have a Grassmann-odd BRST generator \(\Omega = \Omega(z)\) and an Hamiltonian \(H = H(z)\), which are taken to satisfy

\[
\{\Omega,\Omega\} = 0 \quad \text{and} \quad \{H,\Omega\} = 0.
\]

They combine nicely into one Grassmann-odd object \(Q\) [1]

\[
Q(z,\theta) = \Omega(z) + \theta H(z),
\]

which is nilpotent due to Eq. (9):

\[
\{Q,Q\} = 0.
\]

Let us in addition consider the following action functional:

\[
S[z] = \int dt d\theta \left[ V_A(z(t,\theta)) \tilde{z}^A(t,\theta) - Q \right].
\]

where the symplectic potential \(V_A\) is related to the symplectic metric \(\omega_{AB}(z)\) via

\[
\omega_{AB}(z) = (\partial_\theta \tilde{z}^A - (1)^{t\ell} (1)^{t\ell} \partial_\theta \tilde{z}^A)(-1)^{\delta_{AB}}.
\]

As usual, we assume the symplectic form to be exact. We emphasize that this construction is quite different from that of Refs. [1,2]. In particular, in the present formulation the superpartners \(z_i(t)\) cannot be viewed as Pfaffian ghosts. To construct the path integral one must thus in addition specify the formally invariant super Liouville measure, as usual. If in addition to Eq. (11) we also assume that there are no boundary terms from \(Q\)

\[
\int dt d\theta \left[ V_A(z(t,\theta)) \tilde{z}^A(t,\theta) - Q \right] dt = 0,
\]

then it follows from the previous considerations that this action satisfies the Master Equation

\[
(S,S) = 0.
\]

It is simple to integrate out the additional \(\theta\)-variable from Eq. (12), and one finds:

\[
S[z] = \int dt \left[ V_A(z_0(t)) \tilde{z}^A_0(t) - H(z_0(t)) \tilde{z}^A_0(t) \partial_\theta \Omega(z_0(t)) \right].
\]

\[\text{2 See Section 4 for details.}\]
Except for the last term, this is simply the conventional phase-space action if there were no constraints. Let us rewrite this last term:
\[ z_0^{0,4} \partial_t \Omega(z_0(t)) = z_0^{0,4} \partial_t \Omega, \quad (17) \]
where we have defined \( z_0^{0,4} = z_0^{0,4} \partial_B \). This is the general phase space action extended with antifields to satisfy the classical BV Master Equation \( S(S) = 0 \).

Note that the variables \( z_0^{0,4} \) and \( z_0^{0,4} \) are canonically conjugate within the antibracket, but the precise identification of which plays the role of “field” or “antifield” becomes apparent upon identification of ghost number \([4]. All these assignments follow automatically from the superfield approach.

To make these considerations more concrete, let us consider the case of first-class constraints \( T_a(z) \) with the usual algebra:
\[ \{ T_a, T_b \} = C_{\alpha \beta} T_{\alpha \beta}, \quad \{ T_a, H_0 \} = V_{\alpha} T_{\alpha}, \quad (18) \]
where for simplicity we assume all the original phase space variables to be bosonic. According to the BFV prescription \([5] one introduces ghosts \( c^a \) together with their conjugate momenta \( \mathcal{P}_a \), and the BRST charge and the extended Hamiltonian are then given by
\[ \Omega = T_a c^a - \frac{1}{2} \mathcal{P}_{a \beta} C_{\alpha \beta} c^a c^\beta + \ldots, \]
\[ H = H_0 + \mathcal{P}_a V^a c^a + \ldots, \quad (19) \]
where the dots denote the higher order terms in the expansion of \( \Omega \) and \( H \) with respect to the ghost momenta \( \mathcal{P} \). The extended Poisson bracket is given in coordinates by
\[ \{ q^i, p_j \} = \delta^i_j, \quad \{ c^a, \mathcal{P}_\beta \} = \delta^a_\beta. \quad (20) \]

We now allow all the phase space coordinates to depend on \( t \) and \( \theta \). Let us write explicitly their expansion with respect to \( \theta \):
\[ q^i = q^i_0 - \theta p^i_0, \quad p_i = p_i^0 + \theta q^i_0, \]
\[ c^a = c^a_0 - \theta a^a, \quad \mathcal{P}_a = u^a + \theta c^a_0, \quad (21) \]
where we have chosen some defining signs in order to facilitate comparison with the existing literature.

According to our choice \( gh(\theta) = 1 \) the ghost numbers of the new variables read
\[ \text{gh} (q^i_0) = 0, \quad \text{gh} (p^i_0) = 0, \quad \text{gh} (q^i) = -1, \]
\[ \text{gh} (p^i) = -1, \quad \text{gh} (c^a_0) = 1, \quad \text{gh} (u^a) = 0, \]
\[ \text{gh} (c^a) = -2, \quad \text{gh} (u^a) = -1. \quad (22) \]

The action (16) then takes the form
\[ S = \int dt \left[ p_i^0 q_i^0 - H_0 + T_a u^a - p_i^0 \{ p_j^0, T_a \} c^a + q_i^0 \{ q_j^0, T_a \} c^a + u^a \hat{c}^a - u^a V_{\beta} c_0^\beta + \frac{1}{2} \hat{c}^a C_{\alpha \beta} c^\gamma c_0^\beta - u^a C_{\alpha \beta} u^\beta c_0^\beta + \ldots \right], \quad (23) \]
where the dots denote higher order terms in powers of \( u^a \) and \( c^a \). It is easy to see that the first three terms in (23) are nothing but the extended Hamiltonian action
\[ S = \int dt \left( p_i^0 q_i^0 - H_0 + T_a u^a \right), \quad (24) \]
corresponding to the system under consideration. Indeed, these terms enter only with ghost number zero variables, and should thus be understood in the BV formalism as the initial action. Making use of the ghost number assignments (22) it is also easy to infer the gauge generators from Eq. (23): they are precisely the gauge generators of the extended Hamiltonian action (24). It follows that \( u^a \) are simply the Lagrangian multipliers corresponding to the constraints \( T_a \). All other assignments coincide exactly with those of the extended phase space action first identified by Fisch and Henneaux \([4]. We have thus explicitly confirmed the remarkable fact that the whole extended phase-space BV formalism is precisely encoded in this superspace path integral approach.

3. An inverse construction

There are two equivalent ways two perform path integral quantization of Hamiltonian systems with first-class constraints:

(i) via the BFV formalism based on the extended Poisson bracket and BRST charge \( \Omega \),

(ii) via the BV formalism based on the antibracket and the master action corresponding to the extended Hamiltonian action.

As we have shown above, the superfield approach allows one to explicitly derive the BV formulation from the BFV prescription on phase space. The space of field histories (which is the BV configuration space) thus appears as the space of super-paths of the initial BFV phase space. Remarkably, this
space comes with a BV antibracket which originates directly from the BFV Poisson bracket. Similarly, the master action derives directly from the BRST charge $\Omega$ and the BFV extended Hamiltonian.

It is natural to ask if there, conversely, exists a “phase space” description of any Lagrangian gauge theory which is dual to the standard BV description. As we shall now show, the answer to this is affirmative. Moreover, we will again directly arrive at the correct dual description by means of the superfield approach. A quite different superfield formulation of BV Lagrangian quantization was first proposed in Ref. [6].

Let us start with the standard BV formulation of any Lagrangian gauge theory. Let $\mathcal{M}$ be the antisymplectic manifold of the BV configuration space, the antisymplectic structure of which determines the symplectic manifold of the BV configuration space, and the path space will thus have an even symplectic structure see Section 4. The corresponding Poisson bracket is given in coordinates by

$$\{G^A, G^B\} = 0,$$

$$\{G^A, G^B\} = E^{AB}(G^0),$$

and the path space will thus have an even symplectic structure (see Section 4). The corresponding Poisson bracket is given in coordinates by

$$\Omega(\Gamma_0, \Gamma_1) = \int d\theta S(\Gamma(\theta)).$$

By construction $\text{gh} (\Omega) = 1$, and we note that $\Omega$ will be nilpotent:

$$\{\Omega, \Omega\} = 0.$$

In fact, this nilpotency condition is here to be viewed as a Poisson-bracket Master Equation. But it immediately raises the question: Can this $\Omega$ also be formally considered as the BRST charge corresponding to a Hamiltonian system with constraints? Although we do not allow the fields $I^A$ to depend on a new bosonic coordinate “time”, this is in fact the case.

It is not difficult to understand the nature of the associated Hamiltonian system of constraints. Let $S$ be the extended master action of a gauge theory described by an initial action $S_0(q^i)$ and gauge generators $R^i_\alpha$ which for simplicity take to be linearly independent (the discussion can be easily generalized to the reducible case). They form a possibly open algebra

$$[R^i_\alpha, R^j_\beta] = C^\gamma_{\alpha\beta} R^j_\gamma + \ldots,$$

where dots means the terms vanishing on the stationary surface of the action $S_0$. Thus in the BV formulation we need, for the minimal sector, the fields of the initial theory $q^i$, $\text{gh}(q^i) = 0$ (which we for simplicity take to be bosonic), ghosts $c^\alpha$, $\text{gh}(c^\alpha) = 1$, and all their antifields. As usual, we combine fields into $\phi^A$, and antifields into $\phi^*_A$. The BV antibracket and ghost number assignments are

$$\{\phi^A, \phi^*_B\} = \delta^A_B, \text{ gh}(\phi^*_A) = - \text{gh}(\phi^A) - 1.$$
We now allow $\phi^A, \phi^\ast_A$ to depend on $\theta$. The expansion of $\phi, \phi^\ast$ in $\theta$ thus reads

$$q^i = q^i_0 - \theta \gamma^i, \quad q^\ast_i = \pi_i + \theta \rho_i,$$

$$c^a = c^a_0 + \theta \eta^a, \quad c^\ast_a = \rho_a + \theta \mathcal{P}_a.$$  \hspace{1cm} (33)

Moreover, it follows from Eq. (26) that ghost number assignments are:

$$\mathrm{gh}(q^i_0) = \mathrm{gh}(\rho_i) = 0, \quad \mathrm{gh}(c^a) = \mathrm{gh}(\eta^a) = 1, \quad \mathrm{gh}(\mathcal{P}_a) = \mathrm{gh}(\pi_i) = -1, \quad \mathrm{gh}(\eta^a) = 2.$$  \hspace{1cm} (34)

The Poisson bracket (27) becomes explicitly

$$\{q^i_0, p_j\} = \delta^i_j, \quad \{\gamma^i, \pi_j\} = \delta^i_j,$$

$$\{c^a_0, \mathcal{P}_\beta\} = \delta^a_\beta, \quad \{\eta^a, \rho_\beta\} = \delta^a_\beta.$$  \hspace{1cm} (35)

Substituting (32) in (28) and integrating over $\theta$ we arrive at

$$\Omega = -\gamma^i \partial_i S_0 + p_i R^i_{\alpha} c^a_0 - \pi_j R^j_{\alpha} \eta^a + \pi_j \{\gamma^i, \pi_j\} c^a_0 - 4 \mathcal{P}_\gamma C^\gamma_{a\beta} c^a_0 c^\beta_0 - \rho_\beta C^\gamma_{a\beta} \eta^a c^\beta_0 + \ldots,$$  \hspace{1cm} (36)

dots denote higher order terms in the variables $\mathcal{P}, \pi$ and $\rho$. We will see that they are to be identified with ghost momenta. Eq. (36) can formally be identified with the BRST charge of a system with constraints. Using the ghost number assignments it is easy to see that $\gamma^i$ and $c^a_0$ are the ghosts associated with first class constraints $\partial_i S$ and $p_i R^i_{\alpha}$ and $\pi_j, \mathcal{P}_a$ are their conjugate momenta. The variables $\eta^a$ and $\rho_\beta$ are simply the ghosts for ghosts and their momenta associated with the reducible constraints $\partial_i S$. To be precise, the Lagrangian gauge generators $R^i_{\alpha}$ are the reducibility functions for the constraints $\partial_i S_0$ due to the Noether identity $R^i_{\alpha} \partial_i S_0 = 0$. The corresponding term $\pi_j R^j_{\alpha} \eta^a$ indeed enters (36). It should be emphasized that all these identifications are in an algebraic sense only: There is no analogue of the ordinary time coordinate of the Hamiltonian system.

An interesting open question concerns the role of quantum corrections to the master action $S$. Suppose we expand the solution to the full quantum Master Equation

$$\frac{\delta}{\delta \Sigma} (S, S) = i\hbar \Delta S$$  \hspace{1cm} (37)

in powers of $\hbar$, and insert this full solution into the definition (28). The nilpotency condition (29) will then be broken by $\hbar$-corrections on the right hand side. What is the interpretation of the $\hbar$-corrections to the BRST charge $\Omega$? Perhaps this is related to canonical quantization of the Poisson bracket and the corresponding Hamiltonian quantum Master Equation $\{\hat{H}(h), \hat{H}(h)\} = 0$. In any case, the question deserves a more detailed study.

4. Geometry of the super path space

It is useful to clarify the geometrical meaning of the structures entering the above superspace formulations, and view them in greater generality. In particular, it is instructive to see how the antibracket and the usual Poisson bracket enter on equal footing. In this section, which will be a bit more abstract, we find it convenient to even use the same symbol for the two, namely $[\cdots]_\mathcal{A}$. One must of course keep in mind that the odd and even brackets have odd and even Grassmann parities, respectively. Now let $\mathcal{M}$ be a symplectic manifold (which can be even or odd), and let $[\cdots]_\mathcal{A}$ be the corresponding Poisson bracket or antibracket, depending on the Grassmann parity. We denote by $n$ the dimension of $\mathcal{M}$ and $\kappa$ the Grassmann parity of the bracket $[\cdots]_\mathcal{A}$, i.e.,

$$\epsilon([f, g]_\mathcal{A}) = \epsilon(f) + \epsilon(g) + \kappa.$$  \hspace{1cm} (38)

The exchange relation, the Leibniz rule and the Jacobi identity are then neatly summarized, for both brackets, by

$$[f, g]_\mathcal{A} = -(-1)^{(\epsilon(f) + \kappa)(\epsilon(g) + \kappa)} [g, f]_\mathcal{A},$$

$$[f, gh]_\mathcal{A} = [f, g]_\mathcal{A} h + (-1)^{(\epsilon(f) + \kappa)(\epsilon(g) + \kappa)} g[f, h]_\mathcal{A},$$

$$[f, [g, h]]_\mathcal{A} = [[f, g]_\mathcal{A}, h]_\mathcal{A}$$

$$+ [g, [f, h]]_\mathcal{A}(-1)^{(\epsilon(f) + \kappa)(\epsilon(g) + \kappa)}.$$  \hspace{1cm} (39)

for any functions $f, g$ and $h$ on $\mathcal{M}$. In local coordinates $\Gamma^A$ on $\mathcal{M}$ we write generically $E^{A\beta} = [\Gamma^A, \Gamma^\beta]$ for both kinds of brackets.

Let in addition $\Sigma$ be a supermanifold of dimension $k$ and of coordinates $x^i$. We assume for simplicity that it is compact. Let there in addition be a volume form $d\mu(x) = \rho(x) dx = \rho(x) dx^1 \ldots dx^k$ on
We denote by $\mathcal{Z}$ the super-path space, i.e. the space of smooth maps from $\Sigma$ to $\mathcal{M}$. In local coordinates each map is described by the functions $\Gamma^A(x)$. As $\mathcal{M}$ is symplectic, and $\Sigma$ has the above volume form, then $[\;]$ the super path space $\mathcal{Z}$ is also symplectic (see also Section 4.3 of Ref. [1] and Ref. [8]). Indeed, for any functionals $F,G$ we define

$$[F,G]_\mathcal{Z} = \langle \, (-1)^{\epsilon(F) + \epsilon(d\mu)} \, d\mu(x) \rangle \times \left( \begin{array}{c} \frac{\partial}{\partial \Gamma^A(x)} F^{AB} \left( \Gamma(x) \right) \, \frac{\partial}{\partial \Gamma^B(x)} \left( \Gamma(x) \right) \frac{\partial}{\partial \delta \Gamma^A(x)} \left( \Gamma(x) \right) \, \frac{\partial}{\partial \delta \Gamma^B(x)} \left( \Gamma(x) \right) \end{array} \right).$$

(40)

Here we have made use of the following conventions for the functional derivatives: for infinitesimal variation $\delta \Gamma^A(x)$ we write

$$\delta F \Gamma = \int d\mu(x) \delta \Gamma^A(x) \left( \frac{\partial}{\partial \delta \Gamma^A(x)} \right) \left( \frac{\partial}{\partial \Gamma^A(x)} \right) \frac{\delta \Gamma^A(x)}{\delta \Gamma^A(x)} \text{d} \mu(x).$$

(41)

In particular, left and right derivatives are then related by

$$\delta \Gamma^A(x) \left( \frac{\partial}{\partial \delta \Gamma^A(x)} \right) \left( \frac{\partial}{\partial \Gamma^A(x)} \right) \frac{\delta \Gamma^A(x)}{\delta \Gamma^A(x)} \text{d} \mu(x).$$

(42)

where $\epsilon(d\mu)$ is the Grassmann parity of the measure. Note that the Grassmann parity of the functional derivative $\frac{\partial}{\partial \Gamma^A(x)}$ is $\epsilon(\Gamma^A) + \epsilon(d\mu)$.

Let us first state some obvious properties of the bracket structure (40). First, the Grassmann parity $k'$ of the bracket (44) is related to the Grassmann parity of the bracket $[\cdot, \cdot]_\mathcal{Z}$ by $k' = k + \epsilon(d\mu)$. The bracket (44) obviously satisfies (39) with $k$ being the $k'$ and is thus a Poisson bracket or an antibracket, depending on its Grassmann parity. Taking $F$ and $G$ in (40) to be

$$F = \int d\mu(x) f(\Gamma(x)),$$

$$G = \int d\mu(x) g(\Gamma(x)).$$

(43)

for some functions $f,g$ on $\mathcal{M}$ we arrive at

$$[F,G]_\mathcal{Z} = \int d\mu(x) f \, g \left( \Gamma(x) \right).$$

(44)

Let there in addition be given a vector field $q = \frac{\partial}{\partial \Gamma^A(x)}$ on $\Sigma$. We assume that $\text{div}_{\Gamma^A}(q) = 0$, which implies that $\langle d\mu \, q \rangle = 0$ for any function $f$ on $\Sigma$. The vector field $q$ can be lifted to a vector field $Q$ on the super path space $\mathcal{Z}$ [7]. In coordinates we have for any functional $F(\Gamma)$

$$QF(\Gamma) = \int d\mu(x) \times \left( \frac{\partial}{\partial \Gamma^A(x)} \right) \left( \frac{\partial}{\partial \Gamma^B(x)} \right) \frac{\delta F(\Gamma)}{\delta \Gamma^A(x)} \left( \Gamma(x) \right) \frac{\delta F(\Gamma)}{\delta \Gamma^B(x)} \left( \Gamma(x) \right).$$

(45)

An important observation is that $Q$ is an Hamiltonian vector field with respect to the bracket (40). Indeed, let $V_A(x)$ be the symplectic potential on $\mathcal{M}$; for the symplectic 2-form we have

$$E_{AB} = \left( \frac{\partial}{\partial \Gamma^A(x)} e^{\delta \Gamma^A(x)} \right) \left( \frac{\partial}{\partial \Gamma^B(x)} e^{\delta \Gamma^B(x)} \right) \text{d} \mu(x).$$

(46)

where $e$ is the parity of the symplectic form. Then for an arbitrary functional $F(\Gamma)$ we have

$$QF = -[C,F]_\mathcal{Z},$$

(47)

with $e(C) = e(d\mu) + e(q) + k$. Note that if $q$ is an odd nilpotent vector field on $\Sigma$, then the corresponding Hamiltonian $C$ automatically satisfies the classical Master Equation $[C,C]_\mathcal{Z} = 0$. Another important property of $Q$ is that for any functional $F$ of the form (43) we have $QF = -[C,F]_\mathcal{Z} = 0$.

These properties of the super path space bracket allows one to directly construct a BV master action

$$W = \alpha C + \beta F.$$
for some functional $F$ from (43) and any $f$ satisfying $[f, f]_p = 0$. This holds for arbitrary coefficients $\alpha$ and $\beta$. When the Grassmann parity of this $W$ is odd, it simply has the interpretation as a BRST-like charge $Q$. It was shown in ref [7] that the BV master actions corresponding to Chern-Simons theory and 2D topological sigma models have precisely the same structure. In all these cases one chooses $\Sigma$ to be an odd tangent bundle over the even manifold $\Sigma_0$, with the odd nilpotent vector field $q$ being the De Rham differential on $\Sigma_0$. Remarkably, the BV master action of the 2D Poisson sigma model used in [9] for the construction of the Kontsevich star product [10] also has just the form (48).

Surprising relations between the Poisson brackets of Hamiltonian BFV quantization and antibrackets of Lagrangian quantization have recently been discovered for topological gauge theories in a quite different manner [11] (see also Refs. [12–14]). It would also be interesting to consider the isomorphism between the Poisson bracket and the antibracket [15] in the light of this superfield construction.

Finally, let us explicitly make contact to the examples we gave in the previous sections. For the first case we choose $\Sigma$ to be a $(1|1)$ supermanifold with coordinates $t$ and $\theta$. We also choose $\mathcal{M}$ to be an even symplectic manifold, and simply take as measure $d\mu = dt d\theta$, as well as an odd operation $\mathcal{D} = \theta \partial_x$. A smooth map $\Sigma \rightarrow \mathcal{M}$ is given by the set of functions $\Gamma^A(t, \theta) = \Gamma^A_0 + \theta \Gamma^A_1$. Using the representation

$$\Gamma^A_0(t) = \int dt \theta \Gamma^A(t, \theta),$$

$$\Gamma^A_1(t) = \int dt \Gamma^A(t, \theta),$$

(49)

and explicitly integrating over $\theta$ in the definition (40) we arrive at

$$\left( \Gamma^A_0(t), \Gamma^B_0(t') \right) = 0,$$

$$\left( \Gamma^A_0(t), \Gamma^B_1(t') \right) = \omega^{AB}(\Gamma^B_0) \delta(t - t'),$$

$$\left( \Gamma^A_1(t), \Gamma^B_1(t') \right) = \Gamma^A_0 \partial_t \omega^{AB}(\Gamma^B_0) \delta(t - t').$$

(50)

In its turn the odd nilpotent vector field $D$, considered as acting on functionals, is a Hamiltonian vector field with Hamiltonian

$$C = \int dt d\theta V_\alpha \mathcal{D} \Gamma^A.$$

Choosing Darboux coordinates $p, q$ on $\mathcal{M}$ one arrives at the standard form $\{d\mu^0, q_0^i\}$. Thus we see that the Hamiltonian action (12) has precisely the “geometrical” form (48) with $f$ being the super BRST charge $Q + \theta H$. The only difference is that $f$ in this case explicitly depends on $\theta$.

In the case of the inverse construction of Section 3 one chooses $\Sigma$ to be a one dimensional space with Grassmann-odd coordinate $\theta$ and $\mathcal{M}$ as an antisymplectic manifold. Using the general formula (40) one arrives directly at the explicit form of the odd path space Poisson bracket (27). In this case we simply take $\alpha = 0$ and $F$ to be the master action $S$ in Eq. (48). We have thus shown how both of these cases follow directly from the above general framework.

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N = 0 supersymmetry and the non-relativistic monopole

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Abstract

We study some of the algebraic properties of the non-relativistic monopole. We find that we can construct theories that possess an exotic conserved fermionic charge that squares to the Casimir of the rotation group, yet do not possess an ordinary supersymmetry. This is in contrast to previous known examples with such exotic fermionic charges. We proceed to show that the presence of the exotic fermionic charge in the non-supersymmetric theory can nonetheless be understood using supersymmetric techniques, providing yet another example of the usefulness of supersymmetry in understanding non-supersymmetric theories. © 2000 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

The use of supersymmetry to understand non-supersymmetric theories has proven to be a valuable resource in the study of quantum theories [1–3]. In the examples just listed, one typically finds a way to understand a theory that is not supersymmetric by treating it as the restriction of a supersymmetric theory, typically by eliminating the fermionic fields of the supersymmetric theory.

There is, of course, another possibility. Suppose, instead, the supersymmetric theory is the restriction of some non-supersymmetric theory. Can the algebraic structure of the supersymmetric theory nonetheless give an indication as to the behavior of the non-supersymmetric theory? In this letter, we study an example where this occurs.

The importance of this is twofold. First, we obtain some particular insights into the theory of non-relativistic magnetic monopoles and dyons. But, more importantly, we extend the usefulness of supersymmetry in understanding non-supersymmetric theories. The analysis of quantum systems is sufficiently difficult that any new techniques are useful. Because supersymmetry is itself so powerful, any time we can link a non-supersymmetric theory to a supersymmetric one, we have the potential of deeper insights into the non-supersymmetric theories.

In the first section of this paper, we present a non-supersymmetric model that has an exotic conserved fermionic charge. Existing examples of the appearance of such charges have always been in supersymmetric theories [4], and with the supersymmetry algebra invoked in an essential way to explain appearance of such exotic charges [5]. Thus, in this section, we establish a counterexample, in which an exotic fermion charge (one that does not square to the Hamiltonian) can exist in a context in which there is no ordinary supersymmetry, and so the charge cannot be understood in the usual way.

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Then we examine the algebraic structure of this theory in some depth. We find an intriguing set of conjugate operators that lead us to an interesting formulation of the theory, one that affords a straightforward way to motivate the identification of the exotic fermionic charge. We comment that it is also possible that these operators can be used to demonstrate integrability in some cases (for a discussion of integrability in non-relativistic monopole theories, see [6]), but that is beyond the scope of this work.

Finally, we use the algebraic properties of these operators to analyze the appearance of the exotic fermionic charge in the non-supersymmetric theory. We find that it is possible to draw a connection between the non-supersymmetric theory and the supersymmetric theory, and thus explain the appearance of the exotic charge using supersymmetric arguments, even though the theory in question is not supersymmetric! We will thus have established an N = 0 supersymmetry approach to these exotic charges.

2. A non-supersymmetric monopole model

Consider the non-relativistic theory in three spatial dimensions of a spin ½ particle in the presence of a magnetic monopole or dyon. Such a theory has a Hamiltonian

\[
H = \frac{1}{2m} (p - eA)^2 - \frac{1}{2} eF_{ij} S^i S^j + V(r),
\]

where \( A^i \) is the gauge field of a magnetic monopole, \( \frac{1}{2} e F_{ij} \) is the associated magnetic field, \( S^k \) is the spin operator for the particle, \( e \) is its electric charge, and \( V(r) \) is a spherically symmetric potential energy (which will include a Coulomb term if the monopole field arises from a dyon). The radial coordinate \( r = (x^i x^i)^{1/2} \). One typical parametrization of the monopole gauge field in spherical coordinates is

\[
A^{(i)} = \frac{g}{4\pi} \frac{1 - \cos \theta}{r \sin \theta} \hat{\phi} = \frac{g}{4\pi r} \tan \frac{\theta}{2} \hat{\phi}, \quad \theta < \pi
\]

and

\[
A^{(i)} = -\frac{g}{4\pi} \frac{1 + \cos \theta}{r \sin \theta} \hat{\phi} = -\frac{g}{4\pi r} \cot \frac{\theta}{2} \hat{\phi}, \quad \theta > 0,
\]

where \( g \) is the magnetic charge of the configuration. These two expressions differ by a gauge transformation in the region of overlap.

Of course, the angular momentum is conserved in this theory, although it picks up an anomalous term from the monopole field. Defining the covariant momentum operators \( \Pi^i = p^i - eA^i \), we can write the conserved angular momentum operators as

\[
J^i = \epsilon^{ijk} \Pi^j + S^j - e g \frac{x^j}{r}.
\]

It is useful to re-parametrize the spin in terms of a Grassman coordinate \( \psi_j \). The \( \psi_j \) satisfy

\[
\{ \psi_i, \psi_j \} = \delta_{ij},
\]

and they are introduced into the Hamiltonian via the identification

\[
S^j = -\frac{i}{2} \epsilon^{ijk} \psi_k \psi_j.
\]

The \( S^j \) satisfy the spin \( \frac{1}{2} \) commutation relations. These Grassman variables will be central to the analysis throughout this paper.

Suppose now we define the fermionic charge \( \tilde{Q} \) via the expression

\[
\tilde{Q} = \epsilon_{ijk} \left( x^i \Pi^j \psi_k - \frac{i}{3} \psi_j \psi_k \psi_i \right).
\]

It is a straightforward computation to show that

\[
[H, \tilde{Q}] = 0,
\]

leading to the conclusion that there is an exotic conserved fermionic charge in this theory. This charge, however, is not an ordinary supercharge. It does not square to the Hamiltonian (indeed, dimensionally, it could not). Its square is, however, a bosonic charge already known to be present in the theory. One immediately verifies that

\[
\tilde{Q}^2 = \frac{1}{2} \left( J^2 - e^2 g^2 + \frac{1}{4} \right).
\]
This is important, in that it tells us that \( \tilde{Q} \) does not lead to a whole new elaborate symmetry structure; rather, this exotic fermionic charge fits inside the standard symmetry structures in the minimal possible way.

Such a charge was observed in [4] in the context of a supersymmetric theory, and explained as arising from the interplay of the standard supersymmetry and a Killing-Yano structure in the theory, a special instance of [5]. However, here we have found such a charge in the absence of supersymmetry, and so the explanation of [4] cannot be adequate. Thus we are left to seek the origin of this charge in this non-supersymmetric theory. How could we have known to look for it? Why should it appear?

3. Algebraic relations among operators

Before we attempt to establish a connection between the supersymmetric and non-supersymmetric theories, we wish to start by exploring some of the properties of operators that will turn out to be relevant to both theories. From the properties of these operators, we will be able to motivate the discovery and construction of the \( \tilde{Q} \) charge in the non-supersymmetric theory.

Suppose there is an exotic fermion charge in a non-relativistic quantum theory. What will its properties be?

Let us consider the simplest possibility. If there is to be such a charge, the simplest possibility is that there is only one such charge, and hence it must be a scalar under the rotation group. Its square must then be a conserved bosonic scalar charge, and since this must not be the Hamiltonian (we are attempting to construct an exotic fermionic charge, not a conventional supercharge); if we are to add no more structure than necessary, the fermionic charge should square (up to irrelevant constants) to the Casimir of the rotation group. Then, since \( \left[ J^2, f(r) \right] = 0 \) for any function \( f(r) \), we also have \( \left[ \tilde{Q}^2, f(r) \right] = 0 \), and then the Jacobi identity gives

\[
\left\{ \tilde{Q}, \left[ \tilde{Q}, f(r) \right] \right\} = 0. \tag{3.1}
\]

The simplest way for this to be true for any \( f(r) \) is for \( \tilde{Q} \) to commute with \( r \).

Now there turns out to be a fermionic precursor of \( r \) in this theory. Let us define \( \Gamma = x \cdot \bar{\psi} \). Then \( \Gamma^2 = r^2 \), and hence we could adopt the ansatz \( \{ \tilde{Q}, \Gamma \} = 0 \). Then 10 is naturally satisfied.

The Hamiltonian framework makes it natural to consider the conjugate operator to \( \Gamma \), namely \( Q = \Pi \cdot \psi \). For convenience, we also define \( W = \{ Q, \Gamma \} \): one sees readily that \( W \) essentially measures the engineering dimension of an operator.

We know that the essential ingredient of quantum mechanics comes from the action of \( x^i \) on \( \Pi^j \), or equivalent of \( \Pi^j \) on \( x^i \). What happens if we consider the corresponding fermi-contracted operators here?

If we consider the repeated action of \( Q \) on operators, starting with \( \Gamma^2 = r^2 \), one finds

\[
Q : \Gamma^2 \rightarrow \Gamma \rightarrow W \rightarrow Q \rightarrow Q^2 \rightarrow 0. \tag{3.2}
\]

On the other hand, we can switch the roles of \( \Gamma \) and \( Q \), and then

\[
\Gamma : Q^2 \rightarrow W \rightarrow \Gamma \rightarrow \Gamma^2 \rightarrow 0. \tag{3.3}
\]

(In these expressions, we have omitted overall normalization factors that are irrelevant to our argument.) One notices a complete parity between these two chains. Under the action of \( Q \), one starts at \( \Gamma^2 \) and proceeds all the way to \( Q^2 \) before reaching zero; under the action of \( \Gamma \), one starts at \( Q^2 \) and proceeds all the way to \( \Gamma^2 \) before reaching zero. The roles of \( Q \) and \( \Gamma \) are switched. Note, too, that they act in reverse directions, each one (roughly speaking) undoing the action of the other.

Now let us again consider \( \tilde{Q} \). If it is to be square to the Casimir of the rotation group, it should have engineering dimension zero, and thus should be composed of terms that have an equal number of \( x^i \)'s and \( \Pi^j \)'s. Indeed, the charge \( Q \) should be unchanged under the canonical transformation \( x^i \rightarrow \Pi^j \), \( \Pi^j \rightarrow -x^i \). Now we have already argued that it is natural to have \( \tilde{Q} \) commute with \( \Gamma \). Given the symmetry between the algebraic relations 11 and 12, it is natural to conjecture, then, that any conserved charge \( \tilde{Q} \) will also commute with \( Q \).

Thus, a natural way to try to identify any possible exotic fermionic charge is to construct a dimension zero operator, unchanged under the canonical transformation described above, that commutes with \( \Gamma \).
and $Q$. In fact, the charge $\hat{Q}$ defined in the previous section meets all these criteria. As we will see in the next section, it is exactly these properties that forge the link between the supersymmetric and non-supersymmetric theories.

4. Connection to the supersymmetric theory

We can re-write the Hamiltonian in a more compact, and more instructive, form. Using the covariant derivative $\Pi^i$, one can write the Hamiltonian 1 as

$$ H = (\Pi \cdot \psi)^2 + V(r). \quad (4.1) $$

Then using the previously defined operator $Q = \Pi \cdot \psi$, and introducing a parameter $\alpha$ so we can adjust the magnitude of the potential term, we are led to the Hamiltonian

$$ H_{\alpha} = Q^2 + \alpha V(r). \quad (4.2) $$

Note that the case $\alpha = 0$, which describes a spin-$\frac{1}{2}$ particle moving only in the field of a magnetic monopole, is thus an example of a supersymmetric Hamiltonian, with $Q = \Pi \cdot \psi$ as the supercharge [7]. When $\alpha \neq 0$, we recover the generic case this paper has been considering, which has no supersymmetry.

Consider first the theory with Hamiltonian $H_0 = Q^2$, which arises when one sets $\alpha = 0$. This theory automatically has an exotic fermionic charge, as demonstrated in [4]. This is automatic, because the theory has a Killing-Yano tensor. From this, the appearance of the conserved $\hat{Q}$ follows automatically by considering the action of the ordinary supersymmetry on the Killing-Yano tensor [5]. This explanation says that the conserved charge $\hat{Q}$ appears in part because of the existence of the ordinary conserved supercharge.

How do we explain, then, the existence of a conserved $\hat{Q}$ when $\alpha \neq 0$ and there is no ordinary supersymmetry? How does this non-supersymmetric extension of the original theory preserve this one aspect of the symmetry structure?

The answer is that adding the spherically symmetric potential deforms the theory in just the right way. It is true that it is a deformation that violates the supersymmetry invariance of the theory, and thus the Killing-Yano argument for the appearance of the extra fermionic charge breaks down. However, although this deformation violates supersymmetry, it does commute with $\hat{Q}$, and thus we are deforming a theory which has a natural supersymmetric explanation for the conservation of $\hat{Q}$ in a way that, although it violates supersymmetry, respects the $\hat{Q}$ conservation law. Thus the non-supersymmetric extension preserves some of the algebraic structure of the original supersymmetric theory, in particular, the existence of the exotic fermionic conserved charge.

Thus arguments based on the supersymmetry algebra can explain the appearance of $\hat{Q}$ as a conserved charge in the non-supersymmetric theory; we thus have identified another instance of so-called "$N = 0$ supersymmetry," in which supersymmetry is used to determine the properties of a non-supersymmetric theory. In fact, in this case, the term is especially apt. The phenomenon we are witnessing is much like that which occurs when new terms are added to an $N = 2$ supersymmetric theory, breaking one but not both of the supersymmetries, thereby reducing the invariance to $N = 1$ supersymmetry. Here we are seeing the same sort of reduction in the number of supersymmetries, although it is a reduction from $N = 1$ to $N = 0$.

5. Conclusion

We have seen that the appearance of an exotic fermionic conserved charge — one that does not square to the Hamiltonian, as would an ordinary supersymmetry charge — can occur in a non-supersymmetric theory. Heretofore, such exotic charges had been found and explained in explicitly supersymmetric contexts.

At the same time, we have seen that the appearance of this charge in the non-supersymmetric theory can still be understood by examining the algebraic structure of the theory, and in particular by understanding how the non-supersymmetric theory can be viewed as a particular kind of deformation of a supersymmetric theory.

The above is yet another nice example of a way to link the behavior of supersymmetric and non-supersymmetric theories, and indeed it parallels very nicely the discussion of extended superalgebras and topological charges in [2]. In both cases, one has a
quantity (the topological charge or $\bar{Q}$, respectively) that is conserved in the supersymmetric and non-supersymmetric cases; this quantity is conserved in the supersymmetric case due to the interplay of the supercharge and a geometrical quantity (the gauge-like potential or the Killing-Yano tensor, respectively); and the change to the non-supersymmetric case preserves the conservation law (due to its being topological or due to the nature of the deformation, respectively). It would be interesting to see if it is more than coincidental that both these examples revolve around the supermultiplet of geometrical structures associated with symmetries and conservation laws that arise in the presence of monopoles.

There is another possibility raised by the symmetry between the chains of operator transformations $11$ and $12$, namely the possibility of a simple proof of integrability for non-relativistic monopole systems. The symmetry between these chains even when the Hamiltonian does not have a symmetry under the interchange of $\Gamma$ and $Q$ suggests a possible tool for the construction of a second Hamiltonian structure.

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References


\(^2\) There is also a certain similarity to [8], in which a supersymmetric theory is deformed in two different ways, one that respects and one that violates supersymmetry, and the properties of these different theories are related to each other.
No spontaneous breakdown of chiral symmetry in Nambu–Jona-Lasinio model

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Abstract

We argue that the spontaneous breakdown of symmetry in the chirally symmetric Nambu–Jona-Lasinio model which was supposed to illustrate the origin of the low mass of pions in hadron physics does not occur due to strong fluctuations in the $s$–$p$ field space. Although quarks acquire a constituent mass, $s$ and $p$ turn out to have equal heavy masses of the order of the constituent quark mass. © 2000 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

The chirally symmetric Nambu–Jona-Lasinio model [1] was the first theoretical laboratory to illustrate how light pions arise from a spontaneous breakdown of chiral symmetry in hadron physics. The first realistic formulation of the model which included flavored quarks, possessed chiral symmetry $SU(3) \times SU(3)$, and a spectrum of $\sigma, \pi, \rho, A_1$ mesons and their $SU(3)$ partners, was formulated and investigated in 1976 by one of the authors [2], and has been the source of inspiration for many papers in nuclear physics in the past twenty years [3]. By eliminating the Fermi fields in favor of a pair of collective scalar and pseudoscalar fields $\sigma$ and $\pi$, as well as vector and axial vector mesons, a Ginzburg–Landau like collective field action was derived. This had been studied in detail earlier as an effective action guaranteeing all low-energy properties of hadronic strong interactions which were known from current algebra and partial conservation of the axial current (PCAC).

In two important respects, however, the model was unsatisfactory. First it was not renormalizable in four dimensions, but required a momentum space cutoff $\Lambda$ to produce finite results. Moreover, to obtain physical quantities of the correct size, the cutoff had to be rather small, below one GeV, thus limiting the reliability of the predictions to very low energies. Second, if the fermions were identified with quarks, the model could not account for their confinement.

The nonrenormalizability was removed in [2] by replacing the four-fermion interaction by the exchange of a massive vector meson. The different attractive meson channels were obtained by a Fierz...
transformation. The mass of the vector meson took over the role of the cutoff. The energy range of applicability was, however, not increased since the model would still allow for free massive quarks.

The purpose of this note is to point out a much more severe problem with the model which seems to invalidate most conclusions derived from it in the literature: If chiral fluctuations are taken into account in a certain nonperturbative approximation, the spontaneous symmetry breakdown disappears, and the zero-mass pions acquire the same mass as the $\sigma$-mesons, both of the same order as the constituent quark mass. The nonperturbative nature of the argument seems to be the reason why the phenomena has been overlooked until now.

Since the Nambu–Jona-Lasinio model is incapable of accounting for confinement, it gave no reason for introducing colored quarks. It is curious to observe that the restoration of symmetry by chiral fluctuations would offer such a reason, albeit with an unphysical number of colors: the physically desired spontaneous symmetry breakdown conclusion can only be achieved by introducing at least five identical replica of fermions. The existing three colors are insufficient to save the purpose of the model.

The non-perturbative arguments used in this paper are analogous to those applied before in a discussion of the Gross–Neveu model [4] in $2 + \varepsilon$ dimensions [5], where it was shown that this model has two phase transitions, one where quarks become massive and another one where chiral symmetry breaks spontaneously. They have also been applied to explain the experimental observation of two transitions in high-$T_c$ superconductors [6], and to show that directional fluctuations in Ginzburg–Landau theories with spontaneously broken $O(N)$ symmetry disorder the system before size fluctuations of the order field become relevant [7].

2. Nambu–Jona-Lasinio model

Let us briefly recall the relevant features of the Nambu–Jona-Lasinio model for our considerations. The model contains $N_f$ quark fields $\psi(x)$, one for each flavor. Each of them may appear with $N_c$ colors, such that the total number of quarks is $N = N_f \times N_c$. Since the fluctuation phenomenon to be discussed will be caused by the almost massless modes, we may restrict ourselves to the almost massless up and down quarks. We will comment later on the effect of the heavier quarks.

The Lagrangian of the model is given by [3]

$$\mathcal{L} = \bar{\psi}(i\slashed{\partial} - m_0)\psi + \frac{g_0}{N_c} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2 \right],$$

(1)

where an implicit summation over $a = 1, 2, 3$ is assumed. A small diagonal quark mass matrix $m_0$ breaks slightly the $SU(2) \times SU(2)$ part of the chiral symmetry which lifts the mass of the pion to a small nonzero value. We have omitted the flavor symmetric vector gluon exchange used in Ref. [2] which would have given rise, after a Fierz transformation, to additional vector and axial vector interactions, which would not influence the chiral fluctuations to be investigated here. Thus we use the original nonrenormalizable interaction corresponding to an infinite vector gluon mass. The coupling constant in (1) is defined with the number of colors $N_c$ in the denominator, to allow for a finite $N_c \to \infty$ limit of the model at a fixed $g_0$. The $2 \times 2$-dimensional matrices $\lambda_a/2$, with $a = 1, \ldots, 3$ generate the fundamental representation of flavor $SU(2)$, and are normalized by $\text{tr}(\lambda_a \lambda_b) = 2 \delta_{ab}$.

Via a Hubbard–Stratonovich transformation, the Lagrangian (1) is converted into a theory of collective scalar and pseudoscalar fields $\sigma$ and $\pi_a$. Defining the propagator in the presence of the meson fields

$$G = \frac{i}{i\slashed{\partial} - m_0 - \sigma - i\gamma_5 \lambda_a \pi_a},$$

(2)

and adding external quark sources $q, \bar{q}$, one can integrate out the quark fields from the corresponding Lagrangian. Summing over colors, the generating functional of the Green functions takes the well-known form

$$\mathcal{Z} = \int [\mathcal{D}\sigma \mathcal{D}\pi] \exp \left\{ iN_c \left[ -i\text{Tr} \ln iG^{-1} \right. \right.$$

$$\left. - \frac{1}{2g_0} \int d^4x \left( \sigma^2 + \pi_a^2 \right) \right.$$ 

$$+ iN_c^{-1} \int d^4x d^4y \bar{q}(y) G(q) \right\}.$$ 

(3)
The term inside the brackets is the collective field action \( \mathcal{A}[\sigma, \pi] \), whereas the symbol \( \text{Tr}' \) includes both, the functional spacetime "'index'" \( x \), and the internal trace over spin and flavor indices: \( \text{Tr}' \equiv \int d^D x \text{tr}_s \text{tr}_f \).

By extremizing \( \mathcal{A}[\sigma, \pi] \) at zero sources \( \eta, \bar{\eta} \), we obtain the field equation for the collective field \( (\sigma, \pi_s) \):

\[
\text{tr}_s \text{tr}_f \left[ G(x,x) \left( \frac{1}{i\hbar \gamma_5} \right) \right] = \frac{1}{g_0} \left( \sigma(x), \pi_s(x) \right). \tag{4}
\]

For constant fields, this equation becomes a gap equation. Its solutions will be marked by a superscript "s" for "stationary phase approximation". From now on, unless explicitly stated, we shall consider the model with zero mass, \( m_0 = 0 \). The stationary pseudoscalar solutions \( \pi_s \) can always be chosen to be vanishing, while the scalar solutions can be \( \sigma^s = 0 \), or \( \sigma^s \equiv \rho_0 \). In the first case, the ground state is chirally symmetric, in the second the symmetry is spontaneously broken. This is the state of physical interest whose stability will now be discussed.

### 3. Effective potential and gap equation

In the limit \( N_c \to \infty \), the generating functional is given exactly by the extremal field configurations, which will be parameterized as \( (\sigma^s(x), \pi_s(x)) = (\rho_0(x), 0) \). The system has an effective action per quark

\[
\frac{\Gamma(\rho_0, \Psi, \bar{\Psi})}{N_c} = -i \text{Tr}' \ln i G_{\rho_0}^{-1} - \frac{1}{2g_0} \int d^D x \rho_0^2 + \frac{1}{N_c} \int d^D x \bar{\Psi} i G_{\rho_0}^{-1} \Psi_\rho_0,
\]

where \( \Psi = i G_{\rho_0} \eta \) is the expectation value \( \langle \Psi \rangle \) of the quark field, and \( G_{\rho_0} \) its propagator

\[
G_{\rho_0} = \frac{i}{i\slashed{D} - \rho_0}. \tag{6}
\]

This shows that the solution of the gap equation with \( \rho_0 \neq 0 \) describes quarks with a nonzero mass \( M = \rho_0 \), which has been generated by the spontaneous symmetry breakdown, and is referred to as the constituent quark mass. In the present approximation of zero bare mass \( m_0 \), the constituent quark mass is about equal to 300 MeV for up and down quarks (see the discussion in Refs. [2,8]). In either case, the Green function (2) in the stationary field is diagonal in flavor space.

In the absence of external quark sources, the ground state expectation value of a fermion field is always zero, and the expectation value \( \rho_0(x) \) is constant, so that (5) reduces into

\[
\frac{\Gamma(\rho)}{N_c} = -i \text{Tr}' \ln i G_{\rho}^{-1} - \frac{1}{2g_0} \int d^D x \rho^2,
\]

where we have allowed the fields \( \sigma \) and \( \pi_s \) to be nonextremal, defining \( \sigma^2 + \pi_s^2 \equiv \rho^2 \), and reserving the notation \( \rho_0^2 \) for the extremum. This is determined, after a Wick rotation to euclidean momenta \( p_E \) with \( \rho_0 = i p_{E,0} \), \( d^D p \to -d^D p_E \), \( p^2 \to -p_E^2 \), by the gap equation

\[
\frac{1}{g_0} = 2 \times 2^{D/2} \int \frac{d^D p_E}{(2\pi)^D} \frac{1}{p_E^2 + \rho_0^2}. \tag{7}
\]

We have divided the two sides of the gap equation by a common factor \( g_0 \), since we want to study the spontaneously broken phase.

The gap equation must be regularized, which may be done in many ways. Here, we shall use two methods: analytic continuation in the dimension \( D \), and a cutoff \( \Lambda \) in momentum space. The former is mathematically more elegant and has the advantage of relating the properties in four dimensions to those in \( 2 + \epsilon \). It has, however, some unphysical properties which require special attention, as we shall see. Such problems are absent in a cutoff regularization scheme, which exhibits clearly the physical divergences caused by the infinite number of degrees of freedom of the field system. Factorizing the integral in (8) into direction and size of the momentum \( p_E \), we bring the gap equation to the form

\[
\frac{1}{g_0} = 2 \rho_0^{D-2} \frac{\Gamma(1-D/2)}{(2\pi)^{D/2}}. \tag{9}
\]

Denoting by \( \Omega \) the \( D \)-dimensional volume \( \int d^D x \), the volume density \( \varepsilon(\rho) \equiv -\Gamma(\rho)/\Omega \) of the effective action (7) is the effective potential per quark. Performing the internal traces, and subtracting a
divergent constant term associated with the chirally symmetric state with $\rho_0 = 0$, we obtain the condensation energy in euclidean space:

$$\Delta \varepsilon (\rho) = \frac{N_c}{2} \left[ \frac{1}{g_0} \rho^2 - \rho^0 \frac{A^2}{2} \right].$$  

(10)

In an even number of dimensions $D$, both the gap equation (9) and the effective potential (10) are divergent, due to a pole in the factor $\Gamma (1 - D/2)$. Introducing the diverging parameter $b_\varepsilon = 2 \Gamma (1 - D/2) / [D (2\pi)^{D/2}]$, we can rewrite the gap equation and effective potential in the more compact form as

$$\frac{1}{g_0} = D \rho_0^{D-2} b_\varepsilon,$$  

(11)

$$\Delta \varepsilon (\rho) = \frac{N_c}{2} \left[ \frac{1}{g_0} \rho^2 - 2 \rho^0 b_\varepsilon \right].$$  

(12)

In the more physical regularization with a cutoff $\Lambda$ in momentum space, these expressions look more complicated:

$$\frac{1}{g_0} = \frac{2}{(2\pi)^2} \left[ \Lambda^2 - \rho_0^2 \ln \left( 1 + \frac{A^2}{\rho_0^2} \right) \right].$$  

(13)

$$\Delta \varepsilon (\rho) = \frac{N_c}{2} \left[ \frac{1}{g_0} \rho^2 - \frac{2}{(2\pi)^2} \left[ \rho_0^2 \frac{A^2}{2} \right] \right.$$  

$$\left. + \frac{\Lambda^4}{2} \ln \left( 1 + \frac{\rho^2}{\Lambda^2} \right) - \frac{\rho^4}{2} \ln \left( 1 + \frac{A^2}{\rho^2} \right) \right].$$  

(14)

The results (11) and (12) of the analytic regularization scheme can be mapped roughly into the cutoff results (13) and (14) if we recall the special property of dimensional regularization that all integrals over pure momentum powers vanish identically: $\int d^n k k^n = 0$ (Veltman's rule). Thus, arbitrary pure powers of the cutoff $A^{n+D}$ have no counterpart in dimensional regularization. Only logarithmic divergences can be related to diverging pole terms $1/\varepsilon \to \infty$ for $\varepsilon \to 0$. It is therefore inconsistent to relate $\varepsilon$ to $\Lambda$ by setting $\Gamma (\varepsilon/2 - 1) = A^2/\rho_0^2$, as proposed by Krewald and Nakayama [9]. Only the logarithmic divergence in (13) can be mapped to the small-$\varepsilon$ divergence in (11), setting $\Gamma (\varepsilon/2 - 1) = -\ln (1 + A^2/\rho_0^2)$. With their inconsistent identification, Krewald and Nakayama matched $\Lambda$ by an $\varepsilon > 2$ which lies in the wrong region $D < 2$, the physically relevant range being $D \in (2 + \varepsilon, 4 - \varepsilon)$.

Note that the matching of the logarithm at the level of the effective potential leads to the properly matched gap equation, thus having circumvented the unphysical properties of the analytic regularization.

The free use of this scheme in renormalizable field theories relies on the fact that all infinities are eventually absorbed in unobservable bare quantities, such that the artificial zeros of the integrals over pure powers of momenta cannot produce problems. In nonrenormalizable theories, on the other hand, only a cutoff (or a related Pauli–Villars regularization) is physical, and analytic regularization must be treated with caution. This is seen even more dramatically in integrals which do not have logarithmic infinities. For example the condensation energy (10) in $D = 3$ dimensions would be a finite negative number in analytic regularization, while being a linearly divergent positive function of the cutoff.

4. Chiral fluctuations

Since the physical number of quarks $N_c$ is finite, the fields perform fluctuations of magnitude $1/\sqrt{N_c}$ around their extremal value. As long as $N_c$ can be considered as a large number, the deviation from the extremal field configuration $(\sigma', \pi'_c) \equiv (\sigma - \rho_0, \pi'_c)$ are small, and the action can be expanded in powers of $(\sigma', \pi'_c)$. The quadratic terms in this expansion define the propagators of the collective fields $(\sigma', \pi'_c)$. The higher expansion terms of the trace of the logarithm in (3) define the interactions. With this decomposition, the inverse of the quark propagator (2) can be decomposed into a constant and a fluctuating part, setting $iG^{-1} = iG^{-1}_0 - (\sigma' + i\gamma_5 \lambda_c \pi'_c)$, with $G^{-1}_0$ of Eq. (6). Then we have

$$\text{Tr} \ln iG^{-1} = \text{Tr} \ln iG^{-1}_0 - \frac{\lambda}{4} \text{Tr} \ln \left[ 1 + iG^{-1}_0 (\sigma' + i\gamma_5 \lambda_c \pi'_c) \right].$$  

(15)

An expansion of the last term up to the second order in the fields gives an approximate partition function
In for the pseudoscalars, and to 

\[ q \]

extra term \( r \)

In this expression, the gap equation \( 8 \) has been 

collective free field propagators \( G_\sigma, G_\pi \). In momentum 

space, we identify

\[
\mathcal{A}_0[\sigma', \pi'] = \frac{1}{2} \int d^D q \left[ \pi'_x(q) G_\sigma^{-1} \pi'_x(-q) + \sigma'(q) G_\pi^{-1} \sigma'(-q) \right].
\]  

(17)

where

\[
G_{\sigma, \pi}^{-1} = 2 \times 2^{D/2} N_c \int_0^1 dy \int \frac{d^2 p_E}{(2\pi)^2} \frac{q_E^2 + p_E q_E + (2\rho_0^2, 0)}{\left( (q_E^2 + 2 p_E q_E)^2 + p_E^2 + \rho_0^2 \right)^2}. 
\]  

(18)

In this expression, the gap equation \( 8 \) has been 

used to eliminate the term \( 1/g_0 \). The notation \( (2\rho_0^2, 0) \) 

indicates that only the equation for \( \sigma \) contains an 

extra term \( 2\rho_0^2 \).

In four spacetime dimensions, the integral evaluated 

dimensional regularization reduces to \( q_E^2/2 \) for 

the pseudoscalars, and to \( (q_E^2 + 4\rho_0^2)/2 \) for 

the scalars, both with a diverging coefficient. The first 

leads to a zero mass for pions as a manifestation of 

Goldstone’s theorem, the second to a mass equal to 
twice the constituent quark mass for the \( \sigma \)-mesons. 

For a finite result, the integrals must be regularized. 

In \( D = 4 - \epsilon \) dimensions, the inverse euclidean 

propagator is seen to start out for small \( q_E^2 \) like

\[
G_\sigma^{-1} = N_c \left( 1 - \frac{D}{2} \right) \frac{1}{\rho_0} \frac{q_E^2}{\rho_0^2} \frac{q_E^2}{2} 
\]

\[
= Z_\sigma^{(\epsilon)}(\rho_0) q_E^2 + \mathcal{O}(q_E^4).
\]  

(19)

with the same \( b_\epsilon \) as defined above Eq. \( 11 \). If the 

theory is regularized with a cutoff \( \Lambda \) in 

momentum space, this becomes

\[
G_\sigma^{-1} = \frac{N_c}{(2\pi)^2} \left[ \ln \left( 1 + \frac{\Lambda^2}{\rho_0^2} \right) - \frac{\Lambda^2 - \rho_0^2}{\Lambda^2 + \rho_0^2} \right] q_E^2 
\]

\[
= Z_\sigma^{(\Lambda)}(\rho_0) q_E^2.
\]  

(20)

In the right-hand part of the two equations, the 
factors in front of \( q_E^2 \) have been identified as the 

wave function renormalization constants \( Z_\sigma(\rho_0) \) of 

the pion field in the two regularization schemes.

As a consequence of the spontaneous symmetry 

breakdown, the fluctuations of the pseudoscalar fields 

are massless. These fields appear in the \( x \)-space 

version of the action \( 17 \) in a pure gradient form

\[
\mathcal{A}_0[\pi'] = \frac{\beta}{2} \int d^D x \left[ \partial \pi'_x(x) \right]^2.
\]  

(21)

with \( \beta = Z_\sigma \). Due to chiral symmetry, this gradient 

action can be extended to the gradient action of an 
arbitrary field \( (\sigma, \pi) \). Introducing the directional 

unit vector fields \( \nu_i = (\dot{\sigma'}, \ddot{\pi}') \equiv (\sigma', \pi')/\rho \), we 

find:

\[
\mathcal{A}_0[\nu_i] = \frac{\beta}{2} \rho^2 \int d^D x \left( \partial \nu_i(x) \right)^2, \quad i = 1, \ldots, N_c,
\]  

(22)

with \( N_c = 4 \) and

\[
\beta(\rho) = Z_\sigma(\rho).
\]  

(23)

This chirally invariant action describes the massless 

pions with all multipion interactions.\footnote{Only two approximations are involved: the first one consists in freezing the size \( \rho \) of the fluctuations. The second one neglects corrections due to the finiteness of the sigma mass. The latter corrections are expected to be of the order \( f_p^2/(4M^2) = 3\% \).}

The notation \( 2 \) contains an 

tension of the action \( 17 \) in a pure gradient form

\[ 19 \) shows that the stiffness of pion fluctuations 

in \( D = 2 \) dimensions becomes

\[
\beta = \frac{N_c}{2\pi \rho_0^2},
\]  

(24)

thus coinciding with the stiffness calculated in Ref. 

\cite{5} in the Gross–Neveu model (which contained a
factor \( N \) to be identified with the present \( N_c \times N_c = 2N_c \).

With the more physical cutoff regularization in \( D = 4 \) dimensions, the stiffness of directional fluctuations is

\[
\beta = \frac{N_c}{\rho^2} \left\{ \ln \left[ 1 + \left( \frac{A^2}{\rho_0^2 + \Lambda^2} \right)^2 \right] - \frac{\Lambda^2}{\rho_0^2 + \Lambda^2} \right\}.
\] (25)

This is the crucial quantity leading to our fatal conclusions for the restoration of chiral symmetry. The stiffness (25) is far too small to let the directional field settle in a certain direction, required for spontaneous symmetry breakdown. The disordering effect of phase fluctuations is well-known from many model studies of the \( O(4) \)-symmetric Heisenberg model on a lattice. High-temperature expansions and Monte Carlo simulations have shown that there exists a critical stiffness below which the system goes over into a disordered state.

For an analytic estimate of the critical stiffness, we relax the unit vector constraint for the vectors \( n_i \) in (22) by introducing an additional field \( \lambda(x) \) playing the role of a Lagrange multiplier. The \( n_i \) fields can then be integrated out in the partition function, leading to an action

\[
S = \frac{N_c}{2} \text{Tr} \ln \left( -\partial^2 + \lambda(x) \right)
- \beta (\rho^3) \rho^2 \int d^D x \frac{\lambda(x)}{2},
\] (26)

where \( \text{Tr} \) denotes the functional trace (the summation over the fields component has already been performed). For a large number \( N_c \) of components, the fluctuations are suppressed, and the field \( \lambda(x) \) becomes a constant satisfying a second gap equation

\[
\beta = \frac{N_c}{\rho^2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + \lambda}.
\] (27)

If there is a nonzero solution \( \lambda \neq 0 \), this will play the role of a square mass of the \( n_i \)-fluctuations, and represents an order parameter in the directional phase transition. The model has a phase transition at a critical stiffness

\[
\beta_c = \frac{N_c}{\rho^2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + \lambda}.
\] (28)

For a smaller stiffness, the phase fluctuations are so violent that the system goes into a disordered phase with \( \lambda \neq 0 \) giving all fields \( n_i \) a nonzero square mass \( \lambda \). Since the fields \( n_i \) are the normalized \( \sigma \) and \( \pi_a \) fields of the model, this determines an equal nonzero square mass of \( \sigma \) and \( \pi_a \) mesons, and thus a restoration of chiral symmetry.

Note that the quarks are still massive: their constituent mass is a consequence of the formation of the pairs, which are strongly bound for small \( N_c \). The phase transition taking place at the critical value of the stiffness, on the other hand, is related to the Bose–Einstein condensation of the pairs. At small \( N_c \), the two processes are widely separated. This separation of the two transitions (pair formation and pair condensation) can be judged by the simple fluctuation criterion in Ref. [7].

In our model, the number \( N_c \) is equal to four, which is not very large. Fortunately, Monte Carlo studies of the model [13–15] have shown that \( N_c = 4 \) is large enough to ensure the existence of the transition and the quantitative reliability of the theoretical estimate of the critical stiffness (28). From an evaluation of (28) on a lattice, and a comparison with Monte Carlo studies, we estimate that the critical stiffness obtained from (28) is correct to within less than 2\% [14] \(^3\) or 6\% [13,15] \(^4\). The same maximal error is expected if we work in the continuum using a momentum cutoff scheme.

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\(^3\) Simulations of the four-dimensional \( O(4) \) model on a simple-cubic lattice gives \( \beta^\alpha = 0.6090 \) [14]. This is to be compared with the approximation (28) calculated for a simple-cubic lattice, where \( \beta^\alpha = 4 \times 0.1549 \approx 0.6196 \) which is thus correct to within less than 2\%. (For Refs. [13,15], we find \( \beta^\alpha = 0.584 \) (6\%) and \( \beta^\alpha \approx 0.64 \) (3\%), respectively.) The value 0.1549 is taken from the list of lattice Coulomb potentials at the origin in Table 6.4 of the textbook [12] on p. 178.

\(^4\) See footnote 3.
For $N_s = 4$ and a cutoff $\Lambda_p$ in the integral (28) over pion momenta, the critical stiffness is given by

$$\beta_c = \frac{4}{16\pi^2} \frac{\Lambda_p^2}{\rho^2}. \quad (29)$$

By comparing this with the stiffness of the model in (25), we find

$$N_c = \left( \frac{\Lambda_p}{\Lambda} \right)^2 \left( \frac{\Lambda}{\rho_0} \right)^2 \ln \left[ 1 + \left( \frac{\Lambda}{\rho_0} \right)^2 \right] - \frac{(\Lambda/\rho_0)^2}{1 + (\Lambda/\rho_0)^2}^{-1}. \quad (30)$$

This equation determines the number $N_c$ of identical quarks which is necessary to produce a large enough stiffness $\beta$ to prevent the restoration of chiral symmetry. Only if the number of colors exceeds this critical value, will the model possess a phase in which the pion is a massless Goldstone boson, and $\sigma$ a meson with a mass twice as large as that of the constituent quarks. The critical number (30) is plotted as the solid curve in Fig. 1 for $\Lambda_p = \Lambda$. We see that $N_c = 5$ would be the smallest allowed value. This number, however, is incompatible with color SU(3). This suggests that the Nambu–Jona-Lasinio model always remains in the symmetric phase, due to chiral fluctuations. It can therefore not be used to describe the chiral symmetry breakdown of quark physics, as has been claimed by many publications, which have appeared in particular in nuclear physics [3].

Can this conclusion be avoided by a different choice of parameters? To obtain a critical value smaller than $N_c = 3$ would require a pionic cutoff $\Lambda_p \leq 0.8\,\Lambda$. However, the cutoff cannot be chosen at will. Let us study the cutoff dependence more precisely. For this, we refine the previous crude estimate (28), (29) of the critical stiffness, which will henceforth be called Approximation 1, by taking better account of the shorter wavelength fluctuations, replacing the action (22) by

$$\mathcal{A}[\pi] = \frac{\rho^2}{2} \int d^3x \, \pi(x) G_{\pi}^{-1}(-\partial^2) \pi(x), \quad (31)$$

with $G_{\pi}^{-1}(-\partial^2)$ from Eq. (18). This exchanges $1/k^2$ in Eq. (28) by the full pion propagator $G_{\pi}(k^2)/Z_{\pi}(k^2)$ associated with the action (31). The cutoff $\Lambda_p$ makes the integral over pion momenta finite. Its size is fixed by physical considerations. The pion fields in the symmetry-broken phase are composite, and will certainly not be defined over length scales much shorter than the inverse binding energy of the pair wave function, which is equal to $2M = 2\rho_0$. Thus we perform the integral in the modified Eq. (28) up to the cutoff $4M^2$. This is Approximation 2, yielding the solid curve in Fig. 3.

The phase with broken symmetry for three colors would be reached only if the quark loop integration is cut off at $\Lambda^2 \geq 11M^2$. Such a large value, however, is incompatible with the experimental value of the pion decay constant $f_\pi \simeq 0.093$ which is given, in the large-$N_c$ limit of the model, by $f_\pi/M = \Lambda^{1/2}(M)$. For typical estimates of constituent quark masses $m \in (300, 400)\,\text{MeV}$ [2], we find that $\Lambda^2/M^2$ should lie in the range (3.3, 7.3), the highest value corresponding to the lowest possible mass 300 MeV.
The above study has given us only the critical point, where the pion mass goes to zero. We can do more and determine the common nonzero square masses $m_s^2 = m_n^2 = \lambda$ of $\sigma$ and $\pi_\rho$-fields in the phase of restored chiral symmetry. This is the subject of the next section.

5. Meson masses

The chiral fluctuations give rise to a change of the effective potential. They add to $\Delta \mathcal{V}(\rho, \lambda)$ in Eq. (14) an additional energy coming from the stationary point of the action (26) at a constant $\lambda(x) = \lambda$:

$$\Delta_1 \mathcal{V}(\rho, \lambda) = -\frac{i}{2} \chi Z_0 \rho^2 + \frac{N_s}{2} \int_0^{x_{\mu}^2} \frac{d q_E^2}{16 \pi^2} \ln \left[ q_E^2 + \lambda \right].$$

$$\Delta_2 \mathcal{V}(\rho, \lambda) = -\frac{i}{2} \lambda Z_0 \rho^2$$

$$+ \frac{N_s}{2} \int_0^{x_{\mu}^2} \frac{d q_E^2}{16 \pi^2} \ln \left[ G^{-1}(q_E^2)/Z(\rho) + \lambda \right],$$

for Apprs. 1 and 2, respectively, where the latter has $-\sqrt{2}$ replaced by $G_\sigma^{-1}(q^2)/Z(\rho)$. Extremizing $\Delta_1 \mathcal{V}(\rho, \lambda) + \Delta_2 \mathcal{V}(\rho, \lambda)$ yields two coupled gap equations replacing the independent gap equations (13) and (27). Introducing the reduced quantities $\tilde{Z}(x) = \ln(1 + x^{-1}) - (1 + x)^{-1}$, and $x = \rho^2/\Lambda^2$, $y = \lambda/\Lambda^2$, we have for Appr. 1:

$$x_0 \ln(1 + x_0^{-1}) + \frac{y}{2} \frac{d}{dx} \ln \left[ \frac{\tilde{Z}(x)}{Z(x)} \right] = x \ln(1 + x^{-1}),$$

$$N_s x \tilde{Z}(x)$$

$$= \left( \frac{N_s}{4} \right) (\frac{A_\sigma}{A})^2 + y \ln \left[ 1 + \left( \frac{A_\pi}{A} \right)^2 x^{-1} \right].$$

For Appr. 2, the coupled gap equations are more complicated since the full $q^2$-dependence of $Z_n$ has to be taken into account. They read

$$x_0 \ln(1 + x_0^{-1}) + \frac{N_s}{8 N_c} \int_0^{1} \frac{d k^2}{k^2} \left[ \tilde{Z}(x_0) \right]$$

$$\times \frac{d}{dx_0} \left[ \frac{\tilde{Z}(k^2, x_0)}{Z(k^2, x_0)} \right] + \frac{y}{2} \frac{d}{dx} \left[ x \frac{\tilde{Z}(x)}{Z(x)} \right]$$

$$= x \ln(1 + x^{-1})$$

$$- \frac{N_s}{8 N_c} \int_0^{1} \frac{(A_{\pi}^2/\Lambda^2)}{k^2} \left[ \frac{k^2 dk^2}{\tilde{Z}(k^2, x) / Z(k^2, x)} \right] + y,$$

$$N_s x \tilde{Z}(x)$$

$$= \left( \frac{N_s}{4} \right) \int_0^{1} \frac{p^2 dp}{p^2} \int_0^{1} \frac{(1 - z) dz}{p^2 + k^2 z (1 - z) + x^2}.$$
case $\Lambda_s = \Lambda$. Qualitatively, the pictures remain the same for different ratios $\Lambda_s/\Lambda$. Quantitatively, there is only a shift in the critical number of color (solid curve of Fig. 1) to $N^{cr}_c = 3$ as $\Lambda_s/\Lambda$ is lowered to 0.8, while it increases above the given curve if $\Lambda_s/\Lambda > 1$. This is due to the fact that at the critical point corresponding to $\lambda = 0$, one sees from Eq. (30) (or from Eq. (35) with $y = 0$) that $N^{cr}_c \propto (\Lambda_s/\Lambda)^2$.

The dashed curves of Figs. 1 and 2 are explained in the corresponding legends. Here we only remark that the shape of the dashed curves in Fig. 1 can be understood from the gap equations (34) and (35) without solving them, because $x\tilde{Z}(x)$ is maximal at the minimum of $N^{cr}_c$.

Figs. 3 and 4 correspond to Appr. 2, in which the full momentum dependence for the pion normalization constant is taken into account, and in which the pionic cutoff is $L^2 = 4M^2$, for which we get the ratio $(\Lambda_s/\Lambda)^2 = 4x_0$. The solid curve in Fig. 3 gives the critical number of color in this particular case. Although the conclusion is not as strong as in Appr. 1, our result concerning the lack of breaking of chiral symmetry is robust, since the crossing with the line $N_c = 3$ takes place at a cutoff $L^2 = 4M^2 \geq 11$, which lies outside of the admissible range (3.3,7.3) implied by the physical value of the pion decay constant $f_\pi = 93$ MeV, as discussed at the end of the previous section.

Finally, we give in Fig. 5 the stiffness as a function of the number of color for Appr. 1. The three curves depend so weakly on $\rho_0$ that they seem to coincide. To make the $\rho_0$-dependence visible, we have plotted an extra dotted curve for a very small value $\rho_0 = 0.224\Lambda$ (dotted).

Let us emphasize that these conclusions cannot be reached in the dimensional regularization scheme since, as explained at the end of Section 3, the integral in (28) determining the critical stiffness vanishes. Here the unphysical nature of dimensional regularization makes its application impossible.
Before concluding, let us also remark that the cutoff chosen in Appr. 2 is completely different from that in Appr. 1, where the ratio of cutoffs is a constant. In Appr. 2, the ratio of cutoffs is a function of $x_0$: $A_s^2/A^2 = 4x_0$. If we had taken the cutoff in the same way as in Appr. 1 ($A_s^2/A^2 = 1$), the curve giving the critical number of colors would also have had the same shape as in Appr. 1, although the integration would have been much more involved: the minimum number of color would then be 5.2, whatever the value of $A^2/p_0^2$, a value which is even higher than in Appr. 1. We see that Appr. 2 as presented above, with the physically motivated cutoff $A_s^2 = 4M^2$, gives then the lowest critical number of colors.

Our conclusions were derived from a study of only the $\sigma$, $\pi$ fields. The inclusion of other flavors does not help preventing the restoration since the associated pseudoscalar mesons are quite massive, making their fluctuations irrelevant to the described phenomenon.

6. Conclusion

We have shown that within a certain nonperturbative approximation, the Nambu–Jona-Lasinio model does not really display the spontaneous symmetry breakdown for whose illustration it was constructed. The fluctuations of $\sigma$- and $\pi_r$-fields restore chiral symmetry and make $\sigma$ and $\pi$ equally massive. If our conclusion survives more refined approximations, this would invalidate a large number of publications, especially in nuclear physics, which have been based on the existence of a symmetry-broken ground state of the model. In particular, all studies of the temperature dependence of the symmetry-broken state [3] would deal with nonexistent objects, thus calling for further investigations. Finally, we note that our no-go result for the Nambu–Jona-Lasinio model does not imply problems with the effective-action approach to chiral dynamics. Certainly, there exists an effective chiral action for the meson sector of quantum chromodynamics which does contain almost massless pions for $N_f = 3$. It is only the Nambu–Jona-Lasinio model as it stands which is incapable of describing these for such a low number of colors. In fact, a recent paper [16] prompted by a first version of our preprint points out that an extension of the Nambu–Jona-Lasinio model by interactions involving higher-dimensional operators is not subject to our no-go theorem. Another escape is possible by adding gradient and quartic interaction terms for $\sigma$- and $\pi$-fields to the initial action, thus extending the Nambu–Jona-Lasinio model to a linear sigma model [17].

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Testing lepton number violation with the reaction $e^- e^- \rightarrow \mu \nu q \bar{q}$

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Abstract

We investigate the reaction $e^- e^- \rightarrow \mu \nu q \bar{q}$ as a possible place to test the lepton number violating interactions of singly charged scalars $\Delta_L^Z$ belonging to a SU(2)$_L$ triplet. The existence of such scalars is predicted by some majoron models, as well as by the left-right symmetric extension of the Standard Model. We show that this reaction may be observed in $e^- e^-$ collision well below the threshold of $\Delta_L^Z$. For the colliding energy $\sqrt{s} = 500$ GeV the mass of the singly charged Higgs triplet may be excluded up to 2 TeV and even more, depending on values of the appropriate Yukawa couplings.

Recent experimental results of the Super-Kamiokande collaboration on atmospheric neutrinos [1] strongly indicate that neutrinos oscillate and consequently that at least some of neutrino species have nonzero mass. This gives rise to a series of interesting questions concerning the origin and nature (Dirac/Majorana) of those masses. Specifically, it would be interesting and important to know the structure of Higgs sector responsible for the indicated mass and mixing pattern. For that, it is obviously necessary to go beyond the Standard Model (SM) as in the SM neutrinos are massless. If one does not introduce right-handed neutrinos, one possibility is to extend the scalar sector of the SM with an SU(2)$_L$ triplet $\Delta_L$ with hypercharge $Y_L = 2$. This could yield Majorana mass to left-handed neutrinos through spontaneous symmetry breaking, assuming that $\langle \Delta_L \rangle \neq 0$. This is what happens in some majoron models [2], and the triplet Higgses quite naturally appear also in the left-right symmetric model (LR-model) of electroweak interactions [3,4].

The Higgs triplet $\Delta_L$ with hypercharge $Y = 2$ consists of a doubly charged scalar, a singly charged scalar and two neutral scalars. The gauge invariance allows the triplet $\Delta_L = (\Delta^+,\Delta^0,\Delta^-)$ to interact with leptons via a Yukawa coupling that violates the lepton number $L$ by two units. Due to this property the search of these particles is particularly handy and advantageous in electron-electron collisions where one has $L = 2$ in the initial state.

The phenomenology of the doubly charged Higgs particle has been studied, e.g., in [5]. It was shown that $\Delta_L^-$ with a mass up to 800 GeV may be excluded at the coming LHC experiments. In the present paper we shall concentrate on the search of the singly charged member of the Higgs triplet, $\Delta_L^-$ in $e^- e^-$ collisions at a linear collider [6]. Obviously, due to their different charges, the phenomenological
signatures of the singly charged triplet Higgs differ considerably from those of the doubly charged Higgs. The present experimental data allows to restrict the mass of the singly charged bosons to be above roughly 100 GeV [7]. It may thus happen that the singly charged triplet Higgs turns out to be considerably lighter than its doubly charged counterpart. Hence it is worthwhile to examine its experimental signatures and prospects for obtaining information on its properties independently of the properties of the doubly charged scalar.

There is an important restriction that affects the production and decay rates of the triplet Higgs \( \Delta_L \). The vacuum expectation value of its neutral member, \( \langle \Delta_L^0 \rangle = v_L / \sqrt{2} \), is limited to quite small values in order to avoid a violation of the experimentally well established relation \( \rho \equiv M_Z^2 / (M_G^2 \cdot \cos^2 \theta_W) = 1 \). This restriction remains the same for the LR-model if the \( W_L - W_R \) and \( Z_L - Z_R \) mixings are neglected [8]. The present experimental data indicate that \( v_L \leq 15 \text{ GeV} \). Let us note, that there is an option to extend the Higgs sector further so that the equality \( \rho = 1 \) holds at the tree level due to the so-called "custodial" \( \text{SU}(2)_L \times \text{SU}(2)_R \) symmetry [9]. In this scenario, which will not be considered in the following, there would be more than one charged triplet scalars.

In this paper we will show that the process \( e^- e^- \rightarrow \mu^- \nu \bar{q} \bar{q} \) provides a good environment to study the interactions of the singly charged scalar \( \Delta_L^- \), quite independently of the properties of the doubly charged Higgs \( \Delta_L^{--} \). For the anticipated luminosity of electron-electron collider this process may be observed well below the production threshold of the singly charged Higgs.

We consider the standard \( \text{SU}(2)_L \times \text{U}(1)_Y \) model with an additional Higgs triplet field of hypercharge \( Y = 2 \):

\[
\Delta_L = \begin{pmatrix}
\Delta_L^0 / \sqrt{2} & \Delta_L^+ \\
\Delta_L^+ & -\Delta_L^0 / \sqrt{2}
\end{pmatrix}
(3,1,2)
\]

with the vacuum expectation value:

\[
\langle \Delta_L \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 \\
v_L
\end{pmatrix}
. \quad (2)
\]

This kind of model was first suggested in [2] in order to generate Majorana masses for neutrinos, and it also quite naturally arises as an effective low-energy manifestation of the left-right symmetric model [10]. The couplings of \( \Delta_L \) with the gauge fields are provided by the usual kinetic term in the Lagrangian,

\[
L_{\text{kin}} = \frac{1}{2} \text{Tr}(D_\mu \Delta_L)^+(D_\mu \Delta_L), \quad (3)
\]

where

\[
D_\mu \Delta_L = \partial_\mu \Delta_L + ig' B_\mu \cdot \Delta_L + \frac{ig}{2} W_\mu^a \sigma^a \Delta_L,
\quad (4)
\]

and its interactions with fermions are given by a Yukawa coupling of the form

\[
L_{\text{Yuk}} = -h_{L,a\beta} \psi_{aL}^T C \sigma_2 \Delta_L \psi_{bL} + \text{h.c.}, \quad (5)
\]

where \( \psi \) denotes the lepton doublet (\( \nu, l^- \)) and \( \alpha \) and \( \beta \) are flavour indices. The Yukawa interaction gives rise to Majorana mass terms \( m_{\nu, \alpha} = h_{L,a\beta} \cdot v_L \) for the left-handed neutrinos.

The interactions of the \( \Delta_L \) field described above in (3), (5) are the same for both the SM with additional Higgs triplet and the LR-model. Of course, the full Lagrangian should also include self-interactions of the scalar fields, which are governed by the respective scalar potentials. We will overlook these in the following as a first approximation by assuming no mixing between the doublet and triplet Higgses and the absence of any further Higgs fields.

The phenomenologically most important feature of the models that include triplet Higgses is the lepton number non-conservation arising from the Yukawa coupling (5). It makes the interactions of the triplet Higgses quite complementary to the Yukawa interactions of the SM Higgs doublet, and it provides good opportunities for unambiguous tests of \( \Delta_L^- \) production and decay in \( e^- e^- \) collision experiments. The production of like-sign charged \( \ell^- \ell^- \) collision experiments. The production of like-sign charged Higgs bosons via charged vector boson fusion in several electroweak models was considered in [11]. It was shown that cross section strongly depends on the choice of the Higgs representation and on the parameters of the model. The models described above were, however, not considered in that study. It is impossible to
Fig. 1. Energy dependence of the cross section of the $e^+ e^- \rightarrow \mu^+ \mu^- d\bar{d}$ for different values of the masses of singly charged ($\Delta^+_L$) and doubly charged ($\Delta^-_L$) triplet Higgses.
distinguish the lepton number conserving properties of charged Higgs through this kind of process.

In [11] the production of the singly charged Higgses in $e^- e^-$ collisions was assumed to happen in pairs through a $W^+ W^-$ fusion. This process conserves the lepton number. We are interested, in contrast, in processes that probe the lepton number violating Yukawa couplings (5). The pair production, which proceeds through t-channel exchange of Majorana neutrinos and s-channel exchange of $\Delta L^-$, is in this case not a suitable process to study, however. This is because the neutrino exchange is proportional to Majorana mass of the neutrino and hence is suppressed and the $\Delta L^-\Delta L^-\Delta L^-$ vertex depends on the self-couplings of scalar potential whose values are unknown. We consider instead a production of a single $\Delta L^-$ in the process $e^-e^-\rightarrow \Delta L^- W_{\mu}^-$ which proceeds through t-channel exchange of Majorana neutrino and s-channel exchange of $\Delta L^-$. In this case the neutrino exchange is not suppressed as the t-channel neutrino has the same chirality in the both vertices. Moreover, in the s-channel process the strength of the $\Delta L^-\Delta L^-W_{\mu}^-$ vertex does not depend on any unknown parameter of the scalar potential but is determined by the gauge coupling. The experimentally clearest final state to study is the one where $\Delta L^-$ decays to a muon and a muonic neutrino and $W^-_{\mu}$ decays into two quark jets without missing energy. We hence choose the process $e^-e^-\rightarrow \mu \nu \bar{q} \bar{q}$ for further investigation, specifically with the light quarks $d$ and $\bar{u}$ in final state. In addition to the two amplitudes mentioned above, there exist still another amplitude contributing in the process $e^- e^- \rightarrow \mu \nu \bar{q} \bar{q}$, namely the one where $\Delta L^-$ is exchanged in s-channel producing two muons, one of which decaying further into $\nu \nu \bar{d} \bar{u}$. This amplitude does not directly depend on the properties of $\Delta L^-$ and is for that reason undesirable, but its influence is, of course, unavoidable.

In our calculations, whose results will be presented in the following, we have used CompHEP package created in Moscow University [12]. As an input we have used the vertex functions $\sqrt{2} \cdot h_{L\mu\nu}^y \cdot (1 - \gamma_5)/2$ and $-g \cdot \left( p_\mu^+ - p_\mu^- \right)$ for the $e\nu \Delta L^-$ and $W_\mu^\pm \Delta L^+ \Delta L^-$ couplings, respectively, as well as the ordinary electroweak vertices for the couplings of $W^-$ with leptons and quarks. In our calculations we have imposed the following cuts for the final state phase space:

- Each final state particle has energy greater than 10 GeV (including neutrino).
- The transverse energy of each particle (including missing transverse energy) should be greater than 5 GeV.
- The opening angle between two quark jets should be more than 20°.
- Each final state particle should have the outgoing direction more then 10° away from the beam axis.

In Fig. 1 we present the dependence of the cross section of the process $e^- e^- \rightarrow \mu \nu \bar{d} \bar{u}$ on the collision energy for the different values of masses of singly ($M_{\Delta L^-} = 100, 400, 700, 1000$ GeV) and doubly charged ($M_{\Delta L^-} = 100, 400, 700, 1000$ GeV) triplet Higgses. The cross sections are dominated by the resonance at $\sqrt{s} = M_{\Delta L^-}$. To estimate the width of the peak we have chosen $\Gamma_{\Delta L^-} = 10^{-3} \cdot M$ for the two lepton decays and the $\Delta L^- \rightarrow \Delta L^- W_{\mu}^-$ mode was also taken into account [5]. One may conclude that at 0.01 fb level the process $e^- e^- \rightarrow \mu \nu \bar{d} \bar{u}$ may be observed away from the $\Delta L^-$ resonance and even below the $\Delta L^-$ threshold.

In Fig. 2 presents the sensitivity of the reaction $e^- e^- \rightarrow \mu \nu \bar{d} \bar{u}$ on the masses of the triplet Higgs particles $\Delta L^-$ and $\Delta L^-$ for the collision energy $\sqrt{s} = 500$ GeV. We have estimated the values of the running coupling constants at 500 GeV by applying the approximate RG equations of the SM [13]. The influence of the triplet Higgses on the running, which can be expected to be quite small, is not taken into account. Fig. 2a displays the cross section of the $e^- e^- \rightarrow \mu \nu \bar{d} \bar{u}$ process for the different values of the $\Delta L^-$ and $\Delta L^-$ masses, with assuming for the Yukawa couplings their maximal allowed values that
are in accordance with the present phenomenological constraints [5,14]:
\[ h_{ee}^2 < 10^{-5} \cdot M_{\Delta^-} \text{ GeV}, \]
\[ h_{\mu\mu}^2 < 10^{-5} \cdot M_{\Delta^-} \text{ GeV}. \] (6)

If the mass of doubly charged Higgs is considered to be greater than 100 GeV, then \( h_{ee} \cdot h_{\mu\mu} < 0.18 \) or
\[ \sqrt{h_{ee} \cdot h_{\mu\mu} < 0.4}. \]

As can be seen from Fig. 2a, the process \( e^- e^- \rightarrow \mu^- \nu d \bar{n} \) will be well observable at colliding energy 500 GeV for any values of \( \Delta^- \) and \( \Delta^- \) masses below 2 TeV if the Yukawa couplings of the triplet fields have their maximal allowed values. Furthermore, the anticipated luminosity of linear collider (0.03 fb at \( \sqrt{s} = 500 \text{ GeV} \)) allows to state that the value of the product \( h_{ee} \cdot h_{\mu\mu} \), which enters as a common factor in all considered Feynman amplitudes, may be restricted at least one order of magnitude better than the present bound. Nevertheless, it is not so easy to separate the influence of the singly charged Higgs from that of the doubly charged Higgs particle. First of all let us notice, that in the limit of \( M_{\Delta^-} \) going to infinity, the cross section remains finite due to contribution of the s-channel process \( ee \rightarrow \Delta^- \rightarrow \mu^+ \mu^- \) followed by subsequent decay of one of the muons to a \( v d \bar{n} \). This process does not involve the singly charged Higgs. If \( \Delta^- \) in turn is decoupled, the amplitude with the t-channel exchange of Majorana neutrinos would keep the cross section finite. In order to search for the effects of the singly charged triplet Higgs we have studied the difference between the cross sections of the \( e^- e^- \rightarrow \mu^- \nu d \bar{n} \) process at a fixed and an infinite value (in practice \( M_{\Delta^-} = 2 \text{ TeV} \)) of the \( \Delta^- \) mass. In Fig. 2b we show the dependence of the cross section on the \( \Delta^- \) mass in the case that \( \Delta^- \) is effectively decoupled. Supposing that the mass of \( \Delta^- \) is known one can conservatively estimate, by setting for the Yukawa coupling \( h_{\mu\mu} \) the largest phenomenologically allowed value, the contribution of the \( \Delta^- \) mediated processes on the total cross section. When this is subtracted from the total cross section, what is left is the contribution of the t-channel neutrino exchange process alone. This has a threshold behaviour and its strength gives direct information on the product \( h_{ee} \cdot h_{\mu\mu} \) of Yukawa couplings.

In Fig. 2c we display the 0.03 fb (30 events per year) discovery contours on the \((M_{\Delta^-}, M_{\Delta^-})\)-plane, corresponding to the cross section of the isolated t-channel process, for the different values (0.1, 0.4 and 1.0) of “average” Yukawa couplings \( h_{\text{Yuk}} = \sqrt{h_{ee} \cdot h_{\mu\mu}} \). In the plot the collision energy is taken as \( \sqrt{s} = 500 \text{ GeV} \). It is seen from the figure that the process \( e^- e^- \rightarrow \mu^- \nu d \bar{n} \) might probe the mass \( M_{\Delta^-} \) to much larger values than what is the production threshold, providing that the average Yukawa coupling is larger than 0.1 and the collision does not happen in the vicinity of the \( \Delta^- \) pole. If these conditions are not met \( \Delta^- \) would have detectable effects only when it is produced as a real particle.

The constraints ensuing for the average Yukawa coupling are presented in Fig. 3 for different values of the mass of the doubly charged Higgs \( (M_{\Delta^-} = 400, 1000, 1500 \text{ and } 2000 \text{ GeV}) \) at \( \sqrt{s} = 500 \text{ GeV} \) for a light \( \Delta^- \) the constraint is tightest, about \( h_{\text{Yuk}} \leq 0.1 \) and it does not depend on the mass of \( \Delta^- \) the smaller \( M_{\Delta^-} \) the weaker and also the more dependent on the mass of \( \Delta^- \) the bound.

The main SM background to the reaction \( e^- e^- \rightarrow \mu^- \nu d \bar{n} \) is due to the process \( e^- e^- \rightarrow W^- W^- \nu \bar{\nu} \) studied in [15]. Its cross section was estimated to be below 20 fb for colliding energy \( \sqrt{s} \leq 1 \text{ TeV} \). We have recalculated this directly using CompHEP package and obtained the same results as in [15]. Taking into account the branching ratios of \( W \) to the appropriate decay modes we can estimate the background to be about 1 fb. This is typically below the process \( e^- e^- \rightarrow \mu^- \nu d \bar{n} \), but may be much larger than the contribution rate of the singly charged Higgs to the process. Applying the cut \( M(\mu, \nu \bar{\nu}) > 100 \text{ GeV} \) on the muon-antineutrino mass one can reduce the SM background up to 0.003 fb, which is already one order of magnitude below the singly charged Higgs contributions which we considered. This cut should be possible to impose [16] and we have checked that it does not substantially influence the cross section of the process for the cases when \( M_{\Delta^-} > 100 \text{ GeV} \). But even in the case when \( M_{\Delta^-} = M_{\mu} \) it is possible to compare the cross sections of \( e^- e^- \rightarrow \mu^- \nu d \bar{n} \) and \( e^- e^- \rightarrow d \bar{n} s \bar{\tau} \) which should be equal in the SM. Since the c-tagging is possible only at the 60% level (in comparison with the 90% efficiency of b-tagging [16]) it may be useful to compare the cross sections of \( e^- e^- \rightarrow \mu^- \nu q \bar{q} \) and \( e^- e^- \rightarrow \mu^- \nu d \bar{n} \)
Fig. 3. Sensitivity contours of the reaction $e^-e^-\rightarrow \mu^-\nu_\mu d\pi$ on the $(h_{\nu_\mu}, M_{\Delta^-})$-plane corresponding to $\sigma = 0.03$ fb (30 events per year, solid line), $\sigma = 0.3$ fb (300 events per year, dashed line), $\sigma = 3$ fb (30 events per year, dotted line) for different values of $M_{\Delta^-}$. Collision energy is taken to be $\sqrt{s} = 500$ GeV.
\(q \bar{q} q \bar{q}\) (where \(q, \bar{q}\) stand for light quarks \((u,d,c)\) which should as well be equal in the SM [13]. Any substantial difference between these cross sections would be a signal of the new physics. In other words, in order to get rid of the SM background one should consider the ratio of the cross sections of \(e^- e^- \rightarrow d \bar{t} s \bar{c}\) and \(e^- e^- \rightarrow \mu \nu d \bar{u}\) or, in the case of insufficient quality of c-tagging, of \(e^- e^- \rightarrow \mu \bar{\nu} q \bar{q}\) and \(e^- e^- \rightarrow q \bar{q} q \bar{q}\).

In summary, we have shown that the process \(e^- e^- \rightarrow \mu \nu d \bar{u}\) provides a good test for lepton flavor non-conservation of the singly charged scalars. At the collision energy 500 GeV the process may be seen well below \(\Delta_{L-1}\) and/or \(\Delta_{L-2}\) thresholds for a wide range of the lepton number violating Yukawa couplings. The influence of \(\Delta_{L-1}\) contribution (below its threshold) may be extracted from the process, if colliding energy is away from the \(\Delta_{L-1}\) resonance. The present bounds on the Yukawa couplings may be significantly improved.

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Type II seesaw and a gauge model for the bimaximal mixing explanation of neutrino puzzles

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Abstract

We present an extension of the standard model where the bimaximal mixing pattern among the neutrinos naturally arises via the type II seesaw mechanism. This explains both the atmospheric and solar neutrino data via large angle vacuum oscillation among the three known neutrinos. The model does not include righthanded neutrinos but additional Higgs triplets which acquire naturally small vev’s due to the type II seesaw mechanism. The presence of a global $L_L$ and $S_3$ symmetry leads naturally to the desired mass splittings among neutrinos at the one loop level. This model predicts observable branching ratios for $\tau \rightarrow \mu \mu \mu$, which could be used to test the model. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

The observed deficit of the atmospheric muon neutrinos by the Super-Kamiokande [1,2] collaboration has provided strong evidence that there is oscillation among the known neutrino species. The five solar neutrino experiments [3,4] have added to this sense of excitement by their long standing result that there is also a deficit of the solar neutrinos, which receives a simple explanation again in terms of neutrino oscillations [5]. It thus appears certain that neutrinos have mass and they mix among each other.

Although the details are fuzzy on what is the exact nature of the oscillations, several very interesting scenarios exist. In particular, if the only laboratory indication of the neutrino oscillation by the Los Alamos collaboration (LSND) [6] is not included in the picture, there is a very interesting mixing scheme known as the bimaximal mixing, where both the solar and atmospheric data are explained by maximal mixing among the three known neutrinos [7]. In this picture, solar neutrino puzzle could either be solved via the large angle MSW mechanism [8] or via the vacuum oscillation mechanism [9]. In this paper we will assume that solar neutrino deficit is explained by vacuum oscillation between the $\nu_e$ and $\nu_\mu$ which requires that their mass difference square must be $\sim 10^{-10} \text{eV}^2$. The observed electron energy distribu-
tion as well as some hints of bi-annual variation of the solar neutrino flux by the Super-Kamiokande collaboration may be pointing in this direction.

If we accept this resolution of the neutrino puzzles, this poses two major theoretical challenges: one is how does one get naturally a theory that leads to the bimaximal mixing matrix and secondly, how does the same framework explain the tiny mass difference square needed for vacuum oscillation without fine tuning of parameters? Our goal in this paper is to provide a simple model that generates both these features of the neutrino mass matrix. The key ingredient is the type II seesaw mechanism where the vev of a triplet Higgs becomes ultrasmall due to the presence of a high scale in the theory [10].

The presence of additional global symmetries in the model lead to a pattern of neutrino masses that leads to the bimaximal mixing among neutrinos. The presence of the global symmetry also naturally leads to zero flavor mixing among the charged leptons so that the bimaximal pattern dictated by the neutrino mass matrix remains indeed natural.

Using the definition of the mixing matrix as

$$
\begin{pmatrix}
\nu_e \\ \nu_\mu \\ \nu_\tau
\end{pmatrix} = U_{\nu} \begin{pmatrix}
\nu_1 \\ \nu_2 \\ \nu_3
\end{pmatrix},
$$

the bimaximal mixing corresponds to the mixing matrix $U_{\nu}$ given by [7]

$$
U_{\nu} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}.
$$

The neutrino masses may be degenerate or hierarchical. As mentioned, a convincing theoretical explanation of this elegant mixing pattern seems so far to have been elusive, although there exist many interesting attempts [11,12]. The problem becomes even more challenging when we demand that the solar neutrino puzzle be solved by the vacuum oscillation mechanism.

In this Letter, we use the type II seesaw mechanism in conjunction with the global symmetry $S_3 \times U(1)_{-\mu-\tau}$ to show that both these properties can be realized in a natural manner. In this model, we have the neutrino mass pattern $m_{\nu_e} \ll m_{\nu_\mu} = m_{\nu_\tau}$, $\Delta m_{\text{ATM}} = 0.05$ eV and the generalized bimaximal pattern given by:

$$
U_{\nu} = \begin{pmatrix}
1 & -1 & 0 \\
\sqrt{2} & \sqrt{2} & -\sin \theta \\
\sqrt{2} & \sqrt{2} & \cos \theta
\end{pmatrix}.
$$

2. The model

We consider the extension of the standard model where the fermion content is left unaltered but the following Higgs sector is chosen: three doublets $\phi_0, \phi_1, \phi_2$, two triplets with $Y = 2$ denoted by $\Delta_{1,2}$ and a charged isosinglet with $Y = +2$. The model has an $S_3$ symmetry (i.e. permutation group on three elements), under which the particles are assigned as shown in Table 1.

We also impose an $L_e - L_\mu - L_\tau$ symmetry on the model. The Yukawa part of the Lagrangian in the

<table>
<thead>
<tr>
<th>Fields</th>
<th>$S_3$ transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(L_\mu, L_e)$</td>
<td>2</td>
</tr>
<tr>
<td>$(\mu, \tau, \nu)$</td>
<td>2</td>
</tr>
<tr>
<td>$L_\mu, L_\tau$</td>
<td>1</td>
</tr>
<tr>
<td>$(\phi_0, \phi_2)$</td>
<td>2</td>
</tr>
<tr>
<td>$\eta^+$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1
Transformation properties of the fields in the model under the $S_3$ group.
leptonic sector consistent with these symmetries can be written as:

\[ \mathcal{L}_V = h_1 (\bar{L}_\mu \mu_R + \bar{L}_e \tau_R) \phi_0 + h_2 \left( \bar{L}_\mu \mu_R - \bar{L}_e \tau_R \right) \phi_1 \\
+ \left( \bar{L}_\mu \tau_R + \bar{L}_e \mu_R \right) \phi_2 + h_s \bar{L}_e e_R \phi_0 \\
+ f L_\mu (\bar{L}_\mu \Delta_1 + L_\tau \Delta_2) + f^* L_\mu L_\tau \eta + h.c. \quad (4) \]

We will show later by a detailed examination of the Higgs potential for the system that there is a domain of parameters for which we get the following vevs of the fields:

\[ \langle \phi^0_{1,2} \rangle = \nu_{1,2}; \quad \langle \phi^0 \rangle = \nu_0; \]
\[ \langle \phi^0 \rangle = \phi_1; \quad \langle \phi^0 \rangle = 0. \quad (5) \]

Clearly, the pattern of \( \phi \) vevs leads to a diagonal mass matrix for the charged leptons whereas the \( \Delta \) vev’s leads to a Majorana mass for the neutrinos of the form:

\[ M_{\nu} = \begin{pmatrix} 0 & m_1 & m_2 \\
m_1 & 0 & 0 \\
m_2 & 0 & 0 \end{pmatrix}. \quad (6) \]

As a consequence, diagonalization of the neutrino mass matrix is solely responsible for the neutrino mixings and one obtains the pattern given in the generalized bimaximal form (see Eq. (3)), with \( \tan \theta = m_2/m_1 = \nu_{1,2}^2/\nu_1^2 \).

Let us now address the question of neutrino masses. Clearly, to understand the small neutrino masses, one must have a tiny value for the vev of the \( \Delta \) fields. This is achieved by the type II seesaw mechanism [10]. This is a generic mechanism, which can be illustrated by the following simple model that has only one \( \phi \) and one \( \Delta \). Consider the following Higgs potential for this system [10]:

\[ V(\phi, \Delta) = M^2 \Delta^\dagger \Delta - \mu^2 \phi^\dagger \phi + \lambda_\phi (\phi^\dagger \phi)^2 + \lambda_\Delta \Delta^\dagger \Delta \\
+ \lambda_{\phi, \Delta} \Delta^\dagger \phi^\dagger \phi + M_{\phi, \Delta} \phi^\dagger \phi + h.c. \quad (7) \]

Let us choose \( \mu \sim 100 \) GeV and \( M \sim M' \gg \mu \); in this case, the vev of \( \langle \phi^0 \rangle = \mu \) whereas the vev of \( \langle \Delta^0 \rangle \sim \mu^2/M \ll \mu \). This mechanism has been labeled type II seesaw and we see that if \( M = 10^{14} \) GeV, then we get \( \langle \Delta^0 \rangle = 0.14 \) eV. In the presence of more \( \Delta \) fields and extra symmetries that our model has this mechanism still operates and we have a small mass (in the 0.1 eV range) for the \( \nu_e \) and \( \nu_x \).

The \( \nu_e \) and \( \nu_x \) get mixed in the tree level and the mixing angle is near maximal unless we do fine tuning. As far as the \( \nu_e \) and \( \nu_x \) are concerned, they are degenerate; therefore at the tree level their mass difference vanishes. We will show below that their mass difference arises at the one loop level but due to the presence of the high mass scale that gave rise to the type II seesaw mechanism, the mass difference \( \Delta m^2_{\tau - \mu} \) is naturally suppressed to the level of \( 10^{-10} \) eV$^2$ without any unnatural fine tuning. The tree level mass matrix already explains the atmospheric neutrino puzzle due to the type II seesaw.

Let us now turn to the explanation of the one loop contribution to the neutrino mass matrix. This is where the role of the \( \eta \) boson becomes important. In order to generate the mass difference between the \( \nu_e \) and \( \nu_x \), we need nonzero entries for the \( \mu \mu \) or \( ee \) element. Both of them will violate the \( L_e - L_\mu - L_\tau \) symmetry. This breaking is introduced by a soft term in the potential \( \eta \phi_1 \phi_2 \) since \( \eta \) has \( L_e - L_\mu - L_\tau = -2 \) and \( \phi^\dagger \phi \)'s are neutral under this global symmetry. This is a dimensional coupling and in accordance with our principle, we will choose this of order \( M \).

This leads to a one loop graph as in Fig. 1, which produces a neutrino mass matrix as follows:

\[ M_{\nu} = \begin{pmatrix} 0 & m_1 & m_2 \\
m_1 & m_{\mu, \mu} & 0 \\
m_2 & 0 & m_{\tau, \tau} \end{pmatrix}, \quad (8) \]

where

\[ m_{\mu, \mu} = \frac{f h_2 M_{\eta, \phi, \phi}}{16 \pi^2 \frac{M_{\phi, \phi}^2}{M^2}}. \quad (9) \]

For \( M = M_{\eta, \phi, \phi} \) and \( f h_2 \approx 10^{-3} \), we get \( m_{\mu, \mu} \approx 10^{-8} \) eV, leading to \( \Delta m^2_{\tau - \mu} = 10^{-10} \) eV$^2$, as is required to explain the solar neutrino puzzle via vacuum oscillation. Note that the tau neutrino picks up very tiny mass at the one loop (\( \sim 10^{-10} \) eV). This completes the derivation of the main result of our paper. Let us
Fig. 1. One loop correction that generates the diagonal mass term $m_{\mu\mu}$. A similar diagram provides $m_{\tau\tau}$.

Now discuss the phenomenological consequences of this model.

3. Rare tau decays

The present model contains three sources of Flavor Changing Currents, via $\eta$, $\Delta$'s and $\phi_2$. At tree level, $\eta^+$ and $\Delta_{1,2}$ are involved in rare $\mu$ or $\tau$ decay processes that include neutrinos as final products. However, as the masses of those fields are already very heavy ($\sim M$) such processes are highly suppressed. Therefore we will concentrate in those that involve only charged leptons.

The more general tree level diagrams for Flavor Changing Neutral Currents (FCNC) that arise in the model are depicted in Figs. 2 and 3. The internal lines in those figures represent the contributions of the general mixing in the scalar sector that comes from the trilinear and quartic scalar couplings. In both of the figures if the one scalar field is $f$, the $S_3$ symmetry will restrict the involved vevs.

In Fig. 2, for instance, it allows only $\langle \Delta_1 \rangle$, regardless if the other field is either $\phi_0$ or $\phi_1$. Thus, we estimate the squared amplitude of this diagram to be generically suppressed by

$$h^2 h^2 \frac{\langle \Delta_1 \rangle^2 M_{\Delta\phi\phi}^2}{m_\phi^4 m_{\phi,1}^2} = h^2 \cdot 10^{-12} \text{ GeV}^{-4},$$

where we have assumed as before that $M_{\Delta\phi\phi} = M$, $m_\phi = v$ and taking $h_2 = m_\phi/v = 10^{-2}$. Here, $h$ represent the Yukawa coupling of the vertex on the right and is already smaller than $h_2$.

For the diagram in Fig. 3, the vevs could be either $\langle \phi \rangle$ or $\langle \Delta \rangle$. Nevertheless, if we choose $\langle \phi \rangle$, again the $S_3$ symmetry play an important role by constraining one of them to be $\langle \phi_2 \rangle$, which is zero, then the only contribution comes from the $\langle \Delta \rangle$ sector which is more suppressed already than the previous case by an extra $10^{-14}$ times the quartic coupling constant, which being dimensionless may be chosen of the order of one.

Now, we may compare the predictions of our model with the experimental bounds [13] for the decays $\tau \to \mu\mu\mu$ and $\mu\mu\mu$, which are the only allowed modes at tree level. The upper bound for the branching ratios is about $10^{-6}$. From (10), we estimate $B(\tau \to \mu\mu\mu) = 10^{-7}$ and $B(\tau \to \mu\mu\mu) = 10^{-12}$. Therefore, the first mode will be the main decay channel with FCNC in the $\tau$ physics. Then this decay could be observed in the near future.

Let us stress that, as an important signature of the present model, other decay processes as $\tau \to \mu\mu\mu$ and $\mu \to \mu\mu\mu$ are more highly suppressed as they do not appear at tree level, since the only tree level coupling among electron and muon (or tau) also couples $\Delta^+$, which does not mix with $\phi_0$, as required to get three
electrons in the final states. Notice that $\Delta^{++}$ also contributes to $\tau \to \mu ee$ but this is already suppressed by its large mass. To compare with the predictions of other models with an $S_3$ symmetry see Refs. [11,12]

At one loop order the most interesting process again appears at the $\tau$ physics, being the rare decay $\tau \to \mu \gamma$. The corresponding diagram is showed in Fig. 4. Now the amplitude of this diagram is roughly estimated to give

$$\Gamma = \frac{h_1^2 h_2^2 e^2}{16 \pi^2} \langle \Delta_1 \rangle^2 M^2_{\phi \psi} m^2_{H_u} = 10^{-21} \text{GeV}. \quad (11)$$

Then the branching ratio of this process is predicted to be about $\sim 10^{-9}$, which is, on the other hand, below the current experimental bound [13] of $10^{-6}$.

### 4. Analysis of the Higgs potential

Let us show now how $\langle \phi_0^0 \rangle = 0$ may arises naturally from the potential. Given the irreducible representations (irreps) of $S_3$: $2_s = (x_1, x_2)$ and $2_y = (y_1, y_2)$, we may build the following singlets $1_{xy} = x_1 y_1 + x_2 y_2$; $1_{y_1} = x_1 y_2 - x_2 y_1$; and the new doublet $2_{x_1 y_1} = (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1)$. Using this simple rules it is straightforward to find all possible $S_3$ and gauge invariant terms that include the scalar fields of the model. Such potential may be decomposed as $V = V(\phi) + V(\Delta) + V(\phi, \Delta) + V(\phi, \Delta, \eta)$. The last term involves all the expressions containing $\eta$. They do not contribute to the minimization of the potential, thus, it is not necessary to show them explicitly.

As we already discussed above, the type II seesaw formula arises from $V(\Delta) + V(\phi, \Delta)$ by assuming large trilinear couplings. In this case $\langle \Delta \rangle$ becomes smaller than $\langle \phi \rangle$ and then we may neglect those terms for the analysis of the $\phi$ versus Defining $\Phi = (\phi_1, \phi_2)$, the relevant terms of the potential are then represented by

$$V(\phi) = \mu^2 1_{\phi^0} + \mu^2 1_{\phi^0} + \lambda_1 (1_{\phi^0})^2$$

$$+ \lambda_2 (1_{\phi^0})^2 + \lambda_3 1_{\phi^0} 1_{\phi^0} + \lambda_4 1_{\phi^0}$$

$$+ \lambda_5 (1_{\phi^0})^2 + \lambda_6 1_{\phi^0} 1_{\phi^0}, \quad (12)$$

where $1_{\phi^0}$ means the singlet built by using $n \times$ irreps. By examining $V(\phi)$ we may see that all the terms but the last one present an accidental $U(1)_a$ symmetry, which makes itself evident if we parametrize the vevs as

$$\left( \begin{array}{c} \langle \phi_1 \rangle \\ \langle \phi_2 \rangle \end{array} \right) = v \left( \begin{array}{c} \cos \alpha \\ \sin \alpha \end{array} \right). \quad (13)$$

Then, in terms the potential reduces to

$$V(\phi) = V(v, \langle \phi_0 \rangle)$$

$$+ \lambda_n \langle \phi_0 \rangle^2 \sin^2 (3 \cos \alpha - 4 \cos^2 \alpha). \quad (14)$$

Clearly, the last term breaks explicitly that extra symmetry. This potential has extrema for two values of $\cos \alpha$ i.e. $\cos \alpha = \pm 1$ and $\cos \alpha = 1/2$. For $\lambda > 0$, the minimum is at $\alpha = 0$ which is the desired minimum.

It is worth noting that this special effect in the potential does not appear on the $\Delta$ sector, since is just a consequence of the presence of the extra singlet $\phi_0$. As a matter of fact, $V(\Delta)$ is totally $U(1)_a$ symmetric while $V(\phi,\Delta)$ contains several terms that break explicitly such symmetry in a less trivial way, then avoiding a null value of $\langle \Delta_2 \rangle$, and conserving the pattern of neutrino masses.

### 5. Conclusions

We have presented a model of neutrino masses based on the permutation symmetry $S_3$ which predict the bimaximal mixing pattern at tree level which
explain atmospheric oscillations through the hierarchy \( m_{\nu_e} < m_{\nu_{\mu}} = m_{\nu_{\tau}} \), while the tree level mass matrix of the charged leptons is diagonal.

The smallness of the masses of the neutrinos is explained by the type II seesaw formula that produces small vevs for the scalar triplets which gives the neutrino masses. The model contains also a \( U(1)_{\mu - \tau} \) symmetry which avoids diagonal masses at tree level for the neutrino sector. This symmetry is softly broken by the scalar potential through a coupling of the scalar doublets with an odd charged scalar, \( \eta \), which carries \( L_{\mu - \tau} \) lepton number. The breaking is then communicated to the leptonic sector through radiative corrections providing very small diagonal mass terms for \( \nu_{\mu, \tau} \) at the right order of magnitude to explain the solar neutrino deficit via vacuum oscillations.

The present model also predicts the rare decay \( \tau \to \mu \mu \mu \) with a branching ratio closed to the current experimental limits. Then this process could be studied in the near future. Also, the model may be tested at high energies using the direct production and decay of the scalar doublets, which get masses of the order of electroweak scale, for instance, through \( \phi^0_1 \to \tau \mu \).

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References

Neutrino masses and mixings in non-factorizable geometry

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Abstract

We study bulk fermion fields in the localized gravity model with non-factorizable metric recently proposed by Randall and Sundrum, and Gogberashvili. In addition to a tower of weak-scale Kaluza–Klein states we find a zero mode for any value of the fundamental fermion mass. If the fermion mass is larger than half the curvature of the compact dimension, the zero mode can be localized on the “hidden” 3-brane in the Randall–Sundrum model. Identifying this mode with a right-handed neutrino provides a new way for obtaining small Dirac neutrino masses without invoking a see-saw mechanism. Cancellation of the parity anomaly requires introducing an even number of bulk fermions. This naturally leads to a strong hierarchy of neutrino masses and generically large mixing angles. © 2000 Published by Elsevier Science B.V. All rights reserved.

Keywords: Field theories in higher dimensions; Neutrino physics; Beyond Standard Model

1. Introduction

Theories with extra spatial dimensions have received great attention recently, when it was shown that they could provide a solution to the gauge-hierarchy problem. If space-time is a product of Minkowski space with \( n \) compact dimensions, with Standard Model fields localized in the three extended spatial dimensions (i.e., on a 3-brane) and gravity propagating in the extra space, then the strength of gravity on the 3-brane is governed by an effective Planck scale \( M_\text{Pl} \) = \( M^{n+2} V_g \), where \( M \) is the fundamental scale of gravity and \( V_g \) the volume of the compact space [1]. If this space is sufficiently large, the fundamental scale \( M \) can be of order 1 TeV, thus removing the large disparity between the gravitational and the electroweak scales.

An intriguing alternative to the above scenario invokes a non-factorizable geometry with a metric that depends on the coordinates of the extra dimensions [2,3]. In the simplest scenario due to Randall and Sundrum (RS) one considers a single extra dimension, taken to be a \( S^1/Z_2 \) orbifold parameterized by a coordinate \( y = r, \phi \), with \( r, \) the radius of the compact dimension, \( -\pi \leq \phi \leq \pi \), and the points \((x, \phi)\) and \((x, -\phi)\) identified [3]. There are two 3-branes located at the orbifold fixed points: a “visible” brane at \( \phi = \pi \) containing the Standard Model fields, and a “hidden” brane at \( \phi = 0 \). The solution of Einstein’s equations for this geometry leads to the non-factorizable metric

\[
\begin{align*}
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0370-2693(00)00054-X
\end{align*}
\]

\[
d s^2 = e^{-2k r, |\phi|} \eta_{\mu\nu} \, dx^\mu dx^\nu - r_c^2 d \phi^2 , \quad (1.1)
\]
where $x^\mu$ are the coordinates on the four-dimensional surfaces of constant $\phi$, and the parameter $k$ is of order the fundamental Planck scale $M$. (This solution can only be trusted if $k < M$, so the bulk curvature is small compared with the fundamental Planck scale.) The two 3-branes carry vacuum energies tuned such that $V_{\text{vis}} = -V_{\text{hid}} = -24M^3k$, which is required to obtain a solution respecting four-dimensional Poincaré invariance. In between the two branes is a slice of AdS$_5$ space.

With this setup, the effective Planck scale seen by particles confined to four-dimensional space-time is $M_{\text{Pl}}^2 = (M^2/k)(1 - e^{-2kr}/r)$, which is of order the fundamental scale $M$. Unlike the scenarios with large extra dimensions considered in [1], the scale $M$ is therefore not of order the weak scale. However, the “warp factor” $e^{-2kr}/r$ in the metric (1.1) has important implications for the masses of particles confined to the visible brane. The Lagrangian for these particles depends on the induced metric on the brane, $S_{\mu\nu} = e^{-2kr}\eta_{\mu\nu}$, and after field renormalization any mass parameter $m_0$ in the fundamental theory is promoted into an effective mass parameter $m = e^{-kr}\eta_{\mu\bar{\nu}}m_0$, governing the physical properties of particles on the brane [3]. With $kr = 12$ this mechanism produces weak-scale physical masses and couplings from the fundamental four and couplings of order the Planck scale. As a consequence of the warp factor, the Kaluza–Klein excitations of gravitons have weak-scale mass splittings and couplings from the fundamental scale and couplings of order the Planck scale.

In order for this model to provide a viable solution to the hierarchy problem it is important to address the question of how to stabilize the radius $r_c$ of the extra dimension, and the related question of the potentially disastrous cosmology of a visible universe confined to a brane with negative tension [9–11] (see also [12,13]). A mechanism for radius stabilization utilizing a bulk scalar field has been proposed in [14]. In the presence of such a mechanism, one finds a conventional cosmological expansion for temperatures below the weak scale [15]. The couplings of the radion field to Standard Model particles may have interesting implications for collider searches [15,16]. Other phenomenological consequences of the RS model have been discussed in [17], and an alternative model avoiding the negative-tension brane has been proposed in [5].

Resolving the hierarchy problem by introducing extra dimensions poses new challenges. For instance, operators mediating proton decay, lepton-number violation or flavor-changing neutral currents must be sufficiently suppressed. Likewise, the see-saw mechanism for generating small neutrino currents cannot be invoked if the highest energy scale governing physics on the visible brane is the weak scale. However, there is now increasing evidence that the atmospheric neutrino anomaly [18–20] and the solar neutrino problem [21,22] are explained in terms of neutrino oscillations, which require small but non-vanishing neutrino masses. Several four-dimensional alternatives to the see-saw mechanism not requiring a high-energy scale have been proposed, such as radiatively generated neutrino masses [23] and composite models [24]. However, it would be interesting to find new mechanisms that are intrinsically higher dimensional. In the context of models with large extra dimensions ideas in this direction have been presented in [25,26], and some concrete models have been worked out in [27–29]. They contain a massless Standard Model singlet propagating in the bulk of the extra compact space, which serves as a right-handed neutrino. Then the effective four-dimensional Yukawa coupling is suppressed by a volume factor $1/\sqrt{V}$, reflecting the small overlap between the right-handed neutrino in the bulk and the left-handed one on the 3-brane. By construction, this factor provides a suppression of neutrino masses of order $v/M_{\text{Pl}}$, reminiscent of the see-saw mechanism. However, this idea does not work in a scenario with small extra dimensions such as the RS model, simply because of the lack of a volume suppression factor.

In this letter we investigate the possibility of incorporating bulk fermions in the RS model. As in the case of scalars [6] and vector fields [7,8] propagating in the compact dimension we find that the Kaluza–Klein modes have weak-scale masses even though the fermion mass in the fundamental, five-dimensional theory is of order the Planck scale. The
fermion case is more interesting, however, because
the extension of the Dirac algebra to five dimensions
leads to a different propagation of left- and right-
handed modes. After imposing the orbifold boundary
conditions the geometry supports a left-handed or a
right-handed zero mode for any value of the funda-
mental fermion mass, one of which can be localized
on the hidden brane of the RS model. This is differ-
ent from the scalar and vector cases, where zero
modes exist only for vanishing mass in the funda-
mental theory. The localization of a right-handed
zero mode on the hidden brane provides a new
mechanism for obtaining small neutrino masses,
which can be realized by coupling the Higgs and
left-handed lepton fields of the Standard Model,
localized on the visible brane, to a right-handed
fermion in the bulk. The neutrino mass can be tuned
over many orders of magnitude by a small change of
the bulk fermion mass. Moreover, cancellation of the
parity anomaly [30,31] forces us to introduce an even
number of bulk fermions. This naturally leads to a
neutrino mass hierarchy and potentially large mixing
angles.

2. Bulk fermions

Our starting point is the action for a Dirac fermion
with mass $m$ of order the fundamental scale $M$
propagating in a five-dimensional space with the
metric (1.1), which we write in the form [32] 3

\[
S = \int d^4x \int d\phi \sqrt{G} \left( E^A \left( i \frac{1}{2} \bar{\Psi} \gamma^a (\partial_a - \partial_5) \Psi \right) + \frac{\omega_{BC}}{8} (\gamma^a, \gamma^b) \bar{\Psi} \right) - \frac{m}{2} \text{sgn}(\phi) \bar{\Psi} \Psi \right),
\]

where $G = \det(G_{\alpha\beta}) = r^2 e^{-8\phi}$ with \( \sigma = kr_1/|\phi| \) is the
determinant of the metric. We use capital indices
$A, B, \ldots$ for objects defined in curved space, and
lower-case indices $a, b, \ldots$ for objects defined in the
tangent frame. The matrices $\gamma^a = (\gamma^a, i\gamma_5)$ provide
a four-dimensional representation of the Dirac matrices
in five-dimensional flat space. The quantity $E^A_a = \text{diag}(e^\sigma, e^\sigma, e^\sigma, 1/r_0)$ is the inverse vielbein, and
$\omega_{BC}$ is the spin connection. Because in our case the
metric is diagonal, the only non-vanishing entries of
the spin connection have $b = A$ or $c = A$, giving no
contribution to the action in (2.1).

The sign change of the mass term under $\phi \to -\phi$
is necessary in order to conserve $\phi$-parity, as re-
quired by the $Z_2$ orbifold symmetry of the RS
model. Such a mass term can be obtained, e.g., by
coupling the fermion to a pseudoscalar (under $\phi$-
parity) bulk Higgs field. For a single bulk fermion in
five dimensions $\phi$-parity is broken at the quantum
level, giving rise to the so-called parity anomaly
[30,31]. To cancel this anomaly, we will later con-
side an even number of fermion fields.

Using an integration by parts, and defining left-
and right-handed spinors $\Psi_{L,R} = \frac{1}{2}(1 \mp \gamma_5)\Psi$, the
action can be written as

\[
S = \int d^4x \int d\phi \rho \left\{ e^{-3\phi} \left( \bar{\Psi}_L i / \partial \Psi_L + \bar{\Psi}_R i / \partial \Psi_R \right) \right.
\]
\[
- e^{-4\phi} m \text{sgn}(\phi) \left( \bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L \right)
\]
\[
- \frac{1}{2} \left[ \bar{\Psi}_L (e^{-3\phi} \partial_0 + \partial_5 e^{-4\phi}) \Psi_R
\]
\[
- \bar{\Psi}_R (e^{-3\phi} \partial_0 + \partial_5 e^{-4\phi}) \Psi_L \right) \right\},
\]

where we impose periodic boundary conditions
$\Psi_{L,R}(x, \pi) = \Psi_{L,R}(x, -\pi)$ on the fields. The action
is even under the $Z_2$ orbifold symmetry if $\Psi_\mu(x, \phi)$
is an odd function of $\phi$ and $\Psi_\mu(x, \phi)$ is even, or
vice versa. To perform the Kaluza–Klein decomposi-
tion we write

\[
\Psi_{L,R}(x, \phi) = \sum_n \tilde{\psi}^{L,R}_n(x) \tilde{\phi}^{L,R}_n(\phi).
\]

Because of the $Z_2$ symmetry of the action it is
sufficient to restrict the integration over $\phi$ from 0 to
$\pi$. The behavior of the solutions for negative $\phi$
is then determined by their $Z_2$ parity. $(\tilde{f}_n^{L,R}(\phi))$ and
$(\tilde{f}_n^{R,L}(\phi))$ are two complete, orthonormal (with a
scalar product defined below) sets of functions on

---

We do not include a five-dimensional Majorana mass term of
the form $\bar{\Psi}C\Psi$ in the action, because later we will assign lepton
number to the bulk fermion.
the interval $\phi \in [0, \pi]$, subject to certain boundary conditions. We will construct them as the eigenfunctions of hermitian operators on this interval. Inserting the ansatz (2.3) into the action and requiring that the result take the form of the usual Dirac action for massive fermions in four dimensions,

$$S = \sum_n \int d^4x \left( \bar{\psi}_n(x) i\gamma^\alpha \partial_\alpha \psi_n(x) \right) - m_n \bar{\psi}_n(x) \psi_n(x) ,$$

where $\psi = \psi_L + \psi_R$ (except for possible chiral modes) and $m_n \geq 0$, we find that the functions $f_n^{(L,R)}(\phi)$ must obey the conditions

$$\int_0^\pi d\phi \epsilon^{\alpha} f_n^{(L,R)}(\phi) = 0 ,$$

where $n \epsilon = e^{\mp} \in [\epsilon, 1]$ with $\epsilon = e^{\pm kr_\epsilon}$, rescale $f_n^{(L,R)}(\phi) \rightarrow \sqrt{kr_\epsilon} \epsilon f_n^{(L,R)}(t)$, and define the quantities

$$\nu = \frac{m}{k} , \quad x_n = \frac{m_n}{\epsilon k} = \frac{m_n}{k} e^{\pm kr_\epsilon} .$$

The condition of $f_n^{(L,R)}(0) = f_n^{(L,R)}(\pi) = 0$, which follow since either all left-handed or all right-handed functions are $Z_2$-odd, ensure that the differential operators $(\pm \frac{1}{r_\epsilon} \partial_\phi - m)$ are hermitian and their eigenvalues $m_n$ real. (Since the equations are real, the functions $f_n^{(L,R)}(\phi)$ could be chosen real without loss of generality.)

It is convenient to introduce the new variable $t = \frac{m}{k} e^{\pm} \in [1, e]$ with $e^{-1} = e^{\pm kr_\epsilon}$, rescale $f_n^{(L,R)}(\phi) \rightarrow \sqrt{kr_\epsilon} \epsilon f_n^{(L,R)}(t)$, and define the quantities

$$\nu = \frac{m}{k} , \quad x_n = \frac{m_n}{\epsilon k} = \frac{m_n}{k} e^{kr/\epsilon} .$$

The small parameter $\epsilon \sim 10^{-16}$ sets the ratio between the electroweak and the gravitational scales. The two conditions in (2.5) now become

$$\int_0^1 dt f_n^{(L,R)}(t) f_n^{(L,R)}(t) = \delta_{mn} ,$$

$$\left( \pm t \partial_t - \nu \right) f_n^{(L,R)}(t) = -x_n f_n^{(L,R)}(t) ,$$

and the boundary conditions are $f_n^{(L,R)}(0) f_n^{(L,R)}(\pi) = 0$. The system of coupled, first-order differential equations for $f_n^{(L,R)}(t)$ implies the second-order equations

$$\left[ t^2 \partial_t^2 + x_n^2 t^2 - \nu (\nu + 1) \right] f_n^{(L,R)}(t) = 0 .$$

Dimensional analysis shows that the eigenvalues $x_n$ are of order unity, corresponding to weak-scale fermion masses $m_n$ in the four-dimensional theory. The solution of the differential equations is straightforward. We start by looking for zero modes, i.e., solutions with $x_n = 0$. In this case the first-order equations in (2.7) decouple. The properly normalized solutions are

$$f_n^{(L,R)}(t) = f_0^{(L,R)}(1) t^{1/2} \epsilon^{-1/2} ,$$

$$|f_0^{(L,R)}(1)|^2 = \frac{1}{1 - \epsilon^2} \epsilon^{1/2} .$$

Since these are even functions of $\phi$, which do not vanish at the orbifold fixed points, only one of the zero modes is allowed by the orbifold symmetry. This mode exists irrespective of the value of the fermion mass $m$ in the five-dimensional theory. Note that for $\nu > \frac{1}{2}$ the right-handed zero mode has a very small wave function on the visible brane: $f_0^{(R)}(1) \alpha \epsilon^{1/2}$. This property will allow us to obtain small neutrino masses. The presence of fermion zero modes should not come as a surprise, since it is well known that in flat space-time they are associated with domain walls [33]. In our model the domain walls are provided by the 3-branes of the RS model, which separate the regions with a different sign of the fermion mass term. The functions $P_L f_0^{(L)}(t)$ and $P_R f_0^{(R)}(t)$, with $P_{L,R} = \frac{1}{2} (1 \mp \gamma_5)$, can be associated with the “fermionic” and “bosonic” degrees of freedom of a supersymmetric, quantum-mechanical system [34]. The supersymmetry generators are $Q = (\partial_t - \nu t) P_L$ and $Q^\dagger = (\partial_t + \nu t) P_R$, and the Kaluza–Klein modes are the eigenstates of the Hamiltonian $(Q Q^\dagger)$. This explains why left- and right-handed modes have the same eigenvalues $x_n$. The two zero modes correspond to the ground-state solutions of the supersymmetric Hamiltonian. In our case, the requirement of orbifold symmetry allows only one of these solutions to be present.

The solutions of the differential equations (2.8) for the case $x_n > 0$ are Bessel functions. For conve-
nience we assume that $\nu \neq \frac{1}{2} + N$ with an integer $N$. Then the most general solutions can be written in the form

$$f^{L,R}_n(t) = \sqrt{t} \left[ a^{L,R}_n J_{\frac{1}{2} \pm \nu}(x_n t) + b^{L,R}_n J_{-\frac{1}{2} \pm \nu}(x_n t) \right].$$

(2.10)

For the special values $\nu = \frac{1}{2} + N$ the solutions are superpositions of Bessel functions of the first and second kind, which can obtained from our results using a limiting procedure. The two functions $f^L_n(t)$ and $f^R_n(t)$ are not independent, since they are coupled by the first-order differential equations in (2.7), which imply $b^L_n = a^R_n$ and $b^R_n = -a^L_n$. Hence, the solutions take the form

$$f^L_n(t) = \sqrt{t} \left[ a^L_n J_{\frac{1}{2} - \nu}(x_n t) + a^R_n J_{-\frac{1}{2} + \nu}(x_n t) \right],$$

$$f^R_n(t) = \sqrt{t} \left[ a^R_n J_{\frac{1}{2} + \nu}(x_n t) - a^L_n J_{-\frac{1}{2} - \nu}(x_n t) \right].$$

(2.11)

To proceed we must specify the boundary conditions at the locations of the 3-branes. This will give rise to a discrete spectrum of Kaluza–Klein modes. Orbifold symmetry allows two choices of boundary conditions: either all left-handed fields are odd under $\phi$-parity and all right-handed fields are even ("option L"), or all right-handed fields are odd and all left-handed ones even ("option R"). In the first case the boundary conditions are $f^L_n(\epsilon) = f^L_n(1) = 0$, and in the second case $f^R_n(\epsilon) = f^R_n(1) = 0$. Which of these choices is realized in nature is a question that cannot be answered without understanding the physics on the 3-branes, which is beyond the scope of the field-theory model suggested in [3]. The two cases are straightforward to analyze. Using the asymptotic behavior of the Bessel functions, $J_n(x) \sim x^n$ as $x \to 0$, it follows that in the limit $\epsilon \to 0$ only one of the two terms in the wave functions in (2.11) remains. Taking $\epsilon \to 0$ is an excellent approximation unless we were to consider integrals of the functions $f^L_n(t)$ with weight functions that are singular as $t \to 0$.

Table 1 summarizes the results for the eigenvalues and eigenfunctions of the non-zero modes in that limit. The solutions shown correspond to positive $\phi$ and must be extended to negative $\phi$ in accordance with the orbifold symmetry. For option L the results take a different form depending on whether $\nu < \frac{1}{2}$ or $\nu > \frac{1}{2}$, as indicated. The second column in the table shows the equation that determines the eigenvalues $x_n$. In the next two columns we give the values of the coefficients $a^L,R_n$ of the properly normalized solutions. The normalization constants $A^L,R_n(x)$ obey $|A^L,R_n(x)|^2 = 2/[J_n(x)]^2$. The last column shows the chirality of the zero mode. The zero-mode wave functions can be recovered by taking the limit $x_n \to 0$; however, the normalization constants do not apply in this case. In Fig. 1 we show the first few Kaluza–Klein modes for option L and two values of the parameter $\nu$ just below or above the critical value $\nu = \frac{1}{2}$. The important point to notice is the localization of the right-handed zero mode $f^L_0(t)$ on the hidden brane (at $t = \epsilon$) for $\nu > \frac{1}{2}$.

For the special case of integer $\nu$ the exact solutions for the wave functions can be expressed in terms of trigonometric functions. As an example, we quote results for $\nu = 0$ and $\nu = 1$ choosing for the boundary conditions option L, which will be of special relevance to our discussion below. In both cases the non-zero eigenvalues are given by $x_n = n\pi/(1 - \epsilon)$ with an integer $n \geq 1$, and the left-handed solutions are $f^L_n(t) = A\sin[x_n(t - \epsilon)]$, where

<table>
<thead>
<tr>
<th>Option</th>
<th>Eigenvalues $x_n &gt; 0$</th>
<th>$a^L_n$</th>
<th>$a^R_n$</th>
<th>Zero mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>L, $\nu &lt; \frac{1}{2}$</td>
<td>$J_{\frac{1}{2} + \nu}(x_n) = 0$</td>
<td>$J_{\frac{1}{2} - \nu}(x_n)$</td>
<td>$0$</td>
<td>R</td>
</tr>
<tr>
<td>L, $\nu &gt; \frac{1}{2}$</td>
<td>$J_{-\frac{1}{2} + \nu}(x_n) = 0$</td>
<td>$0$</td>
<td>$J_{-\frac{1}{2} - \nu}(x_n)$</td>
<td>R</td>
</tr>
<tr>
<td>R</td>
<td>$J_{\frac{1}{2} + \nu}(x_n) = 0$</td>
<td>$0$</td>
<td>$J_{-\frac{1}{2} + \nu}(x_n)$</td>
<td>L</td>
</tr>
</tbody>
</table>
\[ |\mathcal{M}|^2 = 2/(1 - \epsilon). \] For \( \nu = 0 \), the right-handed solutions are given by
\[ f^R_0(t) = \frac{\mathcal{N}}{\sqrt{2}}, \]
\[ f^L_n(t) = -\mathcal{N} \cos \left[ x_n(t - \epsilon) \right], \]
whereas for \( \nu = 1 \) they take the form
\[ f^R_0(t) = \frac{\mathcal{N} \sqrt{\epsilon}}{\sqrt{2}} \frac{t}{t}, \]
\[ f^R_n(\tau) = \mathcal{N} \left( \frac{\sin \left[ x_n(t - \epsilon) \right]}{x_n} - \cos \left[ x_n(t - \epsilon) \right] \right). \]
(2.12)
(2.13)

3. Yukawa interactions and neutrino phenomenology

We will now show how including a sterile bulk fermion in the RS model can provide a mechanism for obtaining small Dirac neutrino masses, which is quite different from the see-saw mechanism. We focus first on a single fermion generation and consider a scenario where all matter and gauge fields charged under the Standard Model gauge group are confined to the visible brane at \( f_s \), whereas a gauge-singlet fermion field propagates in the bulk. After integration over the compact extra dimension we obtain a tower of four-dimensional Kaluza–Klein fermions in the four-dimensional theory, as shown in (2.4). We choose boundary conditions such that there is a right-handed zero mode (option L) with wave function \( f^R_0(t) \) given in (2.9). Only this choice will lead to an interesting neutrino phenomenology.

Omitting gauge interactions, the action for a Higgs doublet \( H = (\phi_1, \phi_2) \), a left-handed lepton doublet \( L = (\nu_L, e_L) \) and a right-handed lepton \( e_R \) localized on the visible brane is
\[ S = \int d^4x \sqrt{-g_{\text{vis}}} \left( \frac{1}{2} \sum \partial_\mu H^\dagger \partial^\mu H - \frac{1}{2} m^2(H^\dagger - m^2)^2 \right) \]
\[ - \lambda \left( H_0^2 - v_0^2 \right)^2 \]
\[ + \int d^4x \sqrt{-g_{\text{vis}}} \left( \bar{\nu}_L \gamma^\nu \gamma^\rho \nu_L + \bar{e}_R \gamma^\rho \gamma^\nu e_{R0} \right) \]
\[ - \left( y_\nu \bar{L}_0 H_0 e_{R0} + \text{h.c.} \right) \] (3.1)
where \( g_{\text{vis}}^{\mu\nu} = e^{2kr} \eta^{\mu\nu} \) is the induced metric on the brane, \( \sqrt{-g_{\text{vis}}} = \det(-g_{\mu\nu}) = e^{-4kr}, \) and \( \gamma^\nu = E_\nu(\phi = \pi) \gamma^\nu = e^{4kr} \gamma^\nu. \) To restore a canonical normalization of the fields on the brane we must perform the rescalings \( H_0 \to e^{4kr} H, \) \( L_0 \to e^{2kr} L \) and \( e_{R0} \to e^{2kr} e_{R} \), upon which the action takes the form
\[ S = \int d^4x \left( \partial_\mu H^\dagger \partial^\mu H - \lambda \left( H^\dagger - v^2 \right)^2 \right) \]
\[ + \int d^4x \left( \bar{\nu}_L / \partial L + \bar{e}_R / \partial e_R \right. \]
\[ - \left( y_\nu \bar{L}_0 H e_{R0} + \text{h.c.} \right) \] (3.2)
where \( v = e^{-4kr} v_0. \) The remarkable feature noted in [3] is that all dimensionful parameters such as the Higgs vacuum expectation value get rescaled by the warp factor and turned from Planck-scale into weak-scale couplings, whereas dimensionless parameters such as \( \lambda \) and \( y_\nu \) remain unchanged.
We now introduce a Yukawa coupling of the bulk fermion with the Higgs and lepton fields. With our choice of boundary conditions all left-handed Kaluza–Klein modes vanish at the visible brane, so only the right-handed modes can couple to the Standard Model fields on the brane. However, in a more realistic scenario which takes into account a finite width of the 3-branes there will most likely be a non-zero (and indeed sizeable) overlap of the left-handed modes with the Standard Model fields. Hence, in order to avoid weak-scale neutrino masses and lepton-number violating interactions we assign lepton number 3 to the fundamental fermion. The Kaluza–Klein ansatz of the bulk fermion mass matrix $m$ is naturally of order unity. After electroweak symmetry breaking, this Yukawa interaction gives rise to a neutrino mass term $\mathcal{L}_n = Y_n \bar{L}_n \nu L$ + h.c., which in the basis $\nu_L, \nu^c_L, \nu^c_R$ and $\psi_R = (\psi_R^0, \psi_R^1, \ldots, \psi_R^{n-1})$, with $n \to \infty$, takes the form

$$\mathcal{M} = \begin{pmatrix} \nu y_0 & \nu y_1 & \cdots & \nu y_n \\ 0 & m_1 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & m_n \end{pmatrix}.$$  

As a consequence, there will be a mixing of the Standard Model neutrino $\nu_L$ with the heavy, sterile (with respect to the Standard Model gauge interactions) bulk neutrinos $\psi_R$. The Kaluza–Klein excitations of the bulk fermion have masses $m_n$ of order the weak scale $v$. Thus, in order to obtain a light neutrino we need $|y_0| \ll 1$, which requires having a very small wave function of the zero mode on the visible brane, i.e., $|f_0^R(1)| \ll 1$. But this is precisely what happens if the fundamental fermion mass $m$ satisfies the condition $m > k/2$. Since, as mentioned earlier, the curvature $k$ must be smaller than the fundamental scale $M$, this is a natural requirement in the context of the RS model.

In order to study the properties of the physical neutrino states we diagonalize the squared mass matrix $\mathcal{M}^2$. The eigenvalues of this matrix are the squares of the physical neutrino masses, and the unitary matrix $U$ defined such that $U \mathcal{M}^2 U^\dagger$ is diagonal determines the left-handed neutrino mass eigenstates via $\psi^e_L = U \psi^\text{phys}_L$. We denote by $m_n$ the mass of the lightest neutrino $\nu^\text{phys}_L$ and define a mixing angle $\theta_n$ such that $\nu^\text{phys}_L = \cos \theta_n \nu^\text{phys}_L + \ldots$, where the dots represent the admixture of heavy, sterile bulk states. To leading order in the small parameter $|y_0| \ll 1$ we obtain

$$m_n = v |y_0| \cos \theta_n, \quad \tan^2 \theta_n = \sum_{n \geq 1} \frac{\nu^2 |y_n|^2}{m_n^2}. \quad (3.7)$$

Since the mixing angle is constrained by experiment to be very small (see below), it follows from (2.9) and (3.5) that

$$m_n = \sqrt{2 \nu - 1} |Y_n| e^{-\nu^2/2} \sim M \left( \frac{\nu}{M} \right)^{\nu^2/2}; \quad (3.8)$$

This result is remarkable, as it provides a parametric dependence of the neutrino mass on the ratio of the electroweak and Planck scales that is different from the see-saw relation $m_n \sim v^2/M$, except for the special case where $\nu = \frac{1}{2}$. This flexibility allows us to reproduce a wide range of neutrino masses without any fine tuning. For instance, taking $\nu/M = 10^{-16}$, the phenomenologically interesting range of $m_n$ between $10^{-7}$ eV and $10$ eV can be covered by varying $\nu$ between 1.1 and 1.5.

The measurement of the invisible width of the $Z^0$ boson, which yields $n_s = 2.985 \pm 0.008$ for the apparent number of light neutrinos [35], implies that the mixing angle $\theta_n$ must be of order a few percent. For instance, assuming an equal admixture of sterile...
neutrinos for the three generations of light neutrinos, we obtain \( n_\nu = 3 \cos^2 \theta_e \) and hence \( \tan^2 \theta_e = 0.005 \pm 0.003 \). From Table 1 it follows that with our choice of boundary conditions the wave functions of all excited right-handed Kaluza–Klein modes obey \( | f_R^u(1) | = \sqrt{2} \) (for \( e \to 0 \)). We thus obtain

\[
\tan^2 \theta_e = \frac{v^2 |Y_0|^2}{(ek)^2} \sum_{n=1}^{2} \frac{x_n^2}{2^\nu + 1} \frac{v^2 |Y_1|^2}{k^2},
\]

where \( x_n \) are the roots of \( J_{\nu-1/2}(x_n) = 0 \). The infinite sum can be performed exactly and yields \( 1/(2^\nu + 1) \). To satisfy the bound on the mixing angle for \( \nu = O(1) \) requires that \( v_0 |Y_0|^2/k \leq 0.1 \), which is possible without much fine tuning. We emphasize, however, that it would be unnatural to have the dimensionless combination \( v_0 |Y_1|/k \) much less than unity, so a mixing angle \( \theta_e \) not much smaller than the current experimental bound is a generic feature of our scenario, which can be tested by future precision measurements.

So far we have shown how a right-handed bulk fermion can give a small Dirac mass to a Standard Model neutrino. We now generalize this mechanism to three neutrino flavors and more than one bulk fermion. Interestingly, such a generalization is forced upon us by the requirement that the parity anomaly vanishes in terms of neutrino oscillations one needs two very different mass-squared differences: \( \Delta m_{21} \ll \Delta m_{32} \), where \( \Delta m_{1} = m_2^2 - m_1^2 \), and by convention \( m_1 \leq m_2 < m_3 \). This requires a minimum of two massive neutrinos; however, the third neutrino can be massless. In our minimal model this is indeed what happens. Although it is perhaps unconventional to consider a scenario where the number of right-handed neutrinos does not match the number of left-handed ones, we will see that our model explains successfully the known features of the neutrino mass and mixing parameters.

In order to explore this minimal model in more detail we ignore, for simplicity, the heavy Kaluza–Klein excitations of the bulk fermions and focus only on the zero modes. As mentioned above, the admixture of weak-scale sterile neutrino states must be strongly suppressed. It is natural to allow for the possibility that the two bulk fermions have different masses \( m_1 > m_2 \) (of order the Planck scale) in the fundamental theory, and that they couple with similar strength to the three left-handed neutrino flavors.

According to (2.9) and (3.5), the effective Yukawa matrix \( \mathscr{M} \) of the two right-handed zero modes \( \psi_L^{R,1} \) and \( \psi_L^{R,2} \) can then be parameterized as \( x/\epsilon \times \nu \psi_L^{R,1} \) and \( y/\epsilon \times \nu \psi_L^{R,2} \) with \( \nu = m_i/k \) (for \( i = 1,2 \)) and flavor-dependent couplings \( x/\epsilon \) (with \( f = e, \mu, \tau \)) of order unity. Note that the Yukawa couplings of the two zero modes have a very different magnitude: \( x/\epsilon \times \nu \psi_L^{R,1} \) and \( y/\epsilon \times \nu \psi_L^{R,2} \) in the truncated basis \( \psi_L^{R} = (\nu_L^{-}, \nu_L^{L}, \nu_L^{r}) \) and \( \psi_L^{R} = (\psi_L^{R,1}, \psi_L^{R,2}) \) is

\[
\mathscr{M} = \nu \epsilon^{-\nu - 1/2} \begin{pmatrix}
\epsilon^{\nu - 1} x_1 y_1 \\
\epsilon^{\nu - 1} x_2 y_2 \\
\epsilon^{-\nu - 1} x_1 y_1
\end{pmatrix}.
\]

Diagonalizing the matrix \( \mathscr{M} \mathscr{M}^T \) to leading order in \( \epsilon \) we find that the physical neutrino mass eigenstates comprise a massless left-handed neutrino \( \nu_1 \), a very light Dirac neutrino with mass squared

\[
m_{\nu_1}^2 = \nu^2 \epsilon^{-\nu - 1} \left[ \left[ \mu \nu \right]^2 + \left[ \tau \nu \right]^2 + \left[ \tau e \right]^2 \right]/\left[ y_1^2 + y_2^2 + y_3^2 \right] \]

\[
\sim M^2 \left( \frac{\nu}{\langle M \rangle} \right)^{2\nu + 1},
\]

and a light Dirac neutrino with mass squared

\[
m_{\nu_2}^2 = \nu^2 \epsilon^{-\nu - 1} \left[ \left| y_1^2 + y_2^2 + y_3^2 \right| \right] \]

\[
\sim M^2 \left( \frac{\nu}{\langle M \rangle} \right)^{2\nu + 1}.
\]

\(^4\) More complicated models with four or more bulk states are possible. These states could be subject to different boundary conditions. If we impose lepton number, only the right-handed modes can couple to the Standard Model fields. At least two right-handed zero modes are needed for a successful neutrino phenomenology.
In (3.11) we use the short-hand notation \([ij] = x_i y_j - x_j y_i\). Since the lightest neutrino is massless it follows that \(\Delta m^2_{21} = m^2_{1}\) and \(\Delta m^2_{32} = m^2_{1}\), and the ratio \(\Delta m^2_{21}/\Delta m^2_{32} \sim (\nu/M)^2\). An interpretation of the solar neutrino anomaly in terms of neutrino oscillations based on the MSW effect [36] yields values of \(\Delta m^2_{21}\) in the range \(10^{-6} - 10^{-5}\) eV\(^2\), whereas oscillations in vacuum would require a smaller value of order \(10^{-10}\) eV\(^2\) [22]. Such masses can be reproduced in our model by setting \(\nu_i = 1.34 - 1.37\) and \(\nu_i = 1.5\), respectively. An explanation of the atmospheric neutrino anomaly in terms of neutrino oscillations yields \(\Delta m^2_{21}\) in the range \(5 \cdot 10^{-4} - 6 \cdot 10^{-3}\) eV\(^2\) [21], which we can reproduce by taking \(\nu_i = 1.27 - 1.29\). In other words, we can understand the observed hierarchy of the experimentally favored neutrino masses in terms of a small difference of the bulk fermion masses in the fundamental theory. Note that in our minimal model the lightest neutrino is massless. This can be changed by introducing four (or more) bulk fermion states with more than two right-handed zero modes, in which case also the lightest neutrino becomes massive, with \(m^2_i = m^2_{1}\).

Despite the fact that the strong neutrino mass hierarchy is a generic feature of our model, the mixing matrix \(U\) relating the neutrino flavor and mass eigenstates does not contain any small parameter. Defining \(\nu_f = \sum_{i=1}^{3} U_{fi} \nu_i\) we find that all the entries \(U_{fi}\) are of order unity. In the limit \(\epsilon \rightarrow 0\) we obtain

\[
U = \begin{pmatrix}
U_{11} & U_{12} & U_{13} \\
U_{14} & U_{15} & U_{16} \\
U_{17} & U_{18} & U_{19}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{\mu \tau}{N_1} & y_\mu^* \frac{e \mu}{N_1} - y_\tau^* \frac{\tau e}{N_1} & y_\tau \\
\frac{\mu \tau}{N_1} & y_\mu^* \frac{e \mu}{N_1} - y_\tau^* \frac{\tau e}{N_1} & y_\tau \\
\frac{\mu \tau}{N_1} & y_\mu^* \frac{e \mu}{N_1} - y_\tau^* \frac{\tau e}{N_1} & y_\tau
\end{pmatrix}
\]

(3.13)

where \(N_i^2 = [\epsilon \mu]_i^2 + [\mu \tau]_i^2 + [\tau e]_i^2\) and \(N^2 = |y_\mu|^2 + |y_\tau|^2 + |y_\tau|^2\). A mixing matrix of this type, which lacks the strong hierarchy of the quark mixing matrix, can account for the experimental constraints on the neutrino mixing angles. (In fact, it has been pointed out that a fair fraction of random Dirac mixing matrices is consistent with these constraints [37].) The precise form of these constraints depends on how the data are analyzed, and whether the solar and atmospheric neutrino anomalies individually are interpreted in terms of two-neutrino or three-neutrino mixing. Constraints from the CHOOZ reactor experiment [38] combined with the atmospheric neutrino data imply that \(|U_{ee}|^2 \leq \text{few\%}\) [39], which means that \(|y_\mu|\) should be less than \(|y_\tau|\) and \(|y_\mu|\). In the limit \(|y_\mu|^2 \ll |y_\tau|^2 + |y_\tau|^2\), the mixing angles \(\theta_{12}\) and \(\theta_{23}\) responsible for \(\nu_\mu \leftrightarrow \nu_e\) and \(\nu_\mu \leftrightarrow \nu_e\) oscillations obey the approximate relations

\[
\sin^2 2\theta_{12} = \frac{4|\mu \tau|_J^2 (|y_\mu|^2 + |y_\tau|^2)}{|[\mu \tau]^2 (|y_\mu|^2 + |y_\tau|^2) + |[\mu \tau]^2/2}},
\]

\[
\sin^2 2\theta_{23} = \frac{4|y_\mu|^2 |y_\tau|^2}{(|y_\mu|^2 + |y_\tau|^2)^2}.
\]

(3.14)

The atmospheric neutrino anomaly is best explained by near-maximal \(\nu_\mu \leftrightarrow \nu_e\) mixing, such that \(\sin^2 2\theta_{23} > 0.82\) [20]. This implies \(0.64 < |y_\mu/y_\tau| < 1.57\), which clearly does not pose a problem for our model. Likewise, a large-mixing-angle solution to the solar neutrino problem requires \(\sin^2 2\theta_{12} > 0.75\) [22], which yields \(0.58 < |x_\mu|/|y_\mu|^2 + |y_\tau|^2 / |[\mu \tau]| < 1.73\). The small-mixing-angle MSW solution, on the other hand, prefers \(\sin^2 2\theta_{12} \sim 10^{-2}\), which would require that the quantities \(|x_\mu|/|y_\mu|^2 + |y_\tau|^2\) and \(|x_\mu y_\tau - x_\tau y_\mu|\) differ by about a factor 20. This could either be achieved by having \(|x_\mu| \ll |x_\mu|\), or via a near degeneracy of \(x_\mu y_\tau\) and \(x_\tau y_\mu\).

4. Conclusions

We have studied bulk fermion solutions in the localized gravity model with non-factorizable geometry introduced by Randall and Sundrum to solve the gauge-hierarchy problem. Similar to the case of scalar or vector fields propagating in the extra compact dimension, we have found that the Kaluza–Klein
modes have weak-scale masses even if the fermion mass in the fundamental, five-dimensional theory is larger than half the curvature $k$ of the compact space, an appropriate choice of the orbifold boundary conditions leads to a right-handed zero mode localized on the hidden brane, whose wave function on the visible brane is power-suppressed in the ratio of the weak scale to the fundamental Planck scale. Coupling the Higgs and left-handed lepton fields of the Standard Model, localized on the visible brane, with a bulk right-handed neutrino provides a new mechanism for obtaining small neutrino masses. Remarkably, this mechanism leads to a generalization of the see-saw formula with a different parametric dependence on the ratio $v/M$, which can easily reproduce neutrino masses in the range $10^{-5}$ eV to $10$ eV. Without much fine tuning the mixing of the Standard Model left-handed neutrino with sterile, weak-scale Kaluza–Klein excitations of the bulk fermion can be made consistent with experimental bounds. However, a generic prediction of our model is that such a mixing should occur at a level not much below the present bound.

Finally, we have shown that with an even number of bulk fermions one can obtain viable models of neutrino flavor oscillations, which naturally predict a mass hierarchy and a neutrino mixing matrix not containing any small parameter. A minimal implementation of this scenario consists of two right-handed neutrinos, identified with the zero modes of two bulk fermions with slightly different masses in the five-dimensional theory, coupled to the three left-handed neutrinos of the Standard Model. In this model we obtain a massless left-handed neutrino and two massive Dirac neutrinos with a large mass hierarchy and generically large mixing angles.

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Explicit calculation of the running coupling BFKL anomalous dimension

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Abstract

I calculate the anomalous dimension governing the \( Q^2 \) evolution of the gluon (and structure functions) coming from the running coupling BFKL equation. This may be expressed in an exact analytic form, up to a small ultraviolet renormalon contribution, and hence the corresponding splitting function may be determined precisely. Rather surprisingly it is most efficient to expand the gluon distribution in powers of \( a_Q \) rather than use the traditional expansion where all orders of \( a_s \ln(1/x) \) are kept on an equal footing. The anomalous dimension is very different from that obtained from the fixed coupling equation, and leads to a powerlike behaviour for the splitting function as \( x \to 0 \) which is far weaker, i.e. \( \sim x^{-0.2} \). The NLO corrections to the anomalous dimension are rather small, unlike the fixed coupling case, and a stable perturbative expansion is obtained.

1. Introduction

Small \( x \) physics has been an active area of theoretical research in the past few years, largely due to the first data for \( x < 0.005 \) being obtained by the HERA experiments [1,2]. The crux of the debate has been whether the standard DGLAP approach based on renormalization group equations and conventionally ordered simply in powers of \( a_s(Q^2) \), or the BFKL equation [3], which sums leading logarithms in \( (1/x) \), is most effective, and/or whether the two approaches need to be combined in some way. While the DGLAP approach has been relatively successful, albeit with some significant problems (a valence-like input gluon, undershooting for \( x \sim 0.01 \) at the highest \( Q^2 \); see [4]), the original BFKL prediction of a behaviour of the form \( x^{-\lambda} \) at small \( x \), with \( \lambda \sim 0.5 \), was clearly ruled out. A combination of the two approaches, using the BFKL equation to supplement the Altarelli-Parisi splitting functions with higher terms of the form \( a_s^{n+1} \ln^n(1/x) \) had some success (so long as one avoided factorization scheme ambiguities by working in physical quantities [5], but this was difficult to sustain with the most recent data. Moreover, the subject was thrown into some confusion by the calculation of the NLO correction to the BFKL equation [6,7].

In order to illustrate this I begin with a brief discussion of the LO fixed coupling BFKL equation. Working in moment space, i.e. defining the moment of the structure function by

\[
\mathcal{F}(N,Q^2) = \int_0^1 x^{N-1} F(x,Q^2) \, dx,
\]

(1.1)
and similarly for the parton distributions (scaled by $x$), the BFKL equation is
\[
 f(k^2, \bar{\alpha}_s/N) = f_s(k^2, Q_0^2)
\]
\[
 + \frac{\bar{\alpha}_s}{N} \int_0^\infty \frac{dq^2}{q^2} K_0(q^2, k^2) f(q^2),
\]
where $f(k^2, \bar{\alpha}_s/N)$ is the unintegrated gluon four-point function, $f_s(k^2, Q_0^2)$ is the zeroth order input, $\bar{\alpha}_s = (3/\pi) \alpha_s$, and the LO kernel is defined by
\[
 K_0(q^2, k^2) f(q^2) = k^2 \left( \frac{f(q^2) - f(k^2)}{|k^2 - q^2|} + \frac{f(k^2)}{4q^4 + k^4} \right)^{1/2}.
\]

It is convenient to define the input by $f_s(k^2, Q_0^2) = \delta(k^2 - Q_0^2)$, where in the case of deep-inelastic scattering, where one end of the gluon ladder is at a hard scale $Q^2$, while the other end is formally on-shell, $Q_0^2$ is just a collinear regularization which we let $\to 0$ ultimately. The “gluon structure function” is then given by
\[
 \mathcal{F}(Q^2, N) = \int_0^{Q^2} \frac{dk^2}{k^2} f(N, k^2, Q_0^2) \times g_N(N, Q_0^2),
\]
where $g_N(N, Q_0^2)$ is a bare, nonperturbative gluon density in the proton which implicitly absorbs the collinear divergences in $f(k^2)$. The BFKL equation is most easily solved by taking the Mellin transform to $\gamma$-space, i.e.
\[
 \tilde{f}(\gamma, N) = \int_0^\infty \frac{dk^2}{k^2} (k^2)^{\gamma - 1} f(k^2, N),
\]
where it reduces to
\[
 \tilde{f}(\gamma, N) = \tilde{f}_s(\gamma, Q_0^2) + (\bar{\alpha}_s/N) \chi_0(\gamma) \tilde{f}(\gamma, N),
\]
where $\tilde{f}(\gamma, Q_0^2) = \exp(-\gamma \ln(Q_0^2))$ and $\chi(\gamma)$ is the characteristic function
\[
 \chi_0(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma).
\]

A little manipulation leads to the expression
\[
 \mathcal{F}(Q^2, N) = \frac{1}{2\pi i} \int_{\gamma_0}^{1 + \pm i\infty} d\gamma \exp\left( \gamma \ln(Q^2/Q_0^2) \right)
\]
\[
 \times \frac{g_N(N, Q_0^2)}{\gamma(1 - (\bar{\alpha}_s/N) \chi_0(\gamma))}. \tag{1.8}
\]
This inverse transformation is dominated by the leading pole at $1 - (\bar{\alpha}_s/N) \chi_0(\gamma) = 0$, and the solution is
\[
 \mathcal{F}(Q^2, N)
\]
\[
 = \frac{1}{-(\bar{\alpha}_s/N) \chi_0(\gamma_0)} \left( \frac{Q^2}{Q_0^2} \right)^{\gamma_0} g_N(N, Q_0^2). \tag{1.9}
\]

The anomalous dimension $\gamma_0(\bar{\alpha}_s/N)$ may be transformed to $x$-space as a power series in $\bar{\alpha}_s \ln(1/x)$, and has a branch point at $N = 4\ln 2 \bar{\alpha}_s$ (at which $\gamma \to 1/2$) leading to asymptotic small $x$ behaviour for the splitting function
\[
 xP^0(x) \to 0.07 \bar{\alpha}_s \frac{x^{-3}}{(\bar{\alpha}_s \ln(1/x))^{3/2}}. \tag{1.10}
\]

One can, if one ignores the running of the coupling, proceed through exactly the same sort of arguments including the NLO correction to the kernel. In this case the “"intercept”” for the splitting function is shifted from $\lambda = 4\ln 2 \bar{\alpha}_s$ to $\lambda = 4\ln 2 \bar{\alpha}_s(1 - 6.5 \bar{\alpha}_s)$. This is clearly a huge correction, and implies the breakdown of the perturbative expansion for this quantity. Moreover, the power series for the splitting function is dominated by the NLO corrections at all values of $x$ below about $x = 0.01$. For example, using the formulae in [6] the first few terms in the power series for $P(x)$ go like
\[
 xP(x) = \bar{\alpha}_s + 2.4 \bar{\alpha}_s^2 \xi^3/6 + 2.1 \bar{\alpha}_s^3 \xi^4/120 + \cdots
\]
\[
 + 0.43 \bar{\alpha}_s^7 \xi^3 + 1.6 \bar{\alpha}_s^8 \xi^4 + 11.7 \bar{\alpha}_s^9 \xi^5/2
\]
\[
 + 13.3 \bar{\alpha}_s^{10} \xi^6 + 39.7 \bar{\alpha}_s^{11} \xi^7/24
\]
\[
 + 169.4 \bar{\alpha}_s^{12} \xi^8/120 + \cdots, \tag{1.11}
\]
where $\xi = \ln(1/x)$. Clearly, the size of the coefficients more than compensates for the extra power of $\alpha_s$. More careful calculations (discussed a little more
below), including the running coupling but using saddle-point approximations imply the same result, but with the coupling being evaluated at $Q^2$. This does nothing to aid the convergence, particularly at values of $Q^2 \sim 1 \text{GeV}^2$ where the perturbative analysis of structure function evolution does take place.

Hence, this NLO correction left the whole question of how to address the evolution of structure functions at small $x$ in question, and various alternatives have been proposed to help stabilize the calculation. In this paper I demonstrate that the correct way in which to calculate the $Q^2$ evolution at small $x$ is indeed to supplement the conventional AP splitting functions with higher order corrections obtained from the solution to the BFKL equation, but to take the running of the coupling constant fully into account in this equation. The details of this are presented below.

2. BFKL equation for running coupling

Beyond leading order it is impossible to ignore the running of the coupling, since at NLO ultraviolet regularization is required, resulting in a $\ln(k^2/\mu_R^2)$ term where $\mu_R$ is the renormalization scale. Such a term may be eliminated by using the running coupling constant evaluated at the scale $k^2$. Since this is unavoidably involved in our NLO, it seems sensible to consider the fixed coupling LO BFKL equation as just a model which would apply in a conformally invariant world, and more realistically to work with the BFKL equation with running coupling [8,9] from the beginning. Doing this we obtain

$$f(k^2, Q^2; \alpha_s(k^2)/N_f) = f_s(k^2, Q^2)$$
$$+ \frac{\alpha_s(k^2)}{N_f} \int_0^\infty \frac{dq^2}{q^2} K_0(q^2/k^2) f(q^2),$$

(2.1)

where

$$\alpha_s = 1/(\beta_0 \ln(k^2/A^2)).$$

(2.2)

$$\beta_0 = (11 - 2N_f/3)/(4\pi),$$

and $N_f$ is the number of active flavours. One can solve this equation in the same type of way as for the fixed coupling case, i.e. take the Mellin transformation with respect to $(k^2/A^2)$. It is most convenient first to multiply through by $\ln(k^2/A^2)$, and then obtain

$$\frac{df(k, N/\gamma)}{d\gamma} = \frac{df_s(k, Q^2)}{d\gamma} - \frac{1}{\beta_0 N}(\gamma)f(k, N/\gamma),$$

(2.3)

where $\beta_0 = (\pi \beta_0/3)$. Hence, the inclusion of the running coupling has completely changed the form of our double Mellin space equation, turning it into a first order differential equation. This has a profound effect on the form of the solutions. The equation may easily be solved giving,

$$\tilde{f}(\gamma, N) = \exp\left(-X_0(\gamma, N)/(\beta_0 N)\right)$$
$$\times \int_{\gamma}^\infty \frac{df_s(\tilde{\gamma}, N, Q^2)}{d\tilde{\gamma}}$$

$$\times \exp\left(X_0(\tilde{\gamma})/(\beta_0 N)\right)d\tilde{\gamma},$$

(2.4)

where

$$X_0(\gamma) = \int_{1/2}^\gamma X_0(\tilde{\gamma})d\tilde{\gamma}$$
$$= 2\psi(1)(\gamma - \frac{1}{2}) - \ln\left(\frac{\Gamma(\gamma)}{\Gamma(1 - \gamma)}\right).$$

(2.5)

The leading singularity in the $\gamma$ plane for $\exp(-X_0(\gamma)/(\beta_0 N))$, is cancelled by an integral from 0 to $\gamma$ of the integrand depending on $\tilde{\gamma}$ [10,11], and so up to $\mathcal{O}(\Lambda^4/k^2)$ corrections 2.4 simplifies to

$$\tilde{f}(\gamma, N) = \exp\left(-X_0(\gamma)/(\beta_0 N)\right)$$
$$\times \int_{0}^\infty \frac{df_s(\tilde{\gamma}, N, Q^2)}{d\tilde{\gamma}}$$

$$\times \exp\left(X_0(\tilde{\gamma})/(\beta_0 N)\right)d\tilde{\gamma},$$

(2.6)

and hence

$$\mathcal{F}(Q^2, N) = \frac{1}{2\pi i} \int_{\frac{i}{2} + i\varepsilon}^{\frac{i}{2} - i\varepsilon} \exp(\gamma \ln(Q^2/A^2))$$
$$- X_0(\gamma)/(\beta_0 N))d\gamma$$
$$\times \int_{0}^\infty \exp(-\tilde{\gamma} \ln(Q_0^2/\Lambda^2))$$
$$+ X_0(\tilde{\gamma})/(\beta_0 N))d\tilde{\gamma} g_s(Q_0^2/\Lambda^2)$$

$$= \mathcal{F}_s(Q^2, N) \mathcal{G}(Q_0^2, N) g_s(Q_0^2/\Lambda^2).$$

(2.7)
The essential expression, exp\(X_\gamma/\langle \beta_0 N \rangle\), contains singularities at all positive integers, and \(\mathcal{G}_k(Q^2, N)\) is not properly defined, since the integrand has singularities lying along the line of integration. These are due to the divergence of the coupling at low \(k^2\) and can only be removed by some infrared regularization. However, since this factor is independent of \(Q^2\), it does not contribute at all to the solution completely. The function \(\mathcal{G}_k(Q^2, N)\) is determined by the singularities of exp\((-X_\gamma/\langle \beta_0 N \rangle\) in the \(\gamma\) plane. This leads to a fundamental difference between the cases of the fixed and running couplings. Whereas previously the leading singularity was a pole at \((\overline{\alpha}/N)\chi(\gamma) = 1\), i.e. at \(\gamma \to \frac{1}{2}\) as \(N \to 4\ln 2 \overline{\alpha}\), now the leading singularity is an cut at \(\gamma = 0\): there is no powerlike behaviour in \(Q^2\). Similarly, the branch point in the \(N\) plane at \(4\ln 2 \overline{\alpha}\) has become an essential singularity at \(N = 0\): there is no powerlike behaviour in \(x\). The introduction of the running of the coupling has changed the character of the solution completely.

Acknowledging that the only real information contained in \(\mathcal{G}_k(N^2Q^2)\) is on the evolution of the structure function, i.e. defining

\[
\frac{d \ln \mathcal{G}(N, Q^2)}{d \ln (Q^2)} = \frac{d \ln \mathcal{G}_k(N, Q^2)}{d \ln (Q^2)} = \Gamma(N, Q^2).
\]

\[
(2.8)
\]

\(\mathcal{G}_k(N, Q^2)\) gives us an entirely perturbative effective anomalous dimension governing the evolution of the gluon structure function. The time-honoured technique for solving for \(\mathcal{G}_k(N, Q^2)\) is to expand the integrand in 2.7, about the saddlepoint. This results in a contour of integration parallel to the imaginary axis, with real part \(\to \frac{1}{2}\) for the small \(x\) solutions. Using this results in an anomalous dimension

\[
\Gamma(N, Q^2) = \gamma_0(\overline{\alpha}(Q^2)/N)
+ \sum_{n=1}^{\infty} (-\beta_0 \alpha_s(Q^2))^{n} \overline{\alpha}_n(\overline{\alpha}(Q^2)/N),
\]

\[
(2.9)
\]

i.e., the effective anomalous dimension is the naive leading order result with coupling at scale \(Q^2\) plus a series of corrections in increasing powers of \(-\beta_0 \alpha_s(Q^2)\). However, each of the \(\overline{\alpha}(\overline{\alpha}(Q^2)/N)\) is singular at \(N = \lambda(Q^2)\), and the power of the singularity increases with increasing \(n\) [12]. Hence, although the series for the resulting splitting function is in the small quantity \(\alpha_s(Q^2)\beta_0\), the accompanying coefficients are progressively more singular as \(x \to 0\). The saddlepoint approximation is therefore not a reliable result as \(x \to 0\) and explicit investigation reveals that it is only really quantitatively useful when \(\alpha_s(Q^2)\ln(1/x)\) is so small that the effective anomalous dimension is effectively the LO in \(\alpha_s\) part \(xP(x) = \overline{\alpha}(Q^2)\). Therefore any calculations of the anomalous dimension which rely on an expansion about the saddle-point lead to very inaccurate and misleading results for small \(x\).

This instability is not surprising if one examines the integrand along the saddle-point contour of integration, noting that it is very different from the Gaussian form the saddle-point method assumes [12], and also if one notes that it is an expansion obtained from approaching \(\gamma = \frac{1}{2}\) and in terms of functions of \(N\) which are singular at \(N = \lambda(Q^2)\), whereas we know that the full solution no longer sees these points as anything special. In fact, the knowledge of the singularity structure of the integrand implies that \(\gamma = 0\) is a more fruitful point on which to concentrate. Prompted by this we may move the contour of integration to the left and simultaneously use the property that the integrand dies away very quickly at infinity to close the contour so that it simply encloses the real axis for \(\gamma < 0\). It is then useful to express \(\chi_0(\gamma)\) in the form

\[
\chi_0(\gamma) = 1/\gamma + \sum_{n=1}^{\infty} 2\xi(2n+1)\gamma^{2n},
\]

\[
(2.10)
\]

which is strictly valid only for \(|\gamma| < 1\). Doing this we may write

\[
X_0(\gamma) = \ln(\gamma) + \gamma_x + \sum_{n=1}^{\infty} \frac{\xi(2n+1)}{2n+1} \gamma^{2n+1},
\]

\[
(2.11)
\]

and the integrand for \(\mathcal{G}_k(N, Q^2)\) becomes

\[
\gamma^{-1}(\overline{\alpha}(N))/1 \times \exp\left(\gamma t - \frac{1}{(\beta_0 N)} \gamma_x + \sum_{n=1}^{\infty} a_n \gamma^{2n+1}\right),
\]

\[
(2.12)
\]

where \(t = \ln(Q^2/\lambda^2)\) and \(a_n = 2\xi(2n+1)/(2n+1)\). The contribution to the integral from 0 \(\to -\infty +\)
\( \psi_e \) is now the same as that from \(-\infty - i\epsilon \rightarrow 0 \) up to a phase factor, and we may write
\[
\psi_e(N,t) = -\sin \left( \frac{\pi}{(\bar{\beta}_0 N)} \right) \exp \left( -\frac{\gamma_e}{(\bar{\beta}_0 N)} \right) \\
\times \int_{-\infty}^{0} y^{-1/(\bar{\beta}_0 N) - 1} \\
\times \exp \left( \gamma t - \frac{1}{(\bar{\beta}_0 N)} \sum_{n=1}^{\infty} a_n \gamma^{2n+1} \right) dy,
\]
(2.13)
where the integral has to be understood as an analytic continuation, since there are singularities along the real axis, and strictly speaking the integrand is well defined only for \( \gamma > -1 \). Since the factor of \( \exp(\gamma t) \), is present this latter point leads to an ambiguity of order \( \exp(-t) \), i.e. \( \psi(\Lambda^2/Q^2) \) into the value of \( \psi_e(N,t) \).

In order to evaluate the above integral it is convenient to let \( y = \gamma t \), resulting in
\[
\psi_e(N,t) = -\sin \left( \frac{\pi}{(\bar{\beta}_0 N)} \right) \exp \left( -\frac{\gamma_e}{(\bar{\beta}_0 N)} \right) t^{1/(\bar{\beta}_0 N)} \\
\times \int_{-\infty}^{0} y^{-1/(\bar{\beta}_0 N) - 1} \exp(y) \\
\times \exp \left( -\frac{1}{(\bar{\beta}_0 N)} \sum_{n=1}^{\infty} a_n (y/t)^{2n+1} \right) dy.
\]
(2.14)
The latter exponential may be expanded as a power series in \( y/t \) and the integral evaluated using the standard result that
\[
(-1)^n \int (-1/(\bar{\beta}_0 N) + n) \\
= \int_{-\infty}^{0} y^{-1/(\bar{\beta}_0 N) - 1} \exp(y) y^n dy,
\]
and hence
\[
\psi_e(N,t) = -\sin \left( \frac{\pi}{(\bar{\beta}_0 N)} \right) \exp \left( -\frac{\gamma_e}{(\bar{\beta}_0 N)} \right) t^{1/(\bar{\beta}_0 N)} \\
\times \sum_{n=3}^{\infty} A_n (1/(\bar{\beta}_0 N)) t^{-n} (-1)^n \\
\times \Gamma(-1/(\bar{\beta}_0 N) + n).
\]
(2.16)
This result was noted in [13], as was the fact that it may be simplified by using the relationship that as \( N \rightarrow 0 \),
\[
\Gamma(-1/(\bar{\beta}_0 N) + n) \rightarrow \Gamma(-1/(\bar{\beta}_0 N)K - 1/(\bar{\beta}_0 N)N^a.
\]
However, it is important to notice the more general result that for all \( N \)
\[
(-1)^n \Gamma(-1/(\bar{\beta}_0 N) + n) \\
= \Gamma(-1/(\bar{\beta}_0 N)) \Delta_n(-1/(\bar{\beta}_0 N)),
\]
(2.17)
where
\[
\Delta_n(-1/(\bar{\beta}_0 N)) \\
= (-1)^n \sum_{m=1}^{n} (-1)^m d_{mn}(\bar{\beta}_0 N)^{-m},
\]
(2.18)
where \( d_{mn} \) are positive coefficients and \( d_{nn} = 1 \). Hence, ignoring the common factor of \( -\sin(\pi/(\bar{\beta}_0 N)) \Gamma(-1/(\bar{\beta}_0 N)) \exp(-\gamma_e/(\bar{\beta}_0 N)) \), which has no \( t \) dependence, and is irrelevant for the anomalous dimension,
\[
\psi_e(N,t) = t^{1/(\bar{\beta}_0 N)} \left( 1 + \sum_{n=3}^{\infty} A_n (1/(\bar{\beta}_0 N)) t^{-n} \Delta_n \\
\times (-1/(\bar{\beta}_0 N)) \right),
\]
(2.19)
where the \( A_n \) are simply calculable from the expansion of \( \exp(-1/(\bar{\beta}_0 N) \sum_{n=1}^{\infty} a_n (y/t)^{2n+1} \). The common factor of \( t^{1/(\bar{\beta}_0 N)} \) is the well-known double-leading-log result coming from just the LO \( \alpha_e(Q^2)/N \) part of the anomalous dimension. Multiplying this we have an expansion as a power series in \( 1/t \) or equivalently in \( \alpha_e(Q^2) \). In fact
\[
(-1)^n \int t^{-n} (-1/(\bar{\beta}_0 N)) \\
= (\bar{\alpha}_e(Q^2)/N)^n \sum_{m=1}^{n} d_{mn}(-\beta_0 \alpha_e(Q^2))^{n-m} \\
\times (\bar{\alpha}_e(Q^2)/N)^{n-m}.
\]
(2.20)
This explicitly demonstrates that we obtain a set of running coupling corrections to a LO result. Substituting this expression for \( \psi_e(N,t) \) in 2.8 one obtains an expression for the anomalous dimension as a power series in \( \alpha_e(Q^2) \), where at each order we have the leading divergence in \( 1/N \) plus a sum of running coupling correction type terms. With a little work one may regain the whole leading \( \gamma_0(\alpha_e(Q^2)/N) \) (though it is necessary to keep some subleading...
terms in the $\Delta_\epsilon$ to do this), along with a tower of terms which are subleading in powers of $\beta_0 \alpha_s(Q^2)$ to this leading anomalous dimension, i.e. one obtains all the corrections to this naive LO anomalous dimension due to the running of the coupling.

The general features of this full, running coupling BFKL anomalous dimension may be appreciated quite easily. The important fact to note is that although the $\Delta_{\epsilon}(-1/(\bar{\beta}_0 N)) \to (1/(\bar{\beta}_0 N))^n$ as $N \to 0$, the function oscillates with $1/(\bar{\beta}_0 N)$, and remains very much smaller in magnitude than this asymptotic form until very small $N$, roughly until $1/N > n$. This coupled with the accompanying factor of $t^{-n}$ means that for reasonable $t$, i.e. $t > 5$ ($Q^2 \geq 1 \text{GeV}^2$), only the first 5 or so terms in 2.19 contribute for $N > 0.25$. Hence, to a very good approximation

$$G_e(N,t) = t^{1/(\bar{\beta}_0 N)} \left(1 - \frac{2\zeta(3)}{3(\bar{\beta}_0 N)t^3} \Delta_{\epsilon}(-1/(\bar{\beta}_0 N))ight) - \frac{2\zeta(5)}{5(\bar{\beta}_0 N)t^5} \Delta_{\epsilon}(-1/(\bar{\beta}_0 N)), \quad (2.21)$$

and in fact the smallness of the coefficient makes even the $t^{-5}$ term almost negligible. $G_e(N,t)$ initially grows as $N$ falls due to the $t^{1/(\bar{\beta}_0 N)}$ term. However, for $N \approx 0.6$ the negative contribution from the $t^{-3}$ term starts to contribute and ultimately drives the gluon structure function to negative values. The result is shown in Fig. 1. $dG_e(N,t)/dt$ may simply be evaluated also using 2.21, and shows the same general shape, but does not become negative until a slightly lower value of $N$ as also seen in Fig. 1. Hence the anomalous dimension develops a pole at a finite value of $N$, given quite accurately by

$$t^3 = \frac{2\zeta(3)}{3(\bar{\beta}_0 N)} \left(\frac{1}{(\bar{\beta}_0 N)} - \frac{3}{(\bar{\beta}_0 N)^2} + \frac{2}{(\bar{\beta}_0 N)}\right), \quad (2.22)$$

where we use the explicit form of $\Delta_{\epsilon}(-1/(\bar{\beta}_0 N))$. The value of $N$ for this leading pole is shown as a function of $t$ in Fig. 2, and for the sort of values relevant at HERA is $\approx 0.25$. Going to $N < 0.25$ higher order terms become important, and the positive $1/(\bar{\beta}_0 N)^2 t^6 \Delta_{\epsilon}(-1/(\bar{\beta}_0 N))$ term pulls $G_e(N,t)$ back to positive values, and another pole, with opposite sign residue, appears in $\Gamma(N,t)$. At even lower $N$ the analytic expression eventually breaks down, but numerical results show a series of poles becoming closer together. Nevertheless, the position of the leading pole is essentially determined by the first handful of terms in the power series in $\alpha_s(Q^2)$ for $G_e(N,t)$, and hence so is the asymptotic behaviour of the small $x$ splitting function, i.e.
Here I should comment on the limit of the analytic expression. As noted, it involves a series expansion not valid over the whole contour of integration. This is reflected in the fact that the overall magnitude of the $\Delta(-1/(\bar{B}_2 N))$ increases like $n!$ in general. This means that the series in 2.21 is actually asymptotic, although it is an oscillating series, so in principle is unambiguously resummable (i.e. the integral in 2.7 does exist). However, the greatest accuracy may be obtained from 2.21 by truncating the series at order $n_0 \sim t$. In practice I always use $n_0 = 5$. Substituting the truncated expression for $\tilde{\omega}(N,t)$ then results in an infinite series in $\alpha(Q^2)$ for $\tilde{\Gamma}(N,t)$ which is convergent for any given $N$ right of the leading pole. The accuracy of the analytic expression can be found by comparing with results obtained

$\sim x^{-0.25}$. Hence, the introduction of the running coupling has a dramatic effect on the singularity structure of the LO BFKL anomalous dimension, turning the cut into a series of poles, and changing the position of the rightmost singularity by a factor of $\sim 0.4$. This result of the pole in the anomalous dimension has been already noted in [14] using numerical techniques, and in the context of a resummed NLO anomalous dimension. Here I particularly stress the huge modification of the naive LO BFKL anomalous dimension. This huge change is apparent over a wide range of $N$, and in Fig. 3a I show the anomalous dimension as a function of $N$ for all values right of the leading singularity. As one sees, it is rather closer to the simple $\alpha(Q^2)/N$ expression than to the naive BFKL result.
from evaluating 2.7 using numerical integration, and for the gluon structure function for N to the right of the leading pole is found to be rather better than 0.1% for t = 6 and falls like exp(-t).

In order to investigate the quantitative effect of the BFKL anomalous dimension on structure function evolution it is necessary to calculate the BFKL splitting function as a function of x. This is where an analytic expression for the anomalous dimension is particularly useful. A series of numerically obtained values of Γ(N,t) allows an approximate determination of P(x,t), but it is very difficult to be accurate, especially for the wildly oscillating functions of 1/rN which do in fact make up GgN. However, I now have an explicit series for GgN, obtained from the truncated expression for GgN. The N-dependent functions E.. . Fig. 4. a. The splitting functions xP gs(x) and xP LLs(x) plotted as a function of x for t = 6 (Q^2 ~ 8 GeV^2). Also shown is the Ψ(αs(Q^2)) contribution Ψg(Q^2), and the naive LO BFKL splitting function with coupling αs(Q^2). b. The splitting functions xP gLO(x) and xP NLs(x) plotted as a function of x for t = 6.

at each power of αs(Q^2) of course become larger at small N as the series progresses, and to reach small enough x more and more terms are needed. However, at a fixed value of N there is no such growth, and the same is therefore true for fixed x. Hence, one only needs to work to a finite order. Limiting oneself to x > 10^-5 and t > 4.5, the suppression of the Δ,(-1/(B, N)) is quite significant and seventh order in αs(Q^2) is easily sufficient. The splitting function for t = 6 is shown in Fig. 4a. One sees that it is hugely suppressed compared with the naive LO BFKL splitting function, and is even lower than the 2x.. . 0.001. The fact that there is deviation from the standard NLO in αs(Q^2) splitting functions tells us that BFKL influenced structure functions are important. The ultraviolet renormalon contribution is approximated by constructing an x-space function which matches the N-space results for a variety of N. Although this contribution turns out to be a larger fraction of the total than in N-space, it still only makes a very small correction to the evolution. This will be discussed more in a subsequent paper [15].

3. Small x structure functions

It is important to realize that all of the above results are in a sense ambiguous because they deal with a particular way of defining the gluon parton distribution. It is defined in the natural way for a discussion of BFKL physics, but nonetheless is really an intrinsically factorization-scheme-dependent quantity. One may define a real structure function by including a hard scattering cross section at the top of the gluon ladder. This modifies 1.4 to

$$F_i(Q^2, N) = \alpha_s \int_0^\infty \frac{dk^2}{k^2} \sigma_{i,g}(k^2/Q^2) f(N, k^2, Q_0^2) g_b(N, Q_0^2).$$

(3.1)

Pulling out the overall factor of αs and taking the Mellin transformation of 3.1 with respect to (Q^2/Λ^2) leads to the simple expression

$$\tilde{F}_i(\gamma, N) = h_{i,g}(\gamma) \tilde{g}(\gamma, N).$$

(3.2)
Thus we may solve for $\mathcal{F}(N,t)$ in the same way as for $\mathcal{F}(N,t)$ obtaining the same divergent $Q^2$-independent part and a $Q^2$-dependent part given by solving

$$
\mathcal{F}_{E,i}(N,t) = \frac{1}{2\pi i} \oint_{\gamma} \frac{1}{\gamma} \left( \frac{1}{2} + i \gamma \right) \frac{h_{i,s}^{\prime}(\gamma)}{\gamma} \times \exp \left( \gamma t - X_0(\gamma)/\left( \bar{\beta}_0 N \right) \right) d\gamma.
$$

(3.3)

We may proceed as with the gluon structure function by expanding the $h_{i,s}^{\prime}(\gamma)$ which were calculated in [16] as a power series about $\gamma = 0$. This results in an expression

$$
\mathcal{F}_{E,i}(N,t) = t^{1/(\bar{\beta}_0 N)} \left( 1 + \sum_{n=1}^{\infty} B_{i,n} \times \left( 1/(\bar{\beta}_0 N) \right)^{-n} \Delta_n \left( -1/(\bar{\beta}_0 N) \right) \right).
$$

(3.4)

where the $B_{i,n}(1/(\bar{\beta}_0 N))$ are now determined not only by the power series in $\gamma$ obtained from the expansion of $X_0(\gamma)$, but also from the expansion of $h_{i,s}^{\prime}(\gamma)$. In particular they now contain parts at zeroth order in $1/(\bar{\beta}_0 N)$. In order to draw the most direct analogy to the gluon we define

$$
\Gamma_{LL}(N,t) = \frac{d \ln(\mathcal{F}_{E,i}(N,t))}{dt}.
$$

(3.5)

All the terms in the expression for this physical anomalous dimension, determined from 3.4, are part of the standard LO $\Gamma_{LL}^0(\alpha_s(Q^2)/N)$ or are subleading by powers of $\beta_i \alpha_s(Q^2)$ to this, i.e. again we obtain the naive LO result plus running coupling induced corrections to this, and the “coefficient function” $h_{i,s}^{\prime}(\gamma)$ contributes only to the running coupling corrections.

$\Gamma_{LL}(N,t)$ is shown along with the gluon anomalous dimension in Fig. 3a. Clearly the effect of the additional coefficient function, and hence additional running coupling corrections, is to make $\Gamma_{LL}(N,t)$ dip significantly below the simple $\mathcal{F}_{E,i}(Q^2)/N$ and to reduce the value of the intercept compared to the gluon structure function. This is reflected in the effective splitting function $P_{LL}(x,t)$ which is shown in Fig. 4a. This time the dip below the $\mathcal{F}(\alpha_s(Q^2))$ part is far more pronounced. Also, going to low enough $x$, we see that the splitting function turns over again, showing that the subleading poles in the anomalous dimension may have large residues compared to the leading pole, and the increase in $P_{LL}(x)$ with decreasing $x$ is not monotonic. This illustrates that as far as phenomenology at HERA, or any foreseeable collider, is concerned, the value of the intercept for the anomalous dimension is simply not relevant to the evolution of structure functions. I should note that in the case of the physical anomalous dimension and splitting function the power corrections due to the nonconvergence of the series are somewhat large, about 1% in $N$-space for $t = 6$, and lead to a small, but not insignificant effect on the evolution in $x$-space, and must be accounted for. This will be discussed more in [15].

One can follow exactly the same procedure for the other physical anomalous dimension

$$
\frac{\partial \mathcal{F}_{E,i}(N,Q^2)}{\partial \ln Q^2} = \Gamma_{LL}(Q^2;N) \mathcal{F}_{E,i}(N,Q^2).
$$

(3.6)

obtaining qualitatively very similar results. Again the splitting function initially dips as $x$ decreases, but grows again in the same way as $P_{LL}(x,t)$ at lower $x$. This has extremely important implications for the evolution of $\mathcal{F}_{E,i}(x,Q^2)$ in the HERA range, and this will be presented in detail in [15].

4. NLO corrections

So far I have demonstrated that using $\alpha_s(k^2)$ in the BFKL equation, as in 2.1, has a profound effect on the form of the solution for the anomalous dimension. However, it is necessary to check that the results presented are not severely modified by the inclusion of the NLO kernel, i.e. the perturbative calculations are stable, and also to justify that the choice of scale in the coupling is correct, or at least leads to accurate results. The NLO kernel was presented in [6] and the way in which to solve at NLO
with a running coupling was presented in [17]. Writing the NLO equation as

\[ f(k^2, Q^2_0) = f_i(k^2, Q^2_0) \]
\[ + \left[ \alpha_s(k^2) \right] \int_0^\infty \frac{dq^2}{q^2} (K_0(q^2, k^2)) \]
\[ - \alpha_s(k^2) K_i(q^2, k^2) f(q^2), \]

(4.1)

using just the one-loop coupling leads to a 2nd order differential equation in \( \gamma \)-space

\[ \frac{d^2 \bar{f}(\gamma, N)}{d\gamma^2} = \frac{d^2 f_i(\gamma, Q^2_0)}{d\gamma^2} \]
\[ - \frac{1}{\beta_0 N} \frac{d}{d\gamma} \left( \chi_0(\gamma) \bar{f}(\gamma, N) \right) \]
\[ - \frac{\pi}{3\beta_0^2 N} \chi_0(\gamma) \bar{f}(\gamma, N). \]

(4.2)

This can be solved in a very similar way to LO, i.e. it factorizes into the same form as 2.7 with \( Q^2 \)-dependent part given by

\[ \mathcal{G}_{ENLO}(N, t) = \frac{1}{2\pi i} \int_1^{1+\infty} \frac{1}{\gamma} \]
\[ \times \exp(\gamma t - X_{NLO}(\gamma, N)/(\beta_0 N)) d\gamma. \]

(4.3)

However, \( X_{NLO}(\gamma, N) \) is rather more complicated than the previous \( X_0(\gamma) \). It can be expressed in the form

\[ X_{NLO}(\gamma, N) = \int_{1/2}^\gamma X_{NLO}(\tilde{\gamma}, N) d\tilde{\gamma}, \]

(4.4)

where \( X_{NLO}(\gamma, N) \) can be written as a power series in \( N \) beginning at zeroth order with \( \chi_0(\gamma) \). As seen in [17], though here ignoring any resumptions in \( N \), the explicit form is

\[ \chi_{NLO}(\gamma, N) = \chi_0(\gamma) - N \chi_0(\gamma) + N^2 \left( \frac{\chi_0(\gamma)}{\chi_0(\gamma)} \right)^2 \]
\[ - \beta_0 \left( \frac{\chi_0(\gamma)}{\chi_0(\gamma)} \right) + \cdots. \]

(4.5)

where \( \chi_2(\gamma) \) would also appear at order \( N^2 \), had I included it. I shall generally ignore all but the first two terms.

Firstly, I shall address the choice of scale. It was known in [18] that the correct scale was really \( (k - q)^2 \), but that \( k^2 \) could be used, leading to a part of the NLO kernel which is proportional to \( \beta_0 \), i.e. there is a contribution to \( \chi_2(\gamma) \) of the form \( 1/2 \beta_0_0(\chi_3(\gamma) + \chi_0(\gamma)) \). Substituting this into 4.3 leads to a contribution in the integrand of the form \( \exp(\gamma t / (\chi_0(\gamma) + X_0(\gamma))) \). This can be expressed as a power series which at low orders is \( 1 + 1.6\gamma + 1.24\gamma^3 \). Hence, this scale-induced factor has the same form as the \( h_{1/2}(\gamma) \), and not surprisingly results in additional running coupling corrections to the anomalous dimensions. However, the terms in the series do not start until third order, have small coefficients, and have an effect much smaller than \( h_{1/2}(\gamma) \). Hence, the correction for this “incorrect” choice of scale is very small, though in principle it seems as though the factor just considered should really be taken as part of the LO result, since it just gives running coupling corrections only 1.

So the choice of \( \alpha_s(k^2) \) is in practice very reliable, and may easily be corrected for. We must now consider the rest of the NLO correction to the kernel, which is much larger. Here we have an ambiguity in precisely what the NLO calculation means. Do we simply solve 4.1, producing the infinite series in 4.5? Do we truncate \( \chi_{NLO}(\gamma, N) \) after the second term, and if so do we use the whole of \( \exp(1/\beta_0)^1/2 \)

\[ \chi_i(\tilde{\gamma})/\chi_0(\tilde{\gamma}) d\tilde{\gamma} \]

or just expand it out to first order? There are particular problems associated with all choices. I choose the NLO definition such that the anomalous dimension receives only corrections which are one power of \( \alpha_s(Q^2) \) down on the leading order one, i.e. the LO anomalous dimension is of the form \( \Gamma^0(\bar{\alpha}(Q^2)/N, \beta_0 \alpha(\bar{Q}^2)) \) and the NLO corrected one is of the form \( \Gamma^0(\bar{\alpha}(Q^2)/N, \beta_0 \alpha(\bar{Q}^2)) + \alpha_0(\bar{Q}^2) \Gamma^0(\bar{\alpha}(Q^2)/N, \beta_0 \alpha(\bar{Q}^2)) \). Roughly speaking, this involves keeping only the first two terms in 4.5, and expanding \( \exp(1/\beta_0)^1/2 \chi_i(\tilde{\gamma})/\chi_0(\tilde{\gamma}) \).
(\tilde{\gamma})/d\tilde{\gamma})$, out to just first order in $1/B_0$. However, the part of $(\chi(\tilde{\gamma})/\chi_0(\tilde{\gamma}))$ behaving like $1/\gamma$ must be treated very carefully. Details will be presented in [15].

Solving for the NLO anomalous dimension using the same techniques as at LO results in the NLO correction. Unlike the case of fixed coupling, or the naive results of the saddle-point evaluation, these corrections are rather small. The positions of the leading poles in the anomalous dimensions are shown in Fig. 2, and one can see that they change from about 0.21 for $\Gamma_{\pi\pi}$ at LO to 0.16 at NLO, and that the $Q^2$-dependence reduces a little. Similarly the anomalous dimension $\Gamma_{\pi\pi}(N,t)$ over a wide range of $N$ shows only a very small change going from LO to NLO. I use the physical anomalous dimensions to avoid any ambiguity. The results are very similar for the gluon distribution. This is shown in Fig. 3 where the part of the NLO anomalous dimension at first order in $1/\gamma$ is included, since this should properly be included at LO in a combined leading order in $\alpha_s(Q^2)$ and $\alpha_s(Q^2)\ln(1/x)$ expansion scheme. Alternative definitions of NLO lead to very similar results except at very high values of $N$. Note that the NLO correction is actually positive for $N \sim 0.9$ - very different from the case where running coupling corrections are not included and all NLO corrections are negative. I should also note that the ultraviolet renormalon contribution is a far larger proportion of the NLO correction than it is of the LO contribution, being typically 10% for $t = 6$, and needs to be accounted for [15].

One can also make the transformation to $x$-space and calculate the NLO corrected splitting function. This is shown for $t = 6$ in Fig. 4, where the contributions $\alpha\delta(1-x)$ both from the $\theta(\alpha_s(Q^2))$ part and the running coupling corrections to this are absent. The latter of these is a very small contribution. The NLO corrected splitting function is not too different from that at LO, as one can see. However, the real import of the NLO corrections as far as physics is concerned is the effect it has on the evolution of the structure function. This is demonstrated in Fig. 5 where the evolution of a suitable model for the structure function $F_2(x,Q^2)$, i.e. $(1-x)^{p}x^{0.2}$, is shown both for the LO running coupling splitting function, and for the NLO corrected one (all $\delta(1-x)$ contributions other that at first order in $\alpha_s(Q^2)$ one are included). As one sees, at this (rather low) value of $t$, i.e. $Q^2 \sim 8\,\text{GeV}^2$, the effect of the NLO corrections is only of order 10%, and is positive for $x \sim 0.01$. Also shown for comparison is the contribution from the $\pi(\alpha_s(Q^2))/x$ term alone. The running coupling BFKL splitting function leads to slightly quicker evolution than this latter contribution for $x \gtrsim 0.01$, but for $x \sim 0.0001$ the evolution is significantly suppressed.

Hence, the NLO corrections to the running coupling BFKL derived splitting function are well under control, both in terms of the asymptotic powerlike behaviour of the splitting functions and in terms of the evolution in the range currently accessible to
experiments. Beyond the running coupling corrections no further resummations are necessary, or even useful. This is very much in contrast to the case where both ends of the gluon ladder are associated with a hard scale, the so-called "single scale" processes. In this case, as shown in [19] and developed in [17] and [14], the conventional BFKL expansion is fundamentally flawed due to high order poles at \( \gamma = 0 \) and 1, which need to be resummed. Without resummation, all calculations are badly behaved over the full range of \( N \), not just at low \( N \). In the case of deep inelastic scattering the collinear factorization procedure automatically orders the poles at \( \gamma = 0 \) correctly, and the above problem shows up in high order poles at \( \gamma = 1 \) only. The anomalous dimension is totally dominated by the region very close to \( \gamma = 0 \), as this paper shows, and is very insensitive to effects at \( \gamma = 1 \). Including the type of resummation in [19,17] alters results from the NLO corrected case by a very small amount, and is possibly no more influential than the remaining NNLO effects for which it does not account. The corrected treatment at \( \gamma = 1 \) is essential if one is attempting to obtain information about the input form of the gluon, i.e. \( g_s(Q^2,N) \), but this, along with the whole subject of single-scale processes, is also plagued by the infrared divergence problem. A discussion of such issues can be found in [14,20].

5. Conclusions

I have shown that an analytic expression for the anomalous dimensions and splitting functions obtained from the running coupling BFKL equation may be obtained by a power series solution for the \( Q^2 \)-dependent part of the gluon structure function in terms of \( \alpha_s(Q^2) \). This has extremely good accuracy, with only very small errors, which may be interpreted as ultraviolet renormalon contributions, and may be calculated numerically. Moreover, I find the remarkable result that the form of this anomalous dimension and splitting function is almost completely determined by only the first handful (~ 5) of terms in the expansion for the gluon. This is in complete contrast with the case of fixed coupling, where an all orders summation is needed, and it would be interesting to understand the origin of this phenomena.

My results also prove that the effect of the running of the coupling is to weaken the asymptotic powerlike growth of the splitting functions severely compared to the naive results, and even to lower the splitting function below the \( \alpha_s(Q^2)/x \) contribution for \( 0.001 \geq x \geq 0.2 \). It also makes the NLO correction to the splitting functions relatively small, both for the value of the intercept and for the evolution of structure functions for \( x \gg 10^{-3} \), and therefore stabilizes the perturbative series. I also note that a previous conjecture that the effect of the running coupling in the BFKL equation could be accounted for using an \( x \)-dependent scale for the coupling [12], resulting in falling coupling for decreasing \( x \), turns out to be largely correct so long as the change in the scale of the coupling is moderate compared to the scale itself, though it fails when this criterion is not satisfied. In practice this condition is identical to that specifying that diffusion in the fixed coupling BFKL equation is not too large and therefore that the virtualities sampled in the running coupling equation are not too far away from \( Q^2 \). This results in the requirement that \( r^2 \geq 20 \ln(1/x) \) [21]. This covers most of the HERA range for \( Q^2 > 1 \) GeV\(^2\), and means that a phenomenological analysis using the explicit resummation in this paper leads to very similar results to the approach in [12]. In particular, the input \( F_s(x,Q_0^2) \) is of the same shape as \( F_s(x,Q_0^2) \) for \( Q_0^2 \) as low as 1 GeV\(^2\), in contrast to the standard NLO-in-\( \alpha_s(Q^2) \) approach, and the quality of the fit, particularly for small \( x \) data, is far better than using the conventional approach. A more detailed presentation of such a phenomenological study will appear. However, I can say that the inclusion of corrections to the fixed order in \( \alpha_s(Q^2) \) splitting functions from the running coupling BFKL equation not only leads to a stable perturbative expansion but also to a clear improvement in the comparison to data.

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References

Criterion of irreducibility of multi-loop Feynman integrals

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Abstract

The integration by parts recurrence relations allow to reduce some Feynman integrals to more simple ones (with some lines missing). Nevertheless the possibility of such reduction for the given particular integral was unclear. The recently proposed technique for studying the recurrence relations as by-product provides with simple criterion of the irreducibility.

1. Introduction

The growing accuracy of the experimental tests of the Standard Model of particle physics makes necessary calculation of higher order quantum corrections. The latter, in turn, are expressed through so called multi-loop Feynman diagrams (or integrals).

For example, the calculation of the \( O(\alpha_s^2) \) correction to \( R(s) \) (cross-section \( e^+e^- \rightarrow \text{hadrons} \)) demands calculation of the massive 3-loop propagator diagrams. At present such diagrams cannot be evaluated explicitly, but it is possible to construct the reliable approximation, considering their expansions in various kinematic regions [1]. The coefficients in these expansions can be related to a huge number (of order of millions) of 3-loop propagator massless and vacuum massive scalar diagrams, which differ from each other by degrees of their denominators.

Fortunately, one need not calculate all these integrals separately, because it is possible (using the integration by parts algorithm [2]) to find algebraic relations between them. Indeed, suppose we need to evaluate the dimensionally regularized L-loop Feynman integral (\( N = L(L + 1)/2 \)):

\[
B(n) \equiv \int \cdots \int \frac{d^D p_1 \cdots d^D p_L}{D_1^{a_1} \cdots D_N^{a_N}},
\]

where \( p_i \) (\( i = 1, \ldots, L \)) are loop momenta and \( D_a = A_a^\mu \cdot p_\mu - m_a^2 \) (\( a = 1, \ldots, N \)). This maximal number \( N \) of the propagators provides the possibility to express any scalar product of the loop momenta as a linear combination of the factors in the denominator. Possible numerators can be represented as propagators raised to the negative power. If the number of the propagators is greater than \( N \), partial fractioning can be used to deal with integrals with at most \( N \) propagators.

According to general rules of \( d \)-dimensional integration we can integrate by part in this integral or, in other words, the integral of total derivative according...
to loop momenta should vanish. From other side we
can evaluate this derivative as the linear combination
of the $B(n)$ with $n_i$ shifted by $\pm 1$:

$$0 = \int \cdots \int \left( \frac{\partial}{\partial p_i} \right) \cdot p_i \frac{d^n p_1 \cdots d^n p_L}{D_1^{n_1} \cdots D_N^{n_N}}$$

$$= R(I^+, I^-) B(n),$$

where $I^- B(...) \equiv B(...) - 1$ ...

One can use these relations to relate an integral
with some values of $n_i$ to more simple integrals. In
particular, the millions of integrals appeared in the
$O(\alpha^2)$ calculations mentioned above can be related
with 6 massless propagator and 3 massive vacuum
integrals, which should be calculated explicitly.

The most important step in this method is to find
the combination of the relations of (2) type allowing to reduce the given integral to the desirable set of
integrals. In other words, one should represent the
relations (2) in the recursive form, which allows to
reduce each index $n_i$ to the basic value (usually 0 or 1).
This problem is solved for some important cases
1, and the systematic way for the general case is the
subject of the intensive research 4 . The important
subproblem is test of the possibility to reduce the
given integral to simpler integrals. In other words, one should represent the
combination of the relations of 2 type allowing
to reduce each index

$B_n$ should also fit the (3) as the consequence of the
relations (2). Suppose we are able to construct some
specific $s(n)$ with the properties

$$s(n_i) \neq 0, \quad s(n_i) = 0 \quad \text{for} \quad i \in (1, \ldots, k).$$

Such $s(n)$ evidently does not fit (3) and hence (3)
cannot be the consequence of (2). So we got the
sufficient criterion of the irreducibility: the $B(n)$
cannot be represented as the linear combination of integrals from the set \{ $B(n)$, $i \in (1, \ldots, k)$ \} if the partial solution of (2) with properties (4) exists.

In practice, one can construct such solutions using the method of finding the explicit solutions of the
recurrence relations for Feynman integrals, proposed in \[5\]. The key idea is to represent these solutions in
the form of the auxiliary complex integral

$$s(n) = \int \frac{dx_1 \cdots dx_n}{x_1^{n_1} \cdots x_N^{n_N}} g(x_i). \quad (5)$$

The action of the $R(I^+, I^-)$ on (5) leads to

$$R(I^+, I^-) s(n) = \int \frac{dx_1 \cdots dx_n}{x_1^{n_1} \cdots x_N^{n_N}} R(\partial/\partial x_i, x_i) g(x_i)$$

$$+ \text{(surface terms)}. \quad (6)$$

So, if one chooses the integrand as the solution of the

$$R(\partial/\partial x_i, x_i) g(x_i) = 0 \quad (7)$$

and cancels the surface terms by proper choose of the
integration contours, one can fit $R(I^+, I^-) s(n) = 0$.

As it was shown in \[5\], the (7) can be solved for the
general case of the multi-loop Feynman integrals
with arbitrary number of legs and with arbitrary
masses, and the corresponding $g(x_i)$ can be represented
as the product of two polynomials in $x_i$, each
polynomial raised to non-integer degree (see \[5\] for
details).

3. Example

As example, let us consider the master 3-loop
massless non-planar integral. This integral is sup-
posed to be irreducible because of practical failure to
simplify it. To the best of the author’s knowledge, no proof has been found.

So, let us check the possibility to reduce this integral to the linear combination of the simpler integrals (with at least one line missing):

\[
e_{1}(d) + e_{2}(d) + \ldots
\]

where numbers numerate the lines. Let us define denominators as

\[
D_{1} = (l + p + q)^{2}, \quad D_{2} = (l + k + p + q)^{2},
\]

\[
D_{3} = (k + p + q)^{2}, \quad D_{4} = (k + p)^{2},
\]

\[
D_{5} = p^{2}, \quad D_{6} = (l + p)^{2}, \quad D_{7} = k^{2}, \quad D_{8} = l^{2},
\]

\[
D_{9} = 2 l \cdot k.
\]

Let us express the scalar products in terms of denominators:

\[
k^{2} = D_{7}, \quad k \cdot l = \frac{1}{2}D_{9}, \quad k \cdot p = \frac{1}{2}(D_{4} - D_{5} - D_{7}),
\]

\[
l^{2} = D_{8}, \quad p^{2} = D_{5},
\]

\[
k \cdot q = \frac{1}{2}(D_{2} - D_{1} - D_{4} + D_{5} - D_{9}),
\]

\[
l \cdot q = \frac{1}{2}(D_{2} - D_{3} + D_{4} - D_{6} - D_{8}),
\]

\[
l \cdot p = \frac{1}{2}(D_{6} - D_{5} - D_{8}),
\]

\[
p \cdot q = \frac{1}{2}(D_{1} - D_{2} + D_{3} - D_{5} + D_{9} - q^{2}).
\]  

(8)

According to [5], for this particular massless propagator-type case the integrand of (5) will be:

\[
g(x_{i}) = (q^{2})^{-1-d/2} P(x_{i})^{d/2-5/2},
\]

\[
P(x_{i}) = \det_{i_{j}}(p_{k} \cdot p_{i})(x_{j}),
\]  

(9)

where \( p_{k} = (p, k, l, q) \) and \( (p_{k} \cdot p_{i})(x_{j}) \) means (8) with substitution \( D_{i} = x_{i} \).

Let us construct \( s(n) \) with property: \( s(n) = 0 \) if some of \( (n_{1}, \ldots, n_{9}) \) less or equal to 0. The natural way to fit this condition is to choose the integration contours for \( (x_{1}, \ldots, x_{9}) \) as small circles around zero. In this case, according to Cauchy theorem, the integrations will lead to calculation of the \( (n_{1} - 1) \) coefficients in the Taylor expansion of the integrand (here and in the following we omit \( n_{1} \)-independent overall factor):

\[
s(n) \propto \int \frac{dx_{9}}{x_{9}^{n_{9}}} \left[ \frac{\partial^{n_{1}-1} \ldots \partial^{n_{1}-1}}{(n_{1} - 1)! \ldots (n_{8} - 1)!} \right] \times P(x_{i})^{d/2-5/2} \bigg|_{x_{1}, \ldots, x_{8} = 0}.
\]  

(10)

Finally, due to (8), (9), \( P(x_{i})|_{x_{1}, \ldots, x_{8} = 0} \propto x_{9}^{2}(q^{2} - x_{9})^{2} \) and the last integral can be expressed in terms of Pochhammer’s symbols \( (a)_{n} \equiv \Gamma(a + n)/\Gamma(a) \):

\[
\int dx_{9} x_{9}^{d-2}(q^{2} - x_{9})^{2} \propto (d - 4)/2(d - 8).
\]  

(11)

The (10), (11) define \( s(n) \) with desired properties: \( s(n) = 0 \) if some of \( (n_{1}, \ldots, n_{9}) \) less or equal to 0, because in this case the Cauchy integration over the corresponding \( x_{i} \) will lead to zero value; and due to (11) \( s(1, \ldots, 1, 0) \neq 0 \). It means, according to the general statement, that integration by parts relations can not reduce master 3-loop massless non-planar integral \( B(1, \ldots, 1, 0) \) to the linear combination of simpler integrals.

It is instructive to check what happens if we try the same procedure for integral which is known to be reducible, for example the ladder 3-loop massless integral:

\[
\cdot
\]

. In this case (if we choose the \( D_{9} \) as scalar product of the momenta of ”steps”) \( P(x_{i})|_{x_{1}, \ldots, x_{8} = 0} \propto x_{9}^{2} \), so we have only one branching point on \( x_{9} \) complex plane (instead of two for non-planar case) and can not define non-trivial \( x_{9} \) integration, as it should be.

### 4. Final remarks and conclusion

In this paper the sufficient criterion of the irreducibility is proposed: the given integral is irreducible if the specific solution of the recurrence relations (with properties (4)) exists. Formally one can prove the reverse statement with the following
restriction: suppose the given integral \( B(n_a) \) is irreducible to the set \( \{ B(n_i), i \in (1, \ldots, k) \} \) and we are able to construct the procedure which reduces any integral to the linear combination of the given integral and the integrals from this set:

\[
B(n) = c_0(n) B(n_a) + \sum_{i=1}^{k} c_i(n) B(n_i). \tag{12}
\]

Then the coefficient \( c_0(n) \) in front of the given integral will be the solution of the recurrence relations with properties (4). Proof: letting recurrence relations act on both sides of (12) we got zero on the left side, and linear combination of basic integrals on the right side. The coefficient in front of the given integral \( R(I^+, I^-)c_0(n) \) should be zero because otherwise this integral will be the linear combination of the integrals from the set \( \{ B(n_i), i \in (1, \ldots, k) \} \). It means that \( c(n) \) is the solution of the recurrence relations. Then if we consider (12) for \( n = n_i, i \in (1, \ldots, k) \) we got linear relations between \( B(n_i) \) and \( B(n_j) \). If \( B(n_a) \) is irreducible to \( B(n_j) \) then coefficient in front of \( B(n_a) \), namely \( c_0(n_a) \) should be zero. As the result \( c_0(n) \) fits specific properties (4).

So formally the existence of such specific solution is equivalent to irreducibility. But from the practical point of view to prove irreducibility one should explicitly construct this solution. Our experience (up to the 4-loop) shows that the representation (5) provides with such solutions, although at the 4-loop level it demands some efforts to find the proper integration contours.

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Leptogenesis and the small-angle MSW solution

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Abstract

The lepton asymmetry created in the out-of-equilibrium decay of a heavy Majorana neutrino can generate the cosmological baryon asymmetry $Y_B$ when processed through fast anomalous electroweak reactions. In this work I examine this process under the following assumptions: (1) maximal $\nu_\mu - \nu_\tau$ mixing (2) hierarchical mass spectrum $m_3 \approx 5 \times 10^{-2}$ eV $\gg m_2$ (3) small-angle MSW solution to the solar neutrino deficit. Working in a basis where the charged lepton and heavy neutrino mass matrices are diagonal, I find the following bounds on the heavy Majorana masses $M_i$: (a) for a symmetric Dirac neutrino mass matrix no other constraints, a $Y_B$ compatible with BBN constraints can be obtained for $\min(M_2, M_3) > 10^{11}$ GeV (b) if any of the Dirac matrix elements vanishes, successful baryogenesis can be effected for a choice of $\min(M_2, M_3)$ as low as a few $\times 10^9$ GeV. The latter is compatible with reheat requirements for supersymmetric cosmologies with sub-TeV gravitino masses. © 2000 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

The accumulating atmospheric neutrino data from SuperKamiokande [1] has greatly increased the likelihood that neutrinos are massive, and that there is mixing among the neutrino flavor states. Fits to the zenith angle distribution are consistent with (a) maximal $\nu_\mu - \nu_\tau$ mixing and (b) a (mass)$^2$ difference between the two mass eigenstates $|\Delta m^2| = 3 \times 10^{-3}$ eV$^2$. The solar neutrino data, both from SuperK [2] and other experiments [3–5], is as yet less definitive in constraining the neutrino masses and mixing: there exist the small-angle and large-angle solutions [6] of the MSW effect [7], as well as the vacuum oscillation solution. Omitting sterile neutrinos from consideration (perhaps with some unwarranted prejudice), one finds that each of these is not yet a clear favorite: the day-night asymmetry of solar neutrinos, if persistent at higher statistical significance, would disfavor the small-angle MSW and the VO solutions, while the reported recoil electron energy spectrum at SuperK requires significant experimental or theoretical modification at larger recoil energies in order to be compatible with the large-angle solution. It is safe to say that as yet none of the three is ruled out.

These recent advances have renewed interest in leptogenesis as the precursor to the establishment of the cosmological baryon asymmetry $Y_B \equiv n_B / (g^* n_s)$ $\approx 0.6–1.0 \times 10^{-10}$ required for a successful description of nucleosynthesis [8]. (Here $g^*$ is the effective number of spin degrees of freedom.) In the simplest leptogenesis scenario [9], which forms the basis for the discussion in this paper, a $B-L$ asymmetry is established through the $CP$- and $L$-violating out-of-equilibrium decay of the neutral heavy Majorana
lepton which partakes in the see-saw mechanism [10] for the light neutrino masses. (The atmospheric oscillation data indicating very small mass differences can be taken as supportive of the see-saw mechanism.) In the next stage of this scenario, the $B-L$ asymmetry is reprocessed through the fast $(B+L)$-violating anomalous processes [11,12] preceding the electroweak phase transition into the required $Y_B$ [13]. Because the successful completion of this process places non-trivial constraints on the both the Dirac and Majorana sectors of the neutrino mass matrix, it has been extensively discussed in this context in the literature [14]. The approach commonly taken is to explore the implications for leptogenesis of various models or ansatze for the three relevant mass matrices: $m_D = m_{\nu_D}^{\text{Dirac}}, m_\nu = m_{\nu_N}^{\text{Majorana}}, (\ell$ is the charged lepton.) These constitute important exercises in relating the neutrino data and the baryon asymmetry to the underlying Yukawa structures. However, it turns out that in certain experimentally allowed regions of the light neutrino mass matrix the analysis becomes much less dependent on the details of the Yukawa matrices, and compliance with the BBN-consistent $Y_B$ can translate more directly to a constraint on the heavy Majorana masses. This is the work of this paper: I will find that for a hierarchical light neutrino spectrum, maximal $\nu_\mu - \nu_\tau$ mixing, and the small-angle MSW solution to the solar neutrino deficit, agreement with the cosmological $Y_B$ can generally obtained for a (lightest) Majorana mass as small as $10^{11}$ GeV. Moreover, when any of the matrix elements of $m_D$ are zero (in a basis where $m_\nu$ and $M$ are diagonal), this bound can be lowered to $\sim 10^9$ GeV. In the concluding Section 1 discuss the implication of these numbers for the gravitino problem [15] of supersymmetric cosmology.

2. Assumptions

In what follows, I will work under the following assumptions consistent with present neutrino data:

(a) Light neutrino mass hierarchy $|m_3| = 5 \times 10^{-2} \text{ eV} \gg |m_2| \gg |m_1|$;
(b) Maximal $\nu_\mu - \nu_\tau$ mixing, consistent with SuperK atmospheric data [16,17];
(c) Small-angle MSW solution for the solar neutrino deficit;
(d) (Theory) The seesaw mechanism [10] is operative, with a hierarchical structure in the three heavy Majorana masses.

The hierarchical assumption (a) rules out consideration of a nearly-degenerate scenario for neutrino masses in the 2-3 sector. Assumption (c) (the adoption of the small angle MSW solution) is not dictated by observation. Compared to the large-angle MSW, it provides a marginally better (but not good) fit to the recoil electron energy spectrum at SuperK, and a less good fit to the day-night variations [6]. More data, and perhaps a better understanding of the hep neutrino spectrum, will decide the issue. However, it is assumption (c) which allows the study of the constrained system, and that is the reason for its adoption [18]. These assumptions also allow one to ignore renormalization group effects in running from the Majorana to low energy scale [19].

3. Seesaw relation for small angle MSW

Under Assumption (b), the electron neutrino plays no role in the atmospheric neutrino anomaly, and the light neutrino mixing matrix is given by [16,17]

$$U = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta/\sqrt{2} & \cos \theta/\sqrt{2} & 1/\sqrt{2} \\ \sin \theta/\sqrt{2} & -\cos \theta/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \quad (1)$$

where

$$U^T m_\nu U = \text{diag}(e^{i\phi_1} m_1, e^{i\phi_2} m_2, m_3). \quad (2)$$

The phases $(\phi_1, \phi_2)$ are Majorana CP-violating phases, and I have omitted the CP-violating CKM-type phase in $U$. The solar mixing angle is given by $\sin^2 2\theta$, and for the small-angle MSW solution $\sin^2 2\theta = 5 \times 10^{-3}$ [6], so that $\theta = 0.035$.

The seesaw mechanism is expressed by

$$m_\nu = m_D M^{-1} m_D^T \quad (3)$$

$$= \nu^2 \lambda M^{-1} \lambda^T \quad (4)$$

where the matrices $\lambda$ and $M$ are defined through the Lagrangian

$$\mathcal{L} = -\lambda L NH + \frac{1}{2} \lambda^T N M N^T \lambda. \quad (5)$$
so that \( m_{\mu} = \lambda v, \, v = 174 \text{ GeV} \). \( \lambda \) is a \( 3 \times 3 \) complex Yukawa matrix. I will work in a basis where the charged lepton masses and \( M \) are diagonal, so that

\[
M^{-1} = \text{diag}(1/M_1, 1/M_2, 1/M_3).
\]

Even if we were to know all six independent elements of the symmetric matrix \( m_e \) as well as the values of \( M_1, M_2, M_3 \), the seesaw condition (4) provides six equations for the nine complex unknown matrix elements of \( \lambda \). In general, the leptogenesis scenario requires knowledge of the entire matrix \( \lambda \), so that considerable input besides (4) is needed in order to determine \( \lambda \).

The situation improves a great deal for the small-angle MSW solution. I will simplify matters by taking \( \theta = 0 \) in Eq. (1), and \( m_1 = 0, M_1 \gg M_{2,3} \). These are sufficient to decouple \( \nu_e \) from the \( \nu_\mu - \nu_\tau \) seesaw and from the leptogenesis scenario. On input of Eqs. (1) and (2), the seesaw equation (4) (now \( 2 \times 2 \)) reduces to a set of three equations for the four (complex) unknowns \( a, b, c, d \), if we regard \( m_z, m_3, M_2, M_3 \) as ‘known’:

\[
\begin{align*}
(a^2 - c^2)/M_2 + (d^2 - b^2)/M_3 &= 0 \\
(a - c)/M_2 + (d - b)/M_3 &= m_2/v^2 \\
(a + c)/M_2 + (d + b)/M_3 &= m_3/v^2.
\end{align*}
\]

The Dirac Yukawa matrix \( \lambda \) has been parameterized as

\[
\lambda = \begin{pmatrix} a & d \\ c & b \end{pmatrix}
\]

and the Majorana phase \( \phi_2 \equiv \phi \) is incorporated into \( m_z, m_2 = |m_2| e^{i\phi} \).

### 4. The leptogenesis scenario

The leptogenesis scenario has been carefully discussed by many authors [20]. Briefly, the present baryon asymmetry of the universe is calculated in the following manner:

(a) First, the lepton asymmetry \( Y_L \) is given in terms of the decay asymmetry \( \epsilon \) of \( N \), the lightest of the \( N \)‘s, parameterized as follows:

\[
Y_L = \frac{n_L - n_\gamma}{g^* n_\gamma} = \kappa B \epsilon.
\]

where

\[
\epsilon = \frac{3}{16\pi} \frac{1}{(\lambda \lambda^*)_{ii}} \sum_{j \neq i} \text{Im} \left[(\lambda \lambda^*)_{ij} \frac{M_i}{M_j} \right]
\]

on the assumption of a mass hierarchy \( M_i \ll M_{j \neq i} \) [9,21]. The meanings of \( \kappa \) and \( B \) are as follows:

(i) Thermal production: If \( M_{\text{inflaton}} = \min(M_2,M_3) < T_{\text{RH}} \) (\( T_{\text{RH}} \) is the post-inflation reheat temperature) and the inverse decay rate is sufficient to establish equilibrium, then

\[
\kappa = \text{suppression factor due to washout by inverse decay and } 2 \rightarrow 2 \text{ lepton-violating scattering processes}
\]

\[
B = 1/g^*
\]

where \( g^* \) is the effective number of massless spin degrees of freedom at the time of \( N \) decay (\( g^* = 106.75 \) in the Standard Model). The factor \( \kappa \) is determined by numerical integration of the Boltzmann equations [8,22] and depends most sensitively on the ratio

\[
K = \Gamma/T(T = M_1)
\]

\[
= \left( \frac{(\lambda \lambda^*)_{ii} M_i}{8\pi} \right) \left( \frac{1.7\sqrt{G^*} M_{\text{inflaton}}}{M_{\text{inflaton}}} \right)^{-1}
\]

The suppression factor \( \kappa \) reaches its limiting value of 1.0 for \( K \ll 1 \), and drops to 0.01 for \( K = 20 \).

(ii) Non-thermal production via inflaton decay: If \( M_{\text{inflaton}} / T_{\text{RH}} \approx 10 \) and \( 1 \leq K \leq 100 \), then integration of the Boltzmann equations (starting at \( M/T \geq 10 \)) reveals negligible suppression due to inverse decays, and [23]

\[
\kappa = T_{\text{RH}} / M_{\text{inflaton}} = 10^{-3} \left( T_{\text{RH}} / 10^7 \text{ GeV} \right)
\]

\[
B = \text{average number of } N \text{ 's produced in decay of an inflaton}.
\]

(b) Finally, the baryon asymmetry is established when the \( B - L \) asymmetry is processed through the fast \((B + L)-\text{conserving sphaleron processes} [11,12] \)
above the electroweak transition temperature, and is given by [24]

\[ Y_b = -\frac{1}{2} Y_L. \]  

(13)

5. Results

It is clear from (9) that a calculation of \( Y_b \) will involve all the matrix elements of \( \lambda \), so that even the truncated 2 \( \times \) 2 seesaw equations are not quite sufficient to enable a casting of \( Y_b \) in terms of the masses alone. I will give results for the following illustrative constraints on \( \lambda \):

1. \( a, b, c, d, \) respectively, are set = 0;
2. \( c = d \) (symmetric case);
3. \( a = b. \)

Each of these will be worked out for both cases

(i) \( M_1 \gg M_2 \) (ii) \( M_2 \gg M_1 \). In all cases, the hierarchy \( m_3 \gg |m_2| \) will be respected.

The results listed in Table 1 are obtained by inserting these constraints into Eq. (4), solving for the matrix elements, and then utilizing Eqs. (9) and (11) to calculate \( \epsilon \) and the out-of-equilibrium parameter \( K \). The quantities \( x, y, K_0, \) and \( \epsilon_0 \) in the Table are defined as follows:

\[ x \equiv |m_2|/m_1, \]
\[ y \equiv M_{\text{smaller}}/M_{\text{larger}}, \]
\[ K_0 \equiv m_3/m_1, \]
\[ m_0 = (1.7)8\pi\sqrt{G_N} v^2/M_{\nu e} = 1.1 \times 10^{-3} \text{ eV} = 45. \]
\[ \epsilon_0 \equiv \frac{3}{16\pi} \frac{m_3 M_{\text{smaller}}}{v^2} \]
\[ = 10^{-5}(\frac{m_3}{0.05 \text{ eV}})(\frac{M_{\text{smaller}}}{10^{11} \text{ GeV}}). \]  

(14)

Two comments with respect to the results in Table 1 are in order:

1. As noted above, in the case of thermal production, the suppression factor \( \kappa \) depends on \( K \), and was obtained by integration of the rate equations [8,22], subject to the initial conditions \( Y_b(M/T = 0) = 1/g^* \), \( Y_0(M/T = 0) \), where \( Y_0 \equiv n_\nu/g^* n_\nu \). For non-thermal production, \( \kappa \) is given in Eq. (12) above.

I now proceed to calculate \( Y_b \) and require

\[ Y_b = Y_B^{\text{BN}} \geq 0.6 \times 10^{-10}. \]  

(15)

From Eqs. (8), (13) and (14), one obtains (ignoring signs)

\[ Y_B = 3 \times 10^{-8} \kappa (B g^*) (\epsilon/\epsilon_0) \left( \frac{M_{\text{smaller}}}{10^{11} \text{ GeV}} \right). \]  

(16)

where I have taken \( m_3 = 5.0 \times 10^{-2} \text{ eV} \). From (16) and (15), there results a lower bound on the lighter of \( M_2, M_3 \):

\[ \kappa (B g^*) \frac{(\epsilon/\epsilon_0)}{M_{\text{smaller}}} \geq 2 \times 10^6 \text{ GeV}. \]  

(17)

Incorporating the requirements \( x \leq \frac{1}{2} \) (mass hierarchy), \( |\sin \phi| \leq 1 \), the three lines in Table 1 can be addressed in turn for each of the scenarios, and a bound obtained from Eq. (17):

Thermal production:

Line 1:

\[ M_{\text{smaller}} \geq 2 \times 10^{11} \text{ GeV}. \]  

(18)

Line 2: In this case \( \kappa \) depends strongly on \( x \) through the dependence of \( K = 2 x K_0 \), and this is reflected in the bound for \( M_{\text{smaller}} \).

\[ \begin{align*}
&x = 0.25: \quad M_{\text{smaller}} \geq 4 \times 10^{10} \text{ GeV} \\
&x = 0.04: \quad M_{\text{smaller}} \geq 4 \times 10^9 \text{ GeV}
\end{align*} \]  

(19)

Line 3: As noted above, the expression for \( \epsilon/\epsilon_0 \) given in the Table reflects a simplifying constraint \( x \ll \frac{1}{2} y \). The resulting bound is

\[ M_{\text{smaller}} \geq 5.0 \times 10^{10} \text{ GeV} \geq 2 \times 10^{11} \text{ GeV}. \]  

(20)

\[ \begin{array}{|c|c|c|c|c|}
\hline
M_{\text{smaller}} - M_1 & M_{\text{smaller}} - M_2 & K/K_0 & \epsilon/\epsilon_0 & \kappa (M_{\text{smaller}} < T_{\text{RH}}) \\
\hline\hline
a = 0 \text{ or } c = 0 & b = 0 \text{ or } d = 0 & 1 & x\sin \phi & 4.0 \times 10^{-3} \\
\hline
b = 0 \text{ or } d = 0 & a = 0 \text{ or } c = 0 & 2 x & \frac{1}{2}\sin \phi & 1.0 \times 10^{-2} (a = 0.25) \\
\hline
c = d \text{ or } a = b & c = d \text{ or } a = b & 1 & (4.\sin \phi )^{-1/2} & 4.0 \times 10^{-3} \\
\hline
\end{array} \]
if we take $y < \frac{1}{2}$ in order to maintain the hierarchy in the heavy masses.

**Non-thermal production:** Here $\kappa$ is given by Eq. (12). In terms of a scaling factor $\zeta = \left(10^{10} \text{GeV}/T_{RH}\right)(100B)^{-1}$ one finds

- Line 1: For $x \leq \frac{1}{2}$
  
  \[ M_{\text{smaller}} \geq 8 \times 10^{11} \zeta \text{ GeV} \]  
  \[ (21) \]

- Line 2:
  
  \[ M_{\text{smaller}} \geq 4 \times 10^{11} \zeta \text{ GeV} \]  
  \[ (22) \]

- Line 3: With the same restrictions as in the previous section,
  
  \[ M_{\text{smaller}} \geq 1 \times 10^{12} \zeta \text{ GeV} \]  
  \[ (23) \]

### 6. Discussion of results and conclusions

(1) This work has focused on a particular sector of the neutrino mass spectrum (hierarchical) and mixing matrix (maximal $\nu_\mu - \nu_e$ mixing, small-angle MSW). With the seesaw mechanism, this greatly constrains the Dirac Yukawa matrix $\lambda$, effectively decoupling the electron neutrino and one of the heavy Majoranas. As a consequence, a single additional condition on the four complex matrix elements of the effective $\lambda$ allows its determination in terms of light and heavy masses, and a single CP-violating Majorana phase. The lepton asymmetry resulting from out-of-equilibrium decays of the heavy Majorana neutrinos may then be computed in terms of these parameters. Comparing the resulting cosmological baryon asymmetry with the value required from BBN then turns out to place a lower bound on the lighter Majorana mass.

(2) In the case of **thermal production**, for the range of scenarios studied (including several not reported in this paper, such as $b = e$), the lower bounds found for the mass of the lightest heavy Majorana are typically of $\mathcal{O}(10^{11}) \text{GeV}$, well below the inflaton mass of $\sim 10^{13} \text{GeV}$. Thus, the heavy Majorana may be produced during reheating via inflaton decay, without recourse to parametric resonance production [25]. A reheat temperature of $\mathcal{O}(10^{11}) \text{GeV}$ requires a large gravitino mass $\gtrsim 2.5 \text{TeV}$ [26] in order that decays of produced gravitinos not destroy the products of nucleosynthesis. If one of the entries in the Dirac Yukawa is zero, there are scenarios (Line 2 of Table 1), in which the smaller Majorana mass may fall below $10^{10} \text{GeV}$, which is a safe reheat temperature for low gravitino masses. In the case of **non-thermal production**, the lower bounds are a bit higher, and can exceed the inflaton mass $\sim 10^{13} \text{GeV}$ if the reheat temperature is less than $10^9 \text{GeV}$ (see Eq. (21)–(23)). In that case, production via parametric resonance would be necessary.

(3) Various small parameters, such as the solar mixing angle $\theta$ or the $U_{e3}$ element of the mixing matrix, have been set to zero. In principle, small entries for these can compete with the mass hierarchy parameters $x$ and $y$, and cloud the results of this work [27]. As a crude measure, one can limit the present discussion to values of $x, y \geq \theta = 0.03$. However, for the present CHOOZ bound $|U_{e3}|^2 \lesssim 5 \times 10^{-2}$, a similar criterion $x, y \geq |U_{e3}|_{\text{max}} = 0.2$ may be too restrictive. As the data improves, the effects of any small non-zero entries can be assessed.

(4) The discussion presented here is considerably more constrained than previous studies [14] which assume entire textures for both the Dirac and Majorana matrices, and often leave undetermined a good number of parameters. As stated in the introduction, such studies are valuable as links to larger theories of flavor symmetries, and are more flexible in accommodating a changing scenario for the neutrino parameters. The aim here is much more phenomenological, incorporating **ab initio** certain constraints on the light neutrino mass matrix, and leaving to vary only one complex parameter. Of course, increased statistics on the day-night effect could begin to seriously disfavor the small-angle MSW solution, thus removing the basis for the simplification in this work.

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Limited entropic uncertainty as new principle of quantum physics

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Abstract

In this paper the Principle of Limited Entropic Uncertainty (LEU-Principle) for the canonic conjugate variables in quantum physics is proved in a more general form by using Tsallis-like entropies for positive nonextensivities \( q > 0 \). Results on experimental tests of this new principle for the angle-angular momentum variables in the quantum scattering are illustrated by using 49 sets of pion-nucleus phase shifts.

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The Uncertainty Principle (UP) of quantum mechanics discovered by Heisenberg [1] which constitute the corner-stone of quantum physics, asserts that: (UP) “there is an irreducible lower bound on the uncertainty in the result of a simultaneous measurement of non-commuting observables”. An well known example, is the Heisenberg inequality [1] for the usual position-momentum \((\hbar = c = 1)\): 

\[
\frac{1}{\hbar} \leq \Delta x \Delta p,
\]

where \( \Delta y \) denotes the standard deviation of the observable \( y \). The extended version of this inequality for an arbitrary pair of quantum mechanical observable \( A \) and \( B \) is given by: 

\[
\frac{1}{2\exp(S_A)} \left( |\langle A,B \rangle| \right) \leq \Delta A \Delta B,
\]

where the corresponding standard deviations \( \Delta A(\psi) \) and \( \Delta B(\psi) \) for a quantum system in a given normalized state \( |\psi\rangle \) depends on the state \( |\psi\rangle \). In order to avoid this state-dependence many authors [2–7] proposed to use the information entropy as a measure of the uncertainty instead of above standard quantitative formulation of the Heisenberg uncertainty principle. Indeed, the entropy of a quantum state describing physical system is a quantity expressing the uncertainty or randomness of the system. So, an alternative mathematical formulation of the uncertainty principle for position-momentum pair is given by the entropic inequality (see Bialynicki-Birula et al. [2])

\[
\pi \epsilon \leq V_x V_p,
\]

where \( V_x \equiv \exp(S_x) \) and \( V_p \equiv \exp(S_p) \) are the statistical variances which are defined as the exponential of the differential entropies \( S_x \) and \( S_p \) corresponding to the \( (x \text{ and } p) \)-observables. Recently, a real progress in the investigation of the quantum entropy it was achieved not only by proving new entropic uncertainty relations [2–7] for the standard additive systems but also by generalization [7] of such results.
to the nonextensive statistics [8,9]. So, the state independent angle-angular momentum entropic lower bounds are proved [5,7] by using Tsallis-like entropies and Riesz theorem for the quantum scattering of the spinless particles. Moreover, the optimal entropic upper bounds for each Tsallis-like scattering entropy \( S_q \), \( S_l(q) \), \( S_d(q) \), were recently proved [6] in terms of optimal entropies derived from the principle of minimum distance in the space of states [10,11]. In this way, the entropic uncertainty band as a new concept in quantum physics is introduced. The generalized entropy as a measure of quantum uncertainty was also discussed in Ref. [12] in connection with the number-phase entropic uncertainty measure for the coherent states within the Pegg–Barnett quantum theory [13].

In this paper the Principle of Limited Entropic Uncertainty (LEU-Principle), as a new principle in quantum physics, is proved. Then, consistent experimental tests of the LEU-principle, obtained by using the available 49 sets of the pion-nucleus phase shifts \([14]\), are presented for both, extensive \((q = 1)\) and nonextensive \((q = 0.5\) and \(q = 2.0)\), statistics cases.

Moreover, some results obtained by the application of LEU-Principle to the diffraction phenomena are discussed.

**LEU-Principle for quantum scattering.** Here we present a short mathematical proof of the for the Principle of Limited Entropic Uncertainty in quantum scattering. So, we start with the scattering of two spinless particles for which all notation and definition from Ref. [5–7] will be adopted here. The angular distribution of probability \( P(x) \) as well as angular-momentum distribution \( \{ p_l, l = 0, 1, 2, \ldots \} \) are presented in Table 1, where \( \frac{dp}{dx} (x) = |f(x)|^2 \) is the differential cross section, \( x = \cos \theta \), \( \theta \)-being the c.m. scattering angle, \( \sigma_{\text{el}} \) is the integrated elastic cross section, \( f_j \) are the partial amplitudes while \( f(x) = \sum (2l + 1)f_j P_j(x) \) is the two-body scattering amplitude and \( P_j(x) \) are Legendre polynomials. Hence, using the results of Refs. [6,7] for the Tsallis-like scattering entropies, in this paper we prove the following standard quantitive formulation of the LEU-Principle.

**LEU-inequalities for scattering:** Let \( \sigma_{\text{el}} \) and \( \frac{dp}{dx} (y) \) be fixed from experiment for a fixed \( y \in [−1,+1] \).

### Table 1

The canonical conjugate variables (1), optimal distributions (2), (3), reproducing kernels (4), optimal variances (5)–(7), and Uncertainty Bands (8), corresponding to optimal scattering are compared with those of the diffraction on a single slit.

<table>
<thead>
<tr>
<th>No.</th>
<th>Optimal scattering</th>
<th>Diffraction on single slit of width 2a</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>( l = \text{angular momentum and } x = \cos \theta )</td>
<td>( l = k\xi, x = p )</td>
</tr>
<tr>
<td>(2)</td>
<td>( p_l^{(1)} = \frac{x}{k}, l = 0, \ldots, L_0, p_l^{(2)} = 0, ) ( l \geq L_0, N_l = \sum (2l + 1) = (L_0 + 1)^2, ) ( (L_0 + 1)^2 = \frac{2\pi}{\pi} X )</td>
<td>( P_l^* (l_0) = 1 / N_0, N_0 = 2ka )</td>
</tr>
<tr>
<td>(3)</td>
<td>( P_{l_1} (x) = \frac{[K(x,1)]^2}{K(1,1)} )</td>
<td>( X = \text{kap, } P^* (p) = \frac{[K(X,0)]^2}{K(0,0)} )</td>
</tr>
<tr>
<td>(4)</td>
<td>( K(x,0) = \frac{1}{2} \sum (2l + 1) P_j (x) ) ( = \frac{1}{2} \left[ \Phi_{l_0+1} (x) + \Phi_{l_0} (x) \right], ) ( K(1,1) = \frac{1}{4} \sum (L_0 + 1)^2 )</td>
<td>( X = \text{kap, } K(X,0) = \left( \frac{x}{X} \right)^{2\pi}, K(0,0) = \frac{\pi}{\sigma_{\text{el}}} )</td>
</tr>
<tr>
<td>(5)</td>
<td>( V_{l_0}^* = \exp \left[ -\int_{x=0}^{1} dx \frac{[K(x,1)]^2}{K(1,1)} \ln \left( \frac{[K(x,1)]^2}{K(1,1)} \right) \right] ) ( = N_0 )</td>
<td>( V_{l_0}^* = \exp \left[ -\frac{1}{2} \int_{x=0}^{1} dx \left( \frac{x}{X} \right)^2 \ln \left( \frac{x}{X} \right) \right] ) ( = N_0 )</td>
</tr>
<tr>
<td>(6)</td>
<td>( V_{l_0}^* V_{l_0}^* = \exp \left[ S_{l_0}^2 + S_0^2 \right] ) ( = N_0 \exp \left[ -\int_{x=0}^{1} dx \frac{[K(x,1)]^2}{K(1,1)} \ln \left( \frac{[K(x,1)]^2}{K(1,1)} \right) \right] )</td>
<td>( V_{l_0}^* V_{l_0}^* = \exp \left[ S_{l_0}^2 + S_0^2 \right] ) ( = 2 \pi \exp (2(1 - C)) )</td>
</tr>
<tr>
<td>(7)</td>
<td>( 2 &lt; \xi V_{l_0} \leq N_0 V_{l_0}^* )</td>
<td>( \pi \leq \xi V_{l_0} \leq 2 \pi \exp (2(1 - C)) = N_0 V_{l_0}^* )</td>
</tr>
</tbody>
</table>
Let \( V_{\theta L}(q) = \exp[S_{\theta L}(q)] \) be the statistical variances corresponding to the Tsallis-like scattering entropies \( S_{\theta L}(q) \). Then, the LEU-inequalities can be given in the following general form:

\[
\exp \left[ (1 - 2^{-1-q})/(q - 1) \right] \leq V_{\theta L}(q) \leq V_{\theta L}^{\alpha}(q),
\]

for \( q \geq 0 \),\(^\dagger\)

\[
\exp \left[ (1 - 2^{-1-q})/(q - 1) \right] \leq V_{\theta L}^{\alpha}(q) \leq V_{\theta L}(q),
\]

for \( q < 0 \),\(^\dagger\)

where

\[
V_{\theta L}^{\alpha}(q) = \exp \left[ \frac{1}{q - 1} \left( 1 - \sum (2l + 1) \right) \right.
\]

\[
\times \left. \left[ \frac{P_l^2(y)}{2K_{l,y}(y)} \right]^{q + 1} \int \frac{dx}{d} \left[ \frac{K(x,y)}{K(y,x)} \right] \right] \right)^{q/2},
\]

and the reproducing kernels \( K(x,y) \) and \( K(y,y) \) for two-body scattering (see Ref. [10]) are expressed as

\[
K(x,y) = \frac{1}{2} \sum_{l=0}^{L_{xy}} (2l+1) P_l(x) P_l(y)
\]

\[
= \frac{L_{xy} + 1}{2} \left[ \hat{P}_{l,y}(x) P_{l,y}(y) - P_{l,y}(x) \hat{P}_{l,y}(y) \right] / x - y,
\]

and

\[
K(y,y) = \frac{1}{2} \sum_{l=0}^{L_{xy}} (2l+1) P_l(y) P_l(y)
\]

\[
= \frac{L_{xy} + 1}{2} \left[ \hat{P}_{l,x}(y) P_{l,x}(y) - \hat{P}_{l,x}(y) P_{l,x}(y) \right],
\]

and the optimal angular momentum \( L_{xy} \) can be obtained by solving the implicit algebraic equation

\[
\frac{d\sigma}{d\Omega}(y) = K(y,y) \frac{\sigma_{el}}{2\pi},
\]

and

\[
\hat{P}(z) \equiv \frac{d\sigma}{d\Omega}(z)
\]

The results on the LEU-Principle [(2)–(7)], for the particular case \( y = 1 \), are presented in Table 1.

\(^\dagger\) Proof: Therefore, using the Tsallis-like \( S_{\theta L}(q)_1, S_{\theta L}(q)_2, S_{\theta L}(q)_3 \) scattering entropies, given in Ref. [6] by Eq. (6)–(11), for the canonical conjugate variables [\( \cos \theta, \ell \)] we define the statistical entropic variances

\[
V_{\chi}(q) = \exp[S_{\chi}(q)], \chi = \theta, L, \ell, q \in R,
\]

where the entropic index \( q \neq 1 \) controls the degree of nonextensivity of the Tsallis-like entropy reflected in the pseudoadditivity entropy rule: \( S_{\theta L}(q) = S_{\chi}(q) + S_{\chi}(q) + (1 - q) S_{\chi}(q) S_{\chi}(q) \).

Now, an optimal upper bounds on the statistical variances \( V_{\chi}(q), \chi = \theta, L, \ell, q \in R \) can be obtained, as in Refs. [6], by solving the following constrained optimization problem

\[
\max / \min \{ V_{\chi}(q) \} \text{ when } \sigma_{el} \text{ and } \frac{d\sigma}{d\Omega}(y), \text{ are fixed}
\]

or equivalently

\[
\begin{align*}
\text{min} & \quad \lambda_0 V_{\chi} + \lambda_1 \left[ \frac{\sigma_{el}}{4\pi} - \sum (2l + 1) \left| f_l \right|^2 \right] \\
& \quad + \lambda_2 \left[ \frac{d\sigma}{d\Omega} (1) - \sum (2l + 1) f_l P_l(y) \right] \\
& \quad \rightarrow \max / \min,
\end{align*}
\]

where \( \lambda_i = 0, 1, 2 \), are the corresponding Lagrange multipliers. Then, the singular solution \( \lambda_i = 0 \) of the Lagrange function (10) is obtained by solving the minimization problem

\[
\begin{align*}
\left[ \sum (2l + 1) \left| f_l \right|^2 + \alpha \left[ \frac{d\sigma}{d\Omega} (1) \right. \\
& \quad - \left. \sum (2l + 1) f_l P_l(y) \right] \right] \rightarrow \min.
\end{align*}
\]

Thus, the unique solution of the problem (11) exists and is given by

\[
f(x) = f_{\sigma_{el}}(x)
\]

\[
= f(x) \frac{K(x,y)}{K(y,y)}, x, y \in [-1, +1],
\]

where \( f(x) \) is the two-body scattering amplitude and the reproducing kernels functions \( K(x,y) \) and \( K(y,y) \) are given by Eqs. (5) and (6). Consequently,
one of the final result is not only the upper bound (2) for \( q \geq 0 \), but also the inequalities

\[
V_{\theta}(q) \leq V^{\text{o}_0}(q), V_x(q) \leq V^{\text{o}_x}(q),
\]

\[
V_{\theta L}(q) \leq V^{\text{o}_{\theta L}}(q), \quad \text{for} \quad q \geq 0,
\]

(13)

and

\[
V^{\text{o}_0}(q) \leq V_{\theta}(q), V^{\text{o}_x}(q) \leq V_x(q),
\]

\[
V^{\text{o}_{\theta L}}(q) \leq V_{\theta L}(q), \quad \text{for} \quad q \leq 0,
\]

(14)

where \( \{V^{\text{o}_x}(q), X = \theta, L, \theta L\} \) are the \textit{optimal statistical variances} corresponding to the optimal state (12), and are given by

\[
V^{\text{o}_x}(q) = \exp \left[ \frac{1}{q-1} \left( 1 - \int_{-1}^{1} \frac{\left[ K(x,y)^2 \right]^{q-1}}{K(y,y)} \right) \right], \quad q \in R,
\]

(15)

\[
V^{\text{o}_{\theta L}}(q) = \exp \left[ \frac{1}{q-1} \left( 1 - \sum (2l + 1) \right) \times \left[ \frac{p_l^y(y)}{2K(y,y)^{q-1}} \right]^q \right], \quad q \in R,
\]

(16)

while the optimal statistical variance \( V^{\text{o}_{\theta L}}(q) \) is presented in formula (4). We note that, the results (15), (16), are obtained with the aid of the \textit{optimal probability distributions}

\[
p_l^y = \frac{P_l(y)}{[2K(y,y)]^{1/2}}, \quad \text{for} \quad l = 0, 1, \ldots, L_{\theta y},
\]

and \( p_l^y = 0, \quad \text{for} \quad l > L_{\theta y},
\]

(17)

\[
P^{\text{o}_x}(x) = \frac{[K(x,y)^2]}{K(y,y)}, \quad K(y,y) \neq 0,
\]

\[y \in [-1, +1].\]

(18)

Therefore, we can conclude that by Eq. (13) for \( V_{\theta L}(q) \) we proved the upper \textit{bound part of the LEU-Principle}. The proof of the lower bound \( V^{\text{o}_{\theta L}}(q) \) (2), for \( q \geq 0 \), can be obtained from (4) as the limit of \( V^{\text{o}_{\theta L}}(q) \) when \( L_{\theta L} \to 0 \). The \textit{lower bound} \( V^{\text{o}_{\theta L}}(q) \) of the \textit{LEU-Principle} (3), for \( q < 0 \), can be proved as follows. First we prove, via Lagrange multipliers, the bounds: \( [1 - 2^{1-q}]((q-1) \leq S_0(q), 0 \leq S_{\theta L}(q), \) for \( q < 0 \). These bounds are equivalent to: \( 2^{1-q} \leq 1 + (1 - q)S_0(q) \) and \( 1 \leq 1 + (1 - q)S_{\theta L}(q) \), respectively. Then, by combining these two last inequalities we obtain the stated result: \( (1 - 2^{1-q})/(q - 1) \leq S_0(q), \) or, equivalently \( V^{\text{min}}_{\theta L}(q) \leq V^{\text{o}_{\theta L}}(q) \).

\textbf{LEU-inequalities for Diffraction on single slit}. As an example we consider here the diffraction experiment of an incoming beam of monochromatic photons with momentum \( k \), incident on a wall that contains an infinitely long slit of width \( 2a \). Here we preserve all notations and definitions from Ref. [14]. Hence, by choosing the canonical conjugate variables \( l_{\xi} = k\xi \) and \( p \), it is easy to prove the result from Table 1, which are analogous to the results obtained for the \textit{optimal states} of the two body elastic scattering.

Therefore, the \textit{LEU-inequalities} for diffraction on single slit can be given in the following general form:

\[
\exp \left[ \frac{1 - (2\pi)^{1-q}}{(q - 1)} \right] \leq V_{\theta L}(q) \leq V_{\theta L}^{\text{o}}(q),
\]

(19)

\[
\exp \left[ \frac{1 - (2\pi)^{1-q}}{(q - 1)} \right] \leq V_{\theta L}^{\text{min}}(q) \leq V_{\theta L}^{\text{o}}(q),
\]

(20)

for \( q \geq 0 \), where

\[
V_{\theta L}^{\text{o}}(q) = \exp \left[ \frac{1}{q-1} \left( 1 - \frac{1}{N_{\theta L}^{-q-1}} \right) \times \int_{-a}^{a} dp \left[ \frac{K(p, p_{\theta L})^{-q-1}}{K(p, p_{\theta L})} \right] \right],
\]

(21)

where

\[
N_{\theta L} = \int_{-a}^{a} d(k\xi) = 2ka \equiv \text{‘‘number of states’’}
\]

(22)

and the \textit{reproducing kernel} can be given by

\[
K(p, p_{\theta L}) = \frac{1}{2\pi} \int_{-ka}^{ka} \exp(-il_{\xi}p) \exp(il_{\xi}p_{\theta L}) dl_{\xi}
\]

\[
= \frac{ka}{\pi} \left[ \frac{\sin(ka(p - p_{\theta L}))}{ka(p - p_{\theta L})} \right], \quad l_{\xi} = (k\xi).
\]

(23)

where \( \xi \) is the coordinates in \textit{slit} while \( p \) is the corresponding pattern coordinate as defined in Ref. [14].
Proof: The proof can be easily obtained as in preceding section following step by step the definitions and results from the Table 1.

So, we can conclude that the diffraction is an optimal phenomenon which saturates the upper bound of the entropic uncertainty band given in the last row of the Table 1.

Experimental tests of the LEU-Principle. For numerical investigation of our LEU-Principle it is interesting to calculate the scattering variances (8) for extensive statistics $q = 1$, as well as, for nonextensive statistics case (e.g., $q = 0.5$ and $q = 2$) by reconstruction of the pion-nucleus scattering amplitudes using the experimental pion-nucleus phase-shifts [13]. The results obtained in this way are represented in Figs. 1–3 as functions of the optimal angular momentum $L_o$ which is obtained from the same phase shifts by formula from Table 1. Therefore, in Figs. 1–3 we presented the results of the first experimental test of the Principle of Limited Uncertainty (3) in quantum scattering. The grey regions around the optimal variances $V^{opt}_q$ (the full curves in Figs. 1–3) are obtained by assuming an error of $\Delta L_o = \pm 1$ in the estimation of the optimal angular momentum from the experimental data. From comparison of the results from Figs. 1–3, for different entropic index $q$, we see that the results correspond-
Fig. 3. The experimental tests of the entropic bands: \(1 \leq V_1(q) \leq V_1^{\text{min}}(q)\), for the extensive statistics \((q = 1)\) case, as well as for the nonextensive statistics \((q = 0.5\) and 2) cases, calculated from pion-nucleus phase shifts analyses [15] by using Table 1.

The main results and conclusions can be summarized as follows:

(i) In the present paper we introduced a new principle in quantum physics namely the Principle of Limited Entropic Uncertainty (LEU-Principle). This new principle includes in a more general and exact form not only the old Heisenberg uncertainty principle but also introduce an upper limit on the magnitude of the uncertainty in the quantum physics. The LEU-Principle asserts that: "there is an irreducible lower bound as well as an upper bound on the uncertainty in the result of a simultaneous measurement of non-commuting observables for any extensive and nonextensive \((q \geq 0)\) quantum systems.

(ii) Two important concrete realizations of the LEU-Principle are explicitly obtained in this paper, namely, (a) the LEU-inequalities (2)–(5) for the quantum scattering of spinless particles, and, (b) the LEU-inequalities (19)–(23) for the diffraction on single slit of width \(2a\). In particular from the general results (2)–(5), in the limit \(y \to +1\), we recovers in an exact form all the results previously reported in Refs. [5,6]. Here, an experimental illustration of the LEU-Principle is presented in Figs. 1–3 for the cases \(y = 1\) and \(q = 0.5, 1,\) and 2.

(iii) For the nonextensive quantum systems with negative \(q\) we also proved the validity of the state independent entropic uncertainty relations: \(\exp((1-2^{1/q})/(q-1)) \leq V_0(q)\). Moreover, in this case we get [see Eq. (14)] that the optimal Tsallis-like entropies (if they exists for \(q < 0\)) provides only an important improvement of the above state independent entropic uncertainty relations.

References


Quantum transport equations for a scalar field

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Abstract

We derive quantum Boltzmann equations from Schwinger-Dyson equations in gradient expansion for a weakly coupled scalar field theory with a spatially varying mass. We find that at higher order in gradients a full description of the system requires specifying not only an on shell distribution function but also a finite number of its derivatives, or equivalently its higher moments. These derivatives describe quantum coherence arising as a consequence of localization in position space. We then show that in the limit of frequent scatterings coherent quantum effects are suppressed, and the transport equations reduce to the single Boltzmann equation for particle density, in which particles flow along modified semiclassical trajectories in phase space.

1. Introduction

In this letter we present a controlled derivation of dynamical transport equations for a simple complex scalar theory

$$\mathcal{L} = \left( \partial_{\mu} \phi \right)^* \left( \partial^{\mu} \phi \right) - m^2 (x, t) \phi^\dagger \phi + \mathcal{L}_{\text{int}},$$

(1)

where the mass represents coupling to a classical background field which varies in space and time, and $\mathcal{L}_{\text{int}}$ denotes interactions. For definiteness we consider here the simple quartic interaction $\mathcal{L}_{\text{int}} = -\lambda (\phi^\dagger \phi)^2 / 4$. In particular we study the effect of the breakdown of translational invariance with a treatment of the varying background mass at nontrivial order in gradients.

The motivation for this work comes from electroweak baryogenesis at a first order phase transition [1], where one needs to model the departure from thermal equilibrium induced at the phase boundary of a growing bubble of the broken phase. No systematic treatment of such plasma dynamics, adequate to incorporate the crucial CP violating effects, is yet available. This paper is the second in a series in which we attempt to provide a systematic formalism for the treatment of this problem. Including higher order gradients is required since, in realistic cases, the relevant CP violating effects occur typically at higher order in gradients. The formalism we are developing is general and may be applied to other problems which involve analogous physics.

Our treatment of the problem is based on the out-of-equilibrium closed time contour (CTC) formalism. In our derivation we assume a weak coupling limit and the semiclassical approximation. The
semiclassical condition, $kL \gg 1$, states that the de Broglie wave length in the relevant direction must be large in comparison to the corresponding scale of variation of the external field $L$. Since electroweak bubble walls are believed to be thick: $L \sim (10–20)T^{-1}$, and relevant particle species are weakly coupled, these constraints are satisfied for most of particles in the electroweak plasma. Our treatment does not apply to reflecting particles, for which $kL \sim 1$.

In a recent paper [2] we have analyzed the propagator of the scalar field theory in Eq. (1) to non-trivial order in gradient expansion. The main implication of this analysis is that the quasiparticle picture, in which the plasma is treated as a collection of one-particle excitations with a given dispersion relation, breaks down. The reason is that in the absence of translational invariance, states localized in coordinate space mix coherently in momentum space. As a consequence, a self-consistent description of transport of plasma excitations requires additional equations which encode information about quantum coherence. In this letter we derive such a set of quantum transport equations. Further we show that in the limit of frequent scatterings when $\Gamma_\alpha L \gg 1$, where $\Gamma_\alpha$ is the relevant scattering rate, the terms describing coherent quantum effects are suppressed and can be neglected. This can be understood simply as the result of the scattering projecting onto the local semiclassical (quasiparticle) states. In this limit we recover a Boltzmann equation which describes classical semiclassical trajectories in phase space. In this letter we restrict our analysis to a simple scalar theory of the BBGKY hierarchy. In the weak coupling limit it is natural to truncate the hierarchy by substituting all higher than 2-point functions by the perturbative value for the corresponding interaction vertex. For the scalar theory in Eq. (1) the simplest such truncation consists of approximating the four point function by the simplest four point vertex linear in the quartic coupling $\lambda$. As a consequence the two-point self-energies are evaluated at two loops (cf. Fig. 1) and the resulting dynamical equations are then truncated at the order $\lambda^2$ assuming the weak coupling limit.

The complex time ordering formulation in (2)–(3) can be conveniently expressed in terms of the usual time ordered Green functions along the real axis. If we define

$$G^>(x,y) = -i\langle \phi(x)\phi^*(y) \rangle, \quad (4)$$

$$G^<(x,y) = -i\langle \phi^*(y)\phi(x) \rangle, \quad (5)$$

$$G^r(x,y) = \theta(x_0-y_0)(G^>(x,y) - G^<(x,y)), \quad (6)$$

$$G^\omega(x,y) = -\theta(y_0-x_0)(G^>(x,y) - G^<(x,y)). \quad (7)$$

![Fig. 1. Diagrammatic representation of the Schwinger-Dyson equation for the scalar theory with $\mathcal{L}_\text{int} = -\lambda \phi \phi^2/4$.](image)

Keldysh formalism [3] on a closed time contour (CTC)

$$G(x,y) = G^0(x,y) + \int_G dx'^r \int_G dx'^\omega G^0(x',x') \times \Sigma(x',x''), \quad (3)$$

where $\Sigma$ is the self-energy and $G^0$ is the free particle (tree level) propagator. In order to solve $G(x,y)$ from (3) some external information about the self energy function $\Sigma$ must be provided. In general this means coupling infinitely many new equations to (3), leading to the quantum generalization of the BBGKY hierarchy. In the weak coupling limit it is natural to truncate the hierarchy by substituting all higher than 2-point functions by the perturbative value for the corresponding interaction vertex.
where $G^{r,a}$ are the retarded and advanced functions, we can write (3) as an equivalent set of KadanoffBaym equations [4]:

$$G^{-1}_0 - \Sigma^{r,a} \otimes G^{r,a} (x,y) = \delta(x-y),$$  

(8)

$$G^{-1}_0 - \Sigma^{r} \otimes G^{<} (x,y) = \Sigma^{<} \otimes G^{a} (x,y),$$  

(9)

where $G^{-1}_0 = \partial^2 - m^2$ is the inverse of the free propagator and $\otimes$ represents the convolution integral: $A \otimes B(x,y) \equiv \int d^d z A(x,z) B(z,y)$. Functions (4)–(7) are not all independent. In fact they can be reduced to just two independent real functions by using (i) the hermiticity constraint $G^*(x,y) = G^T(y,x)$ and $G^<(x,y) = -G^<(y,x)$, (ii) the relation $G^r - G^a = G^> - G^<$ and (iii) the spectral representation which relates $\text{Im} G^{r,a}$ to $\text{Re} G^{r,a}$.

Transforming to the Wigner representation, $G(k;X) = \int d^3 r e^{ik \cdot r} G(X+r/2, X-r/2)$, and making use of the relation $(\Sigma \otimes G(k;X)) = \exp -i \otimes \{ \Sigma(k;X) \} \{ G(k;X) \}$, the real part of (8) becomes an equation for the propagator

$$\cos \{ \Omega^2 \pm i \omega \Gamma \} \{ G^{r,a} \} = 1$$  

(10)

and the real part of the Eq. (9) yields the quantum Boltzmann equation for the dynamical variable $G^<$:

$$-\sin \{ \Omega^2 \} \{ i G^< \} = \frac{1}{2} \cos \{ \{ \Sigma^< \} \{ G^< \} - \sin \{ i \Sigma^< \} \{ G_k \} \},$$  

(11)

where $\otimes$ is the Poisson bracket operator

$$\{ \phi \{ f \} \} = \int [ \hat{\partial}_k \hat{f} \cdot \hat{\partial}_k \hat{\phi} - \hat{\partial}_k \hat{\phi} \cdot \hat{\partial}_k \hat{f} ]$$  

(12)

and we defined the shorthand notation

$$\Omega^2 = k^2 - m^2 - \Sigma_k.$$  

(13)

The retarded and advanced operators were decomposed as

$$G^{r,a} = G^r \mp i \omega \mathcal{A}, \quad \Sigma^{r,a} = \Sigma^r \mp i \omega \Gamma,$$  

(14)

where $\mathcal{A}$ is the spectral function. Finally, in the quasiparticle limit $\Gamma \to 0$ it will be useful to assume the spectral decomposition of the Wigner functions:

$$i G^< = 2 \mathcal{A} n, \quad i G^> = 2 \mathcal{A} (n+1),$$  

(15)

where the latter definition follows from the first by the condition $\mathcal{A} = i(G^> - G^<)/2$. Eqs. (10) and (11) are the full dynamical equations in the Wigner representation. These are suitable for the description of systems in slowly varying backgrounds, with truncation at a given order in gradients leading typically to a more accurate modeling of the dynamics.

### 2.1. Propagator equation

To the lowest order in gradients Eq. (10) defines the familiar propagators in a spatially constant background. At this order the spectral function $\mathcal{A}$ is singular in the limit $\Gamma \to 0$, and defines the well known projection to the local quasiparticle on-shell.

Given the decomposition (15), the QBE (11) also becomes singular, allowing a reduction of the QBE by integration over momentum (or frequency) to the well known semiclassical Boltzmann equation involving on-shell excitations only [4]. When higher order gradient corrections are included, the on-shell projection becomes more involved because of the coherent quantum effects described by the gradient terms [2]. Before performing the on-shell projection for the QBE, we shall here illustrate the technique using a simple test function.

Solving Eq. (10) iteratively to the lowest nontrivial order in gradients around the lowest order pole gives

$$G^{r,a} \to G_z = \frac{1}{z} + \frac{1}{2} \frac{m^2}{z^3} - \frac{1}{2} \frac{2k_0^2 m^2 + (m^2)^2}{z^4}$$  

(16)

where $z = k_0^2 - k^2$ with $k_0^2 = \omega^2 - k^2 - m^2(x)$ and we have assumed $\Sigma_x = 0$ for simplicity, in addition to taking the quasiparticle limit $\Gamma \to 0$. The spectral integral over the momentum variable $k$ of some function $\mathcal{F}$ can be converted to a contour integral encircling the multiple pole of $G$ at $z = 0$ [2]:

$$\mathcal{F}_\rho [ \mathcal{F} ] = \frac{1}{2\pi} \int_0^\infty dk_z \mathcal{A}_\rho \mathcal{F}$$  

$$\to \text{Res} \left[ \mathcal{F}_\rho \mathcal{F} / \sqrt{k_0^2 - z} \right]_{z=0}$$  

(17)

where $\mathcal{F}_\rho (k_0) = (\delta_\rho \mathcal{F}) (k_0)$, and the coefficients $c_i$ may contain gradients up to $p$th order with respect to $x$. To the leading order in gradients ($p = 0$) the spectral function indeed becomes singular: $\mathcal{A} \to (\pi/2k_0) [\delta(k_0 - k_0) + \delta(k_0 + k_0)]$, projecting sharply on-shell $k_x = \pm k_0$. 


It should be noted that the singularity of the propagator (16) remains at the quasiparticle shell $k_s^2 = k_0^2$. The effect of gradient corrections in $G_p$ is to project out derivatives of $\mathcal{F}$ w.r.t. $k_s$ up to order $p + 1$. That this produces an effective shift of the pole emerges when the contribution from the first order derivative is included in the definition of the projected function. To second order in gradients

$$\mathcal{I}_2[\mathcal{F}] = \frac{\mathcal{F}(k_\omega)}{k_\omega} + c_2 \mathcal{F}^{(2)}(k_0) + c_3 \mathcal{F}^{(3)}(k_0),$$

where

$$k_\omega = k_0 + \frac{1}{8} m^2 \frac{5}{32} \frac{(m^2)^2}{k_0^5}$$

is the space dependent semiclassical momentum which coincides with the standard WKB dispersion relation. The higher order derivative corrections in Eq. (18), however, are of the same order in $\partial k_s$ as the shift. As we will now see in detail this leads to the breakdown of the quasi-particle approximation when the projection is performed on the QBE. It is then not sufficient to describe the system with a single distribution function obtained by projecting on-shell, but some off-shell information, represented by a finite number of derivatives of $n$, describing coherent quantum effects, is necessary.

3. Quantum Boltzmann equation

We will now consider the QBE (11) to the lowest nontrivial order in gradients. We truncate the collision term at leading order in gradients and focus on the structure of the flow term in the presence of a varying background. This approximation amounts to neglecting the derivatives of the self-energies, while retaining those of the background. (A complete consideration that includes all second order terms in the collision term will be given elsewhere [6].) We then have

$$-\big(\vartheta - \frac{1}{6} \vartheta^3\big)\{\Omega^2\} \{iG^-\} = \frac{1}{2} \{\Sigma^\rightarrow G^- - \Sigma^- G^\rightarrow\}. \tag{20}$$

The on-shell projection of the quantum Boltzmann equation is usually done by integrating over frequencies, resulting in an equation on the phase space $(k, t, x)$ [5]. An alternative yet equivalent approach of integrating over momenta is more convenient here, because in the present spatially varying problem the density of states is most conveniently labeled by the conserved energy.

Let us continue with the special case $\Sigma_n = 0$. Inserting the decomposition (15) into (20) and writing the $\vartheta$-terms explicitly gives

$$\left(\omega \vartheta_i + k \cdot \vartheta + \frac{1}{2} \left(\vartheta \vartheta_i k_0^2\right) \vartheta_j - \frac{1}{4} \left(\vartheta \vartheta_i k_0^2\right) \vartheta_j^3\right) \partial^4 n^2$$

$$= -\vartheta \vartheta \partial^4 (i \Sigma^\rightarrow n - i \Sigma^- (n + 1)). \tag{21}$$

Integrating (21) over $k_s$ gives a dynamical equation coupling $n(k_s)$ and its first three derivatives. The information contained in this zeroth moment clearly does not suffice to define a closed solution to the problem. To provide closure we can perform integrals of Eq. (21) weighted by some higher powers of $k_s$, in a manner analogous to the standard derivation of fluid equations by taking moments of the classical Boltzmann equation. There is however one crucial difference. While for the latter case there is no control parameter and hence no natural closure exists, in the former case the equations close at a finite number of independent moments. This is the case because, as shown in Eq. (17) above, in an integral over any smooth test function weighted by $G_p$ only the first $p + 2$ terms are nonzero. In particular at second order in gradient expansion the closure is obtained by the first four moments.

The QBE becomes very complicated when written in terms of $n^{(i)}(k_s)$. It is more convenient to express the equations in terms of the weighted projections of the generalized distribution function $n$

$$f_i = \theta(k_s) f_i^+ + \theta(-k_s) f_i^-, \tag{22}$$

where

$$f_i^+ = \frac{2}{\pi} \int_{0}^{k_+} dk_s k_s^fi^+ n.,$$

$$f_i^- = \frac{2}{\pi} \int_{0}^{k_-} dk_s (-k_s)^i n.. \tag{23}$$

These functions are a straightforward generalization of the distribution function for the spatially constant...
case. In the absence of gradient corrections \( f_i^\pm \rightarrow k_0^{-i-1}n(\pm k_0) \), so that the standard distribution function is then simply \( \theta(k_0)n(k_0) + \theta(-k_0)n(-k_0) \).

Now using moments (23) it is particularly easy to prove the closure. Indeed, at second order in gradients \( \mathcal{A} \rightarrow \mathcal{A}_2 \), and we have the identity

\[
\int_0^\infty dk_x (k_x - k_0)^4 k_x^2 \mathcal{A}_2 n = 0
\]

(24)

and the analogous identity holds for \( k_x < 0 \), so that one immediately obtains

\[
f_{l+4} - 4k_0 f_{l+3} + 6k_0^2 f_{l+2} - 4k_0^3 f_{l+1} + k_0^4 f_l = 0.
\]

(25)

The constraint (25) allows any moment \( f_l \) to be written in terms of the four lowest ones \( f_0, f_1, f_2, f_3 \). A similar binomial constraint holds at \( p \)-th order in gradients, providing closure with \( p + 2 \) moments.

It is curious to observe that the functions \( f_l \) can be interpreted as projections onto different momentums hypersurfaces \( \tilde{k}_l \) of \( n \). Performing the \( k \)-integral in Eq. (23) implies that

\[
f_i^\pm = k_0^{l-2} \kappa_l n(\pm \tilde{k}_l) + \mathcal{O}(\tilde{k}_l^2 n)_{k_i = \pm k_0},
\]

(26)

where the \( l \)th shell momentum \( \tilde{k}_l \) is given by

\[
\tilde{k}_l = k_0 + \frac{(l-1)(l-2)}{16} \frac{m^2}{k_0^3} \left( \frac{m^2}{k_0^3} \right)^2 + \frac{l^2 - 5l + 15}{32} \frac{m^2}{k_0^3}
\]

(27)

and

\[
\kappa_l = k_0 + \frac{(l-1)(l-2)(l-3)}{48} \frac{m^2}{k_0^3} \left( \frac{m^2}{k_0^3} \right)^2 + \frac{(l-1)(l-3)(l-5)}{96} \frac{m^2}{k_0^3}.
\]

(28)

In particular \( \tilde{k}_0 = k_0^2/\kappa_0 = k_{sc} \) is the semiclassical shell represented by the dispersion relation (19). One thus arrives at the following intuitive picture, illustrated in Fig. 2: taking moments of the QBE (11) becomes

\[
\omega \partial_t f_l + \partial_x f_{l+1} - \frac{l}{2} \left( \partial_x \tilde{k}_l^2 \right) f_{l-1} + \frac{l(l-1)(l-2)}{48} \left( \partial_x \tilde{k}_l^2 \right) f_{l-3} = \text{Coll}_l,
\]

(29)

where we dropped the term \( k_0 \cdot \partial_x f_l \) for simplicity (it can be always reinserted by the replacement \( \omega \partial_t \rightarrow \omega \partial_t + k_0 \cdot \partial_x \)). The first four equations (29), with \( l = 0,1,2,3 \), together with the closure condition (25), form a closed set for functions \( f_0, f_1, f_2 \) and \( f_3 \). These equations are our main result. They can be used as a starting point for studying plasma dynamics in out-of-equilibrium situations when gradient approximation applies.

Given the simple 4-point interaction term depicted in Fig. 1, the \( l \)th moment of the collision term appearing in (29) becomes

\[
\text{Coll}_l = -\Gamma^* f_l + \Gamma^+ \left( f_l + k_0^{-l} \kappa_l \right).
\]

(30)
where $\Gamma^>$ and $\Gamma^<$ are the spectral projections of the self-energies $i\Sigma^>$ and $i\Sigma^<$ defined by

$$
\Gamma^> = \frac{\lambda^2}{2} \int_{p,k,k'} (2\pi)^4 \delta^4(k + p - k' - p') \times f_p f_{p' + 1} (f_{k' + 1}),
$$

$$
\Gamma^< = \frac{\lambda^2}{2} \int_{p,k,k'} (2\pi)^4 \delta^4(k + p - k' - p') \times f_p f_{p + 1} (f_{k + 1}),
$$

Multiplying Eq. (29) for $f_0$ by $k_{sc}/\omega$ one obtains

$$
\partial \omega f + \frac{k_{sc}}{\omega} \partial_1 (f + f_{qc1}) = -\frac{\Gamma^>}{\omega} f + \frac{\Gamma^<}{\omega} (f + 1). \tag{33}
$$

This equation already resembles the standard Boltzmann equation, the main difference being coupling to an unknown function $f_{qc1}$. Dividing the $f_1$-equation by $\omega$ and subtracting Eq. (33) one then obtains

$$
(\partial + \Gamma_{qc}) f_{qc1} + \frac{\kappa_2 - k_{sc}}{\omega} \partial_1 f_{qc1} + \frac{\partial_1 \kappa_2}{\omega} (f_{qc1} + f_{qc2})
$$

$$
+ \frac{\kappa_2}{\omega} \partial_1 f_{qc2} = s_1, \tag{34}
$$

where the source

$$
s_1 = \frac{k_{sc} - \kappa_2}{\omega} \partial_1 f + \frac{(\partial_1 \kappa_2^2/2k_{sc}) - \partial_1 \kappa_2}{\omega} f \tag{35}
$$

represents coherent mixing of $f$ and $f_{qc1}$. One can obtain similar equations for the coherent quantum densities $f_{qc2}$ and $f_{qc3}$, but we will not present them explicitly here. The coherent density $f_{qc1}$ is damped at the rate $\Gamma_{qc}$ which reads

$$
\Gamma_{qc} = \frac{\Gamma^> - \Gamma^<}{\omega}. \tag{36}
$$

This is the out-of-equilibrium generalization of the on-shell damping rate (cf. Eq. (31) and [7]). The coherence equations for $f_{qc1}$ are linear and hence can be quite easily solved, and the solution for $f_{qc1}$ inserted into Eq. (33). Requiring that none of $f_{qc1}$ be sourced by the self-energy $\Gamma^<$ defines $f_{qc1}$ uniquely.

We now pause to discuss the validity of the gradient approximation. It is not hard to see that in the model under study the validity criteria reduce to the following conditions

$$
\partial \omega f \ll \omega f, \quad \partial_1 f \ll k_{sc} f, \tag{37}
$$

which are the particular realization of $||O|| \ll 1$ for the on-shell Boltzmann equation.

In order to study how quantum coherence influences Eq. (33), we now make a simple estimate of $f_{qc1}$. Not far from equilibrium $f$ can be approximated by its equilibrium form so that $\partial_1 f \sim m^2/\omega LT$, where $L$ represents the scale on which $m^2$ varies. To
leading order in $m^2$ the source $s_1$ in Eq. (35) can be estimated as

$$s_1 \sim \frac{1}{(Lk_0)} \frac{m^2}{\omega}.$$  \hspace{1cm} (38)

Similar estimate holds for $s_2$ and $s_3$ in the equations for $f_{q_{c1}}$ and $f_{q_{c3}}$. If we want to include the higher order derivatives of $k_{sc}$ in Eq. (33), this estimate implies that one cannot in general neglect $f_{q_{c1}}$ in Eq. (33).

There is however a limit in which it is legitimate to neglect the part of $G$ gradient terms in Eq. 33. We have thus shown that coherent quantum densities are strongly damped and the leading order in $E$ is a function of energy $v$ where, at the leading order in gradients, the collision term is suppressed by an additional plasma velocity, which is often much smaller than unity. For example, the stationary frame the time derivative term is suppressed by $L^2$, the spatial derivative term is suppressed by $L^3$.

To check consistency of this estimate, note that in the efficient scattering limit one expects $\partial_x f_{q_{c1}} \sim 1/L$, so that the spatial derivative term is suppressed by $1/(\Gamma_{q_{c}} L)$ in comparison to the $\Gamma_{q_{c}} f_{q_{c1}}$ term. In the stationary frame the time derivative term is suppressed by an additional plasma velocity, which is often much smaller than unity. For example, the phase boundary speed at the electroweak phase transition is smaller than about 0.3 of the speed of light [8]. Upon inserting Eq. (39) into Eq. (33), we see that the coherent quantum density $f_{q_{c1}}$ is suppressed by $1/(\Gamma_{q_{c}} L)$ in comparison to the other third order terms in Eq. (33). We have thus shown that in the limit $\Gamma_{q_{c}} L \gg 1$ the coherent quantum density $f_{q_{c1}}$ in Eq. (33) can be neglected so that we arrive at the semiclassical Boltzmann equation

$$\partial_t f + \frac{k_{sc}}{\omega} \partial_x f = \text{Coll}[f],$$ \hspace{1cm} (40)

where, at the leading order in gradients, the collision term is given in Eq. (33). Note that this equation still does not have the standard form since we have integrated over momentum $k_x$ and thus obtained a distribution function $f = f(\omega, k_x; t, x) = k_{sc} f_0$ which is a function of energy $\omega$. To recover the semiclassical Boltzmann equation on the usual phase space we make the change of variables defined as follows

$$\frac{d\omega}{k_{sc}(\omega; x)} \rightarrow \frac{dk_x}{\tilde{\omega}(k_x; x)}.$$ \hspace{1cm} (41)

This choice is natural because it gives the local Lorentz covariant measure in the collision integral (31). This change of variables in fact does not leave $\delta^4(k + p - k' - p')$ invariant. The correction induced is, however, second order in derivatives of the distribution function moments in the collision term, and hence it is beyond the approximation considered in this letter. Making use of $d\omega \rightarrow d\tilde{\omega}(k_x; x) = (\partial_k \tilde{\omega}) dk_x$, we then immediately obtain

$$\partial_t \tilde{\omega}(k_x; x) = \frac{k_{sc}(\tilde{\omega}(k_x; x))}{\tilde{\omega}(k_x; x)} = \frac{k_{sc}(\tilde{\omega}(k_x; x))}{\tilde{\omega}(k_x; x)},$$ \hspace{1cm} (42)

which specifies $\tilde{\omega} = \tilde{\omega}(k_x; x)$. Consider now how the flow term transforms under this change of variables. We first have $f(\omega; x, t) \rightarrow f(k_x; x, t)$, so that

$$\frac{k_{sc}}{\omega} \rightarrow v g \equiv \partial_{k_x} \tilde{\omega}, \quad \partial_x \rightarrow \partial_x - (\partial_{k_x} \tilde{\omega})^{-1} (\partial_{k_x} \tilde{\omega}) \partial_{k_x},$$ \hspace{1cm} (43)

and hence Eq. (40) becomes

$$\partial_t f + v g \partial_x f + F_x \partial_{k_x} f = \text{Coll}[f],$$ \hspace{1cm} (44)

with the following canonical velocity and force

$$v_x = \partial_{k_x} \tilde{\omega}, \quad F_x = -\partial_x \tilde{\omega}.$$ \hspace{1cm} (45)

This semiclassical Boltzmann equation is one of our main results. Note that “inversion” of the semiclassical dispersion relation $k_{sc} = k_{sc}(\omega; x)$ as given in Eq. (42) is defined in a rather non-trivial way by invoking a natural transformation of the collision integral measure, resulting in a different form of the semiclassical energy hypersurface $\tilde{\omega} = \tilde{\omega}(k_x; x)$ from what one might naively guess. This is the unique choice which, within the present approximation, renders the canonical form for the semiclassical velocity and force (45). Recall again however, that in the present work the $\tilde{\omega}$-term operating on the collision term and other analogous contributions were neglected. In a completely consistent computation these
5. Discussion

We have generalized the semiclassical Boltzmann equation to include the effects of a slowly varying background to nontrivial order in gradients for a complex scalar field. Our motivation is baryon production at a first order electroweak phase transition, where this generalization is necessary to model CP violating effects which first appear in transport equations beyond leading order in gradient expansion. We found that consistent treatment of the system to nontrivial order in gradients requires the introduction of coherent quantum densities in addition to the usual distribution function, and derived the corresponding dynamical equations. We also showed that in the limit of frequent scatterings coherent quantum effects are suppressed, and the problem reduces to a single Boltzmann equation for the distribution function, containing a classical force which includes gradient corrections to nontrivial order. This is a particularly nice realization of the quantum-to-classical transition which occurs by the dissipative dynamics within the system itself. The dynamics is rendered dissipative by the weak coupling truncation of the Schwinger-Dyson equation at the order $\lambda^2$, and by the truncation of the gradient expansion at third order in gradients, resulting in localized dynamics in the Wigner representation and irreversibility. The rate at which the coherent quantum effects are dissipated is simply the out-of-equilibrium damping rate (36).

We now comment on how to relate our results to electroweak baryogenesis. The relevant CP violating effects emerge generally only at non-trivial order in gradients of the background, which is analogous to the case considered here. To make a connection with the computation of baryogenesis sources, it is instructive to integrate Eq. (33) over $\omega$ and $P_1$ to obtain a continuity equation for the density of particles which contains clearly separated semiclassical and quantum mechanical sources of the form

$$j_{sc} = \int \frac{d\omega d^2p}{(2\pi)^3} \frac{k_{sc}}{\omega} f_{sc}, \quad j_{qc} = \int \frac{d\omega d^2p}{(2\pi)^3} \frac{k_{qc}}{\omega} f_{qc}.$$  

(46)

This should be contrasted with the situation in literature [9–11], where there is no agreement on whether the relevant CP violating sources come from classical or quantum currents. Moreover, in the frequent scattering limit, $\Gamma_{qc} L \gg 1$, we find that the coherent quantum current $j_{qc}$ is suppressed, and the (dominant) semiclassical current then should be calculated making use of the semiclassical transport Eq. (44). This contains a classical force analogous to the one first introduced as a source for baryogenesis in studies of two Higgs doublet models in Ref. [9], and then subsequently applied to the MSSM in Ref. [12].

Here we studied for simplicity a complex scalar field, whereas the physically relevant cases involve mixing scalar or fermion fields. However we believe that the novel features we found, being essentially a consequence of localization in space, are not specific to the complex scalar theory, but generic to all models. How this is realized in detail in other theories is under investigation [5,6].

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References


QED radiative correction to spin-density matrix elements in exclusive vector meson production

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Abstract

QED radiative effects are considered in the case of the measurement of spin-density matrix elements of diffractive \( \rho \)-meson electroproduction. Large radiative correction for \( r_{50} \) is found in the kinematics of collider experiments at HERA.00

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An extension of the kinematical region of lepton-nucleon deep inelastic scattering to the domain of the diffractive processes provides a possibility to study hadronic nature of the photon along with the nucleon structure. A particular emphasis is placed on the case of exclusive vector meson electroproduction

\[
e(k_z) + p(p) \rightarrow e(k_z) + p(p') + p(p_v),
\]

\[
p \rightarrow \pi^+(p_+) + \pi^-(p_-). \tag{1}
\]

The reason is that the process (1) can be viewed as an off-diagonal Compton scattering analytically continued in the virtuality of the photon \( \gamma^* \) to the vector meson mass \( \gamma^* \rho \rightarrow Vp \). It gives access to the whole set of the corresponding helicity amplitudes.

The process (1) is analyzed experimentally by means of spin-density matrix elements. When measured, they describe internal motion of vector meson constituents and their spin-angular structure. The angular distribution of unpolarized vector meson decay is parameterized by fifteen matrix elements \( r_{50}^{ij} \). It was believed for quite a long time that their behavior complies with the s-channel helicity conservation (SCHC) hypothesis. SCHC means that the helicity of the virtual photon is conserved in the s-channel process \( \gamma^* p \rightarrow pp \). In this case ten matrix elements are equal to zero. They correspond to the case when photon and vector meson have different helicities. However, in the recent measurements \( r_{50}^{00} \) was observed to be nonzero [1–3], which was considered as an indication of SCHC violation.

The procedure of the experimental data analysis is based on the correlation between the lepton scattering, vector meson production and decay planes, which are affected by the radiative corrections (RC). Hence it is important to determine whether the mea-
sured \( r^o \)_{VR} can be the result of underestimation of RC coming from non-observed QED effects and real photon emission. Anyway, RC should be taken into account in order to make consistent the data processing of the corresponding experiments [1–3].

Following Schilling and Wolf [4] the observed matrix elements \( r^o_{\text{obs}} \) can be written through vector meson decay angular distribution \( W(\cos \theta, \phi, \Phi) \) and the weight coefficients \( F_{ij}(\cos \theta, \phi, \Phi) \) (see Appendix C of [4] as:

\[
\begin{align*}
&\frac{d^3 \sigma}{d \phi d \theta} = \int_0^1 \frac{d^3 \phi}{d \phi d \theta} W(\cos \theta, \phi, \Phi) F_{ij}(\cos \theta, \phi, \Phi)(1 + \delta) \\
&\int_0^1 \frac{d^3 \phi}{d \phi d \theta} W(\cos \theta, \phi, \Phi) F_{ij}(\cos \theta, \phi, \Phi)(1 + \delta)
\end{align*}
\]

(2)

Here \( \Phi \) is the angle between the lepton scattering plane and \( \rho \)–production plane, \( \phi \) is the angle between \( \rho \)–decay and production plane, \( \theta \) is the polar angle of the direction of flight of the positive decay pion. \( \delta \) is RC obtained as the ratio of next to lowest order cross section of the process (1) to the Born cross section.

For the case of the vector meson considered in the final state as a stable particle, RC was calculated in [5]. It can be presented in the form

\[
\delta = e^{\delta_{\text{VR}}}(1 + \delta_{\text{VR}} + \delta_{\text{inf}}) = \frac{\sigma_F}{\sigma_0}.
\]

(3)

Recall that \( \sigma_0 \) is the Born cross section, \( \delta_{\text{inf}} \) comes from the effects of vacuum polarization by leptons and hadrons. Also the sum of \( \delta_{\text{VR}} \) and \( \delta_{\text{inf}} \) originates from contributions of vertex function and soft photon emission (the exponent is due to the multiple soft photon radiation). \( \sigma_F \) is caused by the hard photon emission.

Let us discuss the angular dependence of RC (3). We remind that the angle \( \Phi \) is described by the vector meson momentum and non-measured vector \( q \). This vector is determined by measured momenta of initial and scattered leptons. It is clear that the radiation of unobserved real photon changes the vector \( q \) into \( q - k \) (\( k \) is the real photon momentum). In fact, it leads to reorientation of production and scattering planes. Therefore the hard photon contribution \( \sigma_F \) to RC can be significantly dependent on \( \Phi \). The quantities \( \delta_{\text{VR}} \) and \( \delta_{\text{inf}} \) depend only slightly upon \( \Phi \). We note that this dependence has a kinematic origin. It means that if the integration region over photon variable is divided into soft and hard parts \(^1\), the splitting parameter could be chosen in such a way that the mentioned dependence would be completely cancelled.

Strictly speaking, \( \delta \) has been found in [5] as the ratio of four-fold cross sections \( d^3 \sigma / dx dy dt d\Phi \). However, if, as in our case, \( \rho \)-meson decays into \( \pi^+ \pi^- \) and RC is denoted by a ratio of seven-fold cross sections, it can be shown that the results of [5] can be applied unchanged. All one has to do is to show that momenta of \( \pi \)-mesons appear in the correction (3) not separately, but only as \( p_+ + p_- = p_Y \).

In this case, there would be no any scalar products of the four momenta of pions which can produce dependence on \( \cos \theta \) and \( \phi \).

Really, \( \delta_{\text{VR}} \) and infrared finite part of vertex function contributing to \( \delta_{\text{VR}} \) are determined by \( Q^2 \) only. Other contributions to \( \delta \)‘s in (3), coming from the infrared divergence cancellation, can depend on scalar products of vector \( A^2 \) (\( A_k, A_{k_1}, A_{k_2} \)) along with the kinematical variables \( Q^2, W^2, t \). Therefore the angular dependence of \( \delta \)’s is denoted only by \( \Phi \). The real photon phase space is also specified by \( A \), but not by \( p_+ \) or \( p_- \). Finally, if we assume naturally [7] that all structure functions except \( \sigma_L, \sigma_F \) vanish in the hadronic tensor, the hard photon contribution \( \sigma_F \) to RC is found to be free of \( \cos \theta, \phi \) – dependence. Hence RC depends only upon \( \Phi \) and consequently is reduced to the one calculated in [5].

According to Eq. (2) the QED corrections (\( \Delta r = r_{\text{obs}} - r_{\text{Born}} \)) to the matrix elements are

\[
\begin{align*}
\Delta r_{00}^{04} &= -e I_2 r_{10}^1 + aI_1 r_{00}^5, \\
\Delta \text{Re} r_{10}^{04} &= -e I_2 \text{Re} r_{10}^1 + aI_1 \text{Re} r_{10}^5, \\
\Delta r_{11}^{04} &= -e I_2 r_{11}^1 + aI_1 r_{11}^5, \\
\Delta r_{00}^{01} &= -2I_2 r_{00}^1 + eI_4 r_{00}^5, \\
\Delta r_{00}^{00} &= \frac{1}{e} \left[ -2I_2 r_{00}^1 + eI_4 r_{00}^5 - a(I_1 + I_3) r_{00}^5 \right].
\end{align*}
\]

\(^1\) In this case we would come to the formulae analogous to ones of traditional approach of Mo and Tsai [6] for deep inelastic scattering.

\(^2\) \( \Lambda = p^2 + k^2 - k^2 - p_+ - p_- \) is the four-momentum of the system of unobserved particles.
\[ \Delta r^1_{11} = \frac{1}{\epsilon} [I_2(r_{10}^{04} - 1) + \epsilon I_4 r_{11}^{1} - a(I_1 + I_3) r_{11}^{3}] , \]

\[ \Delta \text{Re} r_{10}^1 = \frac{1}{\epsilon} \left[ -2I_2 \text{Re} r_{10}^{04} + \epsilon I_4 \text{Re} r_{10}^1 - a(I_1 + I_3) \text{Re} r_{10}^3 \right] , \]

\[ \Delta r_{1-1}^{1} = \frac{1}{\epsilon} [ -2I_2 r_{1-1}^{04} + \epsilon I_4 r_{1-1}^{1} - a(I_1 + I_3) r_{1-1}^{3}] , \]

\[ \Delta \text{Im} r_{10}^1 = -I_4 \text{Im} r_{10}^{2} + \frac{a}{\epsilon} (I_1 + I_3) \text{Im} r_{10}^{6} , \]

\[ \Delta \text{Im} r_{1-1}^{1} = -I_4 \text{Im} r_{1-1}^{2} + \frac{a}{\epsilon} (I_1 + I_3) \text{Im} r_{1-1}^{6} , \]

\[ \Delta r_{00}^{5} = \frac{1}{a} [2I_1 r_{00}^{04} + aI_2 r_{00}^{1} - \epsilon(I_1 + I_3) r_{00}^{3}] , \]

\[ \Delta r_{11}^{5} = \frac{1}{a} [I_2(1 - r_{11}^{04}) + aI_3 r_{11}^{1} - \epsilon(I_1 + I_3) r_{11}^{3}] , \]

\[ \Delta \text{Re} r_{10}^{5} = \frac{1}{a} \left[ I_1 \text{Re} r_{10}^{04} + aI_4 \text{Re} r_{10}^{1} - \epsilon(I_1 + I_3) \text{Re} r_{10}^{3} \right] , \]

\[ \Delta r_{1-1}^{5} = \frac{1}{a} [I_1 r_{1-1}^{04} + aI_4 r_{1-1}^{1} - \epsilon(I_1 + I_3) r_{1-1}^{3}] , \]

\[ \Delta \text{Im} r_{10}^{6} = -I_2 \text{Im} r_{10}^{2} + \frac{\epsilon}{a} (I_1 + I_3) \text{Im} r_{10}^{10} , \]

\[ \Delta \text{Im} r_{1-1}^{6} = -I_2 \text{Im} r_{1-1}^{2} + \frac{\epsilon}{a} (I_1 + I_3) \text{Im} r_{1-1}^{10} . \quad (4) \]

The polarization parameter of the virtual photon density matrix $\epsilon = \frac{\alpha}{1 - \gamma - \gamma^2}$ is close to $1$ at HERA kinematics, $a = \sqrt{2 \epsilon (1 + \epsilon)}$.

\[
I_n = \int_0^{2\pi} \frac{d\Phi}{2\pi} \cos n\Phi \frac{\delta(\Phi)}{\delta(\Phi)} , \quad n = 0, \ldots, 4 .
\]

Thus the absolute radiative correction to spin-density matrix elements is linear in the lowest order of $r^{04}_n, r^{04}_n$. The dependence on $\delta = \delta(\Phi)$ is included in the coefficients $I_n$.

It is clear that Born matrix elements can be easily extracted from formulae (4) without using any model for $r's$. For a realistic radiative correction procedure the system of Eq. (4) can be solved in a traditional way. To do it, one needs to perform the iteration procedure, where the extracted matrix element at the $n$ step is calculated via $r's$ estimated at the $n-1$ step as

\[ r_{\text{ext}}^{(n)} = r_{\text{obs}} - \Delta r(r_{\text{ext}}^{(n-1)}) . \quad (6) \]

Note that the value of RC $\Delta r$ is expected to be small with respect to correction to the cross section. The reason is that only contributions of higher harmonics $I_n$ survive, however the large contribution of $I_0$ vanishes. The quantities $I_{1,2,3,4}$ are shown in Fig. 1. $I_1$ is of order $1-2\%$, $I_2$ is less then $1\%$ while $I_{3,4}$ are practically negligible in the considered kinematical region. It follows that only those $\Delta r$ would be significantly different from zero, which are proportional to non-vanished matrix elements with relatively large coefficient $I_n$.

In order to treat the radiative effects numerically, the model [8] reasonably reproducing experimental data is used. We estimate the relative RC $\delta r = \Delta r/r$ for those matrix elements which should be non-zero according to SCHC. We found (see Fig. 2) that $\delta r$ do not exceed $1\%$ for HERA kinematics. Let us

![Fig. 1. The dependence of $I_n, n = 1 \ldots 4$ on $Q^2$ under the kinematical conditions of H1/ZEUS experiments: $\sqrt{s} = 300$ GeV, $W = 75$ GeV.](image)
emphasise that if SCHC is true, $\delta r_{00}^{04}$ would be identically equal to zero, since according to SCHC it is proportional to zero matrix elements $r_{10}^1$, $r_{00}^5$.

For the majority of the matrix elements vanishing in the SCHC limit, radiative corrections turn out to be not greater than 1%. However, there are two of them, namely, $\text{Re} r_{10}^{04}$ and $r_{00}^5$, in which RC appears to be substantial (see Fig. 3). One can see that corrections $\Delta \text{Re} r_{10}^{04}$ and $\Delta r_{00}^5$ may reach $\sim 20\%$.

The last result is interesting from the point of view of SCHC violation: the radiative correction procedure reduces the observed effect.
For illustration let us follow the origin of the radiative effect within the experimental data processing. The matrix element $r_{50}^3$ is defined experimentally by fitting of vector meson decay $\Phi$ distribution ($W(\cos \theta, \phi, \Phi)$ integrated over $\cos \theta, \phi$)

$$W(\Phi) \sim 1 - \epsilon \cos 2\Phi (2r_{11}^1 + r_{00}^1) + \cos \Phi (2r_{11}^3 + r_{00}^3).\quad (7)$$

Based on SCHC hypothesis, this distribution would be flat, which corresponds to zero $r_{50}^3$. But this is true only for matrix elements in the lowest order of QED. It can be seen (Fig. 4, see also [9]), that the theoretical radiative corrected $\Phi$-distribution deviates from flat. It has the form similar to the experimental distribution. Thus the observed effect comes not only from SCHC violation but from the radiative corrections as well.

It follows that if RC procedure is included in the data processing, it would lead (as one can see from Figs. 3 and 4) to reduction of the found SCHC violation by almost 20%.

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References

Energy dependence of the near-threshold total cross-section for the $pp \rightarrow pp\eta'$ reaction

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Abstract

Total cross sections for the $pp \rightarrow pp\eta'$ reaction have been measured in the excess energy range from $Q = 1.53$ MeV to $Q = 23.64$ MeV. The experiment has been performed at the internal installation COSY-11 [1] using a stochastically cooled proton beam of the COoler SYnchrotron COSY [2] and a hydrogen cluster target [3,4]. The determined energy dependence of the total cross section weakens the hypothesis of the S-wave repulsive interaction between the $\eta'$ meson and the proton [5,6]. New data agree well with predictions based on the phase-space distribution modified by the proton-proton final-state-interaction (FSI) only. © 2000 Published by Elsevier Science B.V. All rights reserved.

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Recently, total cross sections for the production of the $\eta'$ meson in the collision of protons close to the reaction threshold have been published [7,8] for the first time. Two independent experiments performed at the accelerators SATURNE and COSY have delivered consistent results. The data has triggered off an interest in explaining the unknown dynamics of the $pp \rightarrow pp\eta'$ reaction [9–14]. The determined total cross sections are about a factor of thirty smaller than the ones for the $pp \rightarrow pp\eta'$ reaction [7,15–18] at the corresponding values of excess energy. Trying to explain this large difference Hibou et al. [7] showed that calculations within a one-pion exchange model, where the parameters were adjusted to fit the total cross section for the $pp \rightarrow pp\eta'$ reaction, underestimate the $\eta'$ cross sections by about a factor of two.
This discrepancy suggests that short-range production mechanisms as for example heavy meson exchange, mesonic currents [9], or more exotic processes like the production via a fusion of gluons [19] may contribute significantly to the creation of $\eta$ and $\eta'$ mesons [13]. Such effects are likely, since the momentum transfer required to create these mesons is much larger than for the pion production, and already in case of the $pp \rightarrow pp\pi^0$ reaction a significant short-range heavy meson exchange contribution is necessary in order to obtain agreement with experimental results [20,21]. On the other hand, Sibirtsev and Cassing [12] concluded that the one-pion exchange model, including the proton-proton final state interaction (pp-FSI), reproduces the magnitude of the experimental data and hence, the other exchange currents either play no role or cancel each other.

It is well established that the $\eta$ meson is predominantly produced via the excitation of an intermediate baryonic resonance $S_{11}(1535)$ [15,22–26]. Both, the large difference in the production cross sections for $\eta$ and $\eta'$ mesons, and the lack of experimentally established baryonic resonances, which would decay into $\eta'$, suggest that the $pp \rightarrow pp\eta'$ reaction occurs without an excitation of the colliding protons. Indeed, as demonstrated by Gedalin et al. [11], the magnitude of the close-to-threshold $\eta'$ production can be explained without a resonant production term. However, for the $\eta'$-photoproduction off protons [27–29] the excitation function is described by an assumed coherent excitation of two possible resonances [27] ($S_{11}(1897)$ and $P_{11}(1986)$), which decay into $\eta'$ and proton. Anticipating this hypothesis, recently Nakayama et al. [9] have shown that it is also possible to explain the magnitude and energy dependence of the close-to-threshold total cross section for the $pp \rightarrow pp\eta'$ reaction assuming a dominance of these resonances and choosing an appropriate ratio of pseudoscalar to pseudovector coupling. However, the mesonic and nucleonic currents alone can describe the data as well [9].

The ambiguities in the description of the $pp \rightarrow pp\eta'$ reaction mechanisms, which are partly due to the poorly known coupling constants, indicate that the theory of the $\eta'$ meson creation is still far from delivering a complete and univocal picture of the process and call for further theoretical and experimental effort. A possible gluonium admixture in the $\eta'$ meson makes the study even more complicated but certainly also more interesting. Albeit the quark content of $\eta$ and $\eta'$ mesons is very similar a fusion of gluons emitted from the exchanged quarks of the colliding protons [30] would contribute primarily to the creation of the $\eta'$ meson which is predominantly a flavour singlet state due to the small pseudoscalar mixing angle ($\Theta_{ps} = -15^\circ$) [31].

Another complication in understanding the production mechanism is the unknown $\eta'$-proton interaction, which is of course in itself an interesting issue to be studied. One of the remarkable features of the published results on $\eta$ and $\eta'$ production is that the energy dependence of the total cross section appears not to follow the predictions based on the phase-space volume folded by the proton-proton final state interaction, which is the case in the $\pi^0$ meson production [32,33]. Moreover, for $\eta$ and $\eta'$ mesons the deviations from such predictions were qualitatively different: The close-to-threshold cross sections for the $\eta$ meson are strongly enhanced compared to the model comprising only the proton-proton interaction [15], opposite to the observed suppression in case of the $\eta'$ [5,6]. The energy dependence of the total cross section for the $pp \rightarrow pp\eta$ reaction can be described when the $\eta$-proton attractive interaction is taken into account [34,35]. Although the $\eta$-proton interaction is much weaker than the proton-proton one (compare the scattering length $a_{\rho\eta} = 0.751$ fm + $i$ 0.274 fm [36] with $a_{\rho p} = -7.83$ fm [37]) it becomes important through the interference terms between the various final pair interactions [35]. By analogy, the steep decrease of the total cross section when approaching the kinematical threshold for the $pp \rightarrow pp\eta'$ reaction could have been explained assuming a repulsive $\eta'$-proton interaction [5,6]. This interpretation, however, should rather be excluded now in view of the new COSY-11 data reported in this letter.

The experiment has been performed at the cooler synchrotron COSY-Jülich [2], using the COSY-11 facility [1] and the $H_2$ cluster target [3,4] installed in front of one of the regular COSY dipole magnets. The target, which is realized as a beam of $H_2$ molecules grouped to clusters of up to $10^6$ atoms, crosses perpendicularly the beam of $\sim 2 \cdot 10^{10}$ protons circulating in the ring. The beam of accelerated protons is cooled stochastically during the measure-
Table 1

Total cross sections for the $pp \rightarrow pp\eta$ reaction with respect to the excess energy in the center-of-mass system. Only statistical errors are quoted. In addition, there is an overall systematic uncertainty of 15% in the cross section and 0.44 MeV in energy.

<table>
<thead>
<tr>
<th>Excess energy [MeV]</th>
<th>Total cross section [nb]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.53 ± 0.05</td>
<td>5.0 ± 0.68</td>
</tr>
<tr>
<td>2.11 ± 0.20</td>
<td>6.9 ± 1.4</td>
</tr>
<tr>
<td>5.80 ± 0.06</td>
<td>27.9 ± 3.3</td>
</tr>
<tr>
<td>7.57 ± 0.07</td>
<td>43.5 ± 4.3</td>
</tr>
<tr>
<td>9.42 ± 0.09</td>
<td>46.8 ± 5.6</td>
</tr>
<tr>
<td>10.98 ± 0.12</td>
<td>67.4 ± 8.2</td>
</tr>
<tr>
<td>14.21 ± 0.13</td>
<td>82. ± 13.</td>
</tr>
<tr>
<td>23.64 ± 0.20</td>
<td>140. ± 19.</td>
</tr>
</tbody>
</table>

...ment cycle. Longitudinal and vertical cooling enables to keep the circulating beam practically without energy losses and without a spread of its dimensions when passing $1.6 \times 10^6$ times per second through the $10^{14}$ atoms/cm$^2$ thick target during a 60 minutes cycle. The beam dimensions are determined from the distribution of elastically scattered protons and are found to be 2 mm and 5 mm in the horizontal and vertical direction, respectively [38]. Quoted values denote standard deviations of an assumed Gaussian beam density distribution. The $pp \rightarrow pp\eta$ reaction has been investigated at eight different energies of a proton beam corresponding to excess energies ranging from $Q_s = 1.53$ MeV to $Q_s = 23.64$ MeV as listed in Table 1. The total integrated luminosity obtained during two weeks of the experiment amounts to $1.4 \text{ pb}^{-1}$, and was monitored by the simultaneous measurement of elastically scattered protons. A comparison of the measured differential distributions with results from the literature [39] determines the absolute luminosity with the statistical accuracy of 2.5% for each excess energy.

If at the intersection point of the cluster beam with the COSY proton beam the collision of protons results in the production of a meson, then the ejected protons – having smaller momenta than the beam protons – are separated from the circulating beam by the magnetic field. Further they leave the vacuum chamber through a thin exit foil and are registered by the detection system consisting of drift chambers and scintillation counters [1,18]. The hardware trigger, based on signals from scintillation detectors, was adjusted to register all events with at least two positively charged particles [40]. Tracing back trajectories from the drift chambers through the dipole magnetic field to the target point allowed for the determination of the particle momenta. From momentum and velocity, the latter measured using scintillation detectors, it is possible to identify the mass of the particle. Fig. 1 shows the squared mass of two simultaneously detected particles. A clear separation is seen into groups of events with two protons, two pions, proton and pion and also deuteron and pion. This spectrum enables to select events with two registered protons. The knowledge of the momenta of both protons before and after the reaction allows to calculate the mass of an unobserved particle or system of particles created in the reaction. Fig. 2a depicts the missing mass spectrum obtained for the $pp \rightarrow ppX$ reaction at an excess energy of $Q = 5.8$ MeV above the $\eta$ meson production threshold. Most of the entries in this spectrum originate from the multi-pion production [8,40], forming a continuous background to the well distinguished peaks accounting for the creation of $\omega$ and $\eta$ mesons, which can be seen at mass values of $782$ MeV/c$^2$ and $958$ MeV/c$^2$, respectively. The signal of the $pp \rightarrow pp\eta$ reaction is seen as well.

**Fig. 1.** Squared masses of two positively charged particles measured in coincidence. Pronounced peaks are to be recognized when two protons, proton and pion, two pions, or pion and deuteron were registered. Note that the number of events is shown in logarithmic scale.
Fig. 2. Mass spectrum of the unobserved particle or system of particles in the $pp \rightarrow ppX$ reaction determined at $Q = 5.8$ MeV above the $\eta$ production threshold.

The reaction is better to be seen in Fig. 2b, where the missing mass distribution in the vicinity of its kinematical limit is presented. Fig. 3a shows the missing mass spectrum for the measurement at $Q = 7.57$ MeV together with the multi-pion background (dotted line) as combined from the measurements at different excess energies [38]. Subtraction of the background leads to the spectrum with a clear signal at the mass of the $\eta$ meson as shown by the solid line in Fig. 3b. The dashed histogram in this figure corresponds to Monte Carlo simulations where the beam and target conditions were deduced from the measurements of elastically scattered protons [38].

The magnitude of the simulated distribution was fitted to the data, but the consistency of the widths is a measure of understanding of the detection system and the target-beam conditions. Histograms from a measurement at $Q = 1.53$ MeV shown in Fig. 3c,d demonstrate the achieved missing-mass resolution at

Fig. 3. Missing mass distribution with respect to the proton-proton system: (a),(b) measurements at $Q = 7.57$ MeV and (c),(d) at $Q = 1.53$ MeV. Background shown as dotted lines is combined from the measurements at different energies shifted to the appropriate kinematical limits and normalized to the solid-line histogram. Dashed histograms are obtained by means of Monte Carlo simulations.
the COSY-11 detection system, when using a stochastically cooled proton beam. The width of the missing mass distribution (Fig. 3d), which is now close to the natural width of the η' meson ($\Gamma_{\eta'} = 0.203 \text{ MeV}$ [41]), is again well reproduced by the Monte Carlo simulations. The broadening of the width of the η' signal with increasing excess energy (compare Fig. 3b and 3d) is a kinematical effect discussed in more detail in Ref. [18]. The decreasing signal-to-background ratio with growing excess energy is due to the broadening of the η' peak and the increasing background (see Fig. 2b) when moving away from the kinematical limit. At the same time, the shape of the background, determined by the convolution of the detector acceptance and the distribution of the two- and three-pion production [40], remains unchanged within the studied range of beam momenta from 3.213 GeV/c to 3.283 GeV/c. The signal-to-background ratio changes from 1.8 at $Q = 1.53 \text{ MeV}$ to 0.17 at $Q = 23.64 \text{ MeV}$. The geometrical acceptance, being defined by the gap of the dipole magnet and the scintillation detector most distant from the target [1,18], decreases from 50% to 4% within this range of excess energies. However, in the horizontal plane the range of polar scattering angles is still unlimited. The calculated acceptance depends on the angular distribution of the reaction products, which was assumed to be defined by the three body phase-space and the interaction of the outgoing protons. Calculating acceptance, the proton-proton FSI was taken into account by weighting phase-space generated events by the square of the proton-proton $^1S_0$-wave amplitude, $|A|^2$. The enhancement, $|A|^2$, from the proton-proton FSI was estimated as an inverse of the squared Jost function, with Coulomb interaction being taken into account [42]. Generally, the attractive proton-proton FSI lowers the angle between outgoing protons, increasing the acceptance. However, at the same time the efficiency for the reconstruction of both proton trajectories decreases. For the first five measurements denoted in Table 1 both effects are in the order of 3% and cancel each other. An increase of the overall efficiency is crucial only for the last two points listed in Table 1 and amounts to 9% and 25% for $Q = 14.21 \text{ MeV}$ and $Q = 23.64 \text{ MeV}$, respectively. In order to estimate a systematical error due to the inaccuracy of the pp-FSI, we calculated the acceptance using another prescription for $|A|^2$, which was obtained from the phase-shifts [43] calculated according to the modified Cini-Fubini-Stanghellini formula with the Wong-Noyes Coulomb correction [37,44,45]. Now the obtained efficiency was 13% and 34% larger as compared to the pure phase-space calculations for $Q = 14.21 \text{ MeV}$ and $Q = 23.64 \text{ MeV}$, respectively. Thus, for the highest energy there is a 9% difference depending on the applied prescription. The second main source of the systematical error is the inaccuracy of the determination of the two-track reconstruction efficiency. This was established to be 9% at $Q = 1.53 \text{ MeV}$ [40] and close to zero at $Q = 23.64 \text{ MeV}$. This uncertainty decreases with increasing Q, since at higher excess energy the probability that the tracks of the protons will be too close to be unresolved by the drift chambers is reduced. In addition to the discussed sources of the systematical error, which add up to 9% inaccuracy independent of energy, there are further systematical uncertainties with respect to i) the geometry of the detection system (2%), ii) the estimated losses due to the multiple scattering or nuclear reactions (1%) and iii) the luminosity determination (3%) [40]. Hence, the overall systematical error of the cross section values, including the normalization uncertainty, amounts to 15%.

Fig. 4 shows the compilation of total cross sections for the η' meson production. The data reported here are shown as filled circles. The absolute value of the excess energy was determined from the position of the η' peak in the missing mass spectrum, which should correspond to the mass of the meson η'. The systematical error of the excess energy established by this method equals to 0.44 MeV and constitutes of 0.14 MeV due to the uncertainty of the η' meson mass [41] and of 0.3 MeV due to the inaccuracy of the detection system geometry [46], with the largest effect originating from the inexactness of relative settings of target, dipole and drift chambers.

The solid line depicts calculations of the total cross section assuming that the primary production amplitude is constant and that only a proton-proton interaction significantly influences the exit channel. The magnitude was fitted to the data and the obtained $\chi^2$ value per degree of freedom amounts to 1.6. An inclusion of the η'-proton interaction in the
scattering length approximation, by factorizing p-p and η'-p FSI, resulted in a rather modest estimation of the real part of the η'-proton scattering length: |Re a_{p}^{\eta'}| < 0.8 fm. The proton-proton scattering amplitude was computed according to the formulas from Ref. [42]. The obtained energy dependence (solid line in Fig. 4) agrees within a few line thicknesses with the model developed by Fäldt and Wilkin [47].

The present data show that the phase-space volume weighted by the proton-proton FSI describes the near-threshold energy dependence of the total cross section for the pp → pp η' reaction quite well. The influence of the η'-proton FSI on the energy dependence of the total cross section is too weak to be seen within the up-to-date experimental accuracy. Based on the energy dependence of the total cross section only, it is impossible to decouple effects from η'-proton FSI and primary production amplitude. As shown by Nakayama et al. [9] the variation of the energy dependence of the total cross section, due to the production mechanism in the discussed energy range, can be in the order of 10%. To learn more about the η'-proton interaction a determination of differential cross sections is required.

It is interesting to note that in proton-proton collisions at much higher momenta (450 GeV/c) the η and η' mesons seem to have a similar production mechanism which differs from that of the π^0 one [48]. However, close to threshold the data show similarities between η' and π^0 mesons rather than between the η and η'.

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References

A study of the $f_0(1370)$, $f_0(1500)$, $f_0(2000)$ and $f_2(1950)$ observed in the centrally produced $4\pi$ final states

WA102 Collaboration

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Abstract

The production and decay properties of the $f_0(1370)$, $f_0(1500)$, $f_0(2000)$ and $f_2(1950)$ have been studied in central pp interactions at 450 GeV/c. The $dP_T$, $\phi$ and $|t|$ distributions of these resonances are presented. For the $J = 0$ states, the $f_0(1370)$ and $f_0(2000)$ have similar $dP_T$ and $\phi$ dependences. These are different to the $dP_T$ and $\phi$ dependences of the $f_0(980)$, $f_0(1500)$ and $f_0(1710)$. For the $J = 2$ states the $f_2(1950)$ has different dependences to the $f_2(1270)$ and $f_2(1520)$. This shows that the $dP_T$ and $\phi$ dependences are not just $J$ phenomena. © 2000 Published by Elsevier Science B.V. All rights reserved.
The WA102 collaboration has recently published a study of the centrally produced $4\pi$ final states [1]. In this paper the production and decay properties of the resonances observed in these channels will be presented. In previous publications the properties of the $f_0(1285)$ [2], $\eta_0(1645)$ and $\eta_0(1870)$ [3] have already been presented. In this paper the properties of the $f_0(1370)$, $f_0(1500)$, $f_0(2000)$ and $f_2(1950)$ will be discussed.

In previous analyses it has been observed that when the centrally produced system has been analysed as a function of the parameter $dP_T$, which is the difference in the transverse momentum vectors of the two exchange particles [4,5], all the undisputed $q\bar{q}$ states (i.e. $\eta$, $\eta'$, $f_0(1285)$ etc.) are suppressed at small $dP_T$ relative to large $dP_T$, whereas the glueball candidates $f_0(1500)$, $f_0(1710)$ and $f_2(1950)$ are prominent [6].

In addition, an interesting effect has been observed in the azimuthal angle $\phi$ which is defined as the angle between the $p_T$ vectors of the two outgoing protons. For the resonances studied to date which are compatible with being produced by DPE, the data [7] are consistent with the Pomeron transforming like a non-conserved vector current [8]. In order to determine the $\phi$ dependence for the resonances observed, a spin analysis has been performed on the $\pi^+\pi^-\pi^+\pi^-$ and $\pi^+\pi^-\pi^0\pi^0$ channels in four different $\phi$ intervals each of 45 degrees. As an example, Fig. 1 shows the $J^{PC}=0^{++} \rho\rho$ wave from the $\pi^+\pi^-\pi^+\pi^-$ channel in the four intervals. The waves have been fitted in each interval with the parameters of the resonances fixed to those obtained from the fits to the total data as described in Ref. [1]. The distributions found are consistent for the two channels and the fraction of each resonance as a function of $\phi$ from the $\pi^+\pi^-\pi^+\pi^-$ channel is plotted in Fig. 2. The distributions observed for the $f_0(1370)$ and $f_0(1500)$ are similar to what was found in the analysis of the $\pi^+\pi^-$ final state [9].

In order to calculate the contribution of each resonance as a function of $dP_T$, the waves have been fitted in three $dP_T$ intervals with the parameters of the resonances fixed to those obtained from the fits to the total data as described in Ref. [1]. Table 1 gives the percentage of each resonance in three $dP_T$ intervals together with the ratio of the number of events for $dP_T<0.2$ GeV to the number of events for $dP_T>0.5$ GeV for each resonance considered. The dependences found for the $f_0(1370)$ and $f_0(1500)$ are similar to what was found in the analysis of the $\pi^+\pi^-$ final state [9].

The fact that the $f_0(1370)$ and $f_0(1500)$ have different $\phi$ and $dP_T$ dependences confirms that these are not simply $J$ dependent phenomena. This is also true for the $J=2$ states, where the $f_2(1950)$ has different dependences to the $f_2(1270)$ and $f_2(1520)$ [9].

In order to determine the four momentum transfer dependence ($|t|$) of the resonances observed in the $\pi^+\pi^-\pi^+\pi^-$ channel the waves have been fitted to the total data as described in Ref. [1]. Fig. 2 shows the four momentum transfer from one of the proton vertices for these resonances. The distributions have been fitted with a single exponential of the form $\exp(-b|t|)$ and the values of $b$ found are given in Table 2. The values of $b$ for the $f_0(1370)$ and

![Image of graphs](image-url)
Fig. 2. The $f_0$ and four momentum transfer squared ($|t|$) distributions for a), b) the $f_0(1370)$, c), d) the $f_0(1500)$, e), f) the $f_0(2000)$ and g), h) the $f_0(1950)$.

$f_0(1500)$ are similar to what was found in the analysis of the $\pi^+\pi^-$ final state [9].

The $\phi$ distribution, the $dP_T$ and $t$ dependence of the $f_0(1950)$ are different to what has been observed for other $J^{PC} = 2^{++}$ resonances [9] but are similar to what was observed for the $\phi\phi$ [10] and $K^+(892)\bar{K}^-(892)$ [11] final states which were both found to have $J^{PC} = 2^{++}$. In order to see if the $\phi\phi$

<table>
<thead>
<tr>
<th>$dP_T$</th>
<th>0.2 $\leq dP_T$</th>
<th>$dP_T$ $\geq 0.5$ GeV</th>
<th>(R = \frac{d\Gamma(dP_T)}{dP_T}$</th>
<th>0.2 $\leq dP_T$</th>
<th>$dP_T$ $\geq 0.5$ GeV</th>
<th>(R = \frac{d\Gamma(dP_T)}{dP_T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0(1370)$</td>
<td>11.0 $\pm$ 2.0</td>
<td>32.9 $\pm$ 3.0</td>
<td>56.1 $\pm$ 4.9</td>
<td>0.19 $\pm$ 0.04</td>
<td>11.0 $\pm$ 2.0</td>
<td>32.9 $\pm$ 3.0</td>
</tr>
<tr>
<td>$f_0(1500)$</td>
<td>23.8 $\pm$ 2.5</td>
<td>47.3 $\pm$ 4.5</td>
<td>28.8 $\pm$ 2.9</td>
<td>0.83 $\pm$ 0.12</td>
<td>23.8 $\pm$ 2.5</td>
<td>47.3 $\pm$ 4.5</td>
</tr>
<tr>
<td>$f_0(2000)$</td>
<td>11.9 $\pm$ 1.3</td>
<td>37.7 $\pm$ 3.2</td>
<td>50.2 $\pm$ 4.1</td>
<td>0.23 $\pm$ 0.03</td>
<td>11.9 $\pm$ 1.3</td>
<td>37.7 $\pm$ 3.2</td>
</tr>
<tr>
<td>$f_0(1950)$</td>
<td>27.4 $\pm$ 2.4</td>
<td>45.5 $\pm$ 5.1</td>
<td>27.1 $\pm$ 2.4</td>
<td>1.01 $\pm$ 0.12</td>
<td>27.4 $\pm$ 2.4</td>
<td>45.5 $\pm$ 5.1</td>
</tr>
</tbody>
</table>

Table 2

The slope parameter $b$ from a single exponential fit to the $|t|$ distributions.

<table>
<thead>
<tr>
<th>$f_0(1370)$</th>
<th>$f_0(1500)$</th>
<th>$f_0(2000)$</th>
<th>$f_0(1950)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.8 $\pm$ 0.5</td>
<td>5.1 $\pm$ 0.4</td>
<td>5.6 $\pm$ 0.4</td>
<td>5.9 $\pm$ 0.4</td>
</tr>
</tbody>
</table>

and $K^+(892)\bar{K}^-(892)$ final states could be due to the $f_0(1950)$, the parameters of the $f_0(1950)$ have been used as input to a Breit-Wigner function which has been modified to take into account the different thresholds.

Superimposed on the $\phi\phi$ mass spectrum in Fig. 3a) is the distribution that could be due to the $f_0(1950)$. As can be seen, although the $f_0(1950)$ can describe most of the spectrum, there is an excess of events in the 2.3 GeV mass region. Including a Breit-Wigner to describe the $f_2(2340)$, which has previously been observed decaying to $\phi\phi$ [12], with $M = 2330 \pm 15$ MeV and $I = 130 \pm 20$ MeV gives the distribution in Fig. 3b). Assuming that the $f_0(1950)$ has a $\phi\phi$ decay mode then correcting for the unseen decay modes the branching ratio of the $f_0(1950)$ to $f_2(1270)\pi\pi/\phi\phi$ was found to be 72 $\pm$ 9.

Fig. 3. a) and b) The $\phi\phi$ and c) the $K^+(892)\bar{K}^-(892)$ mass spectra with fits described in the text.
Superimposed on the $K^{*0} K^{*0}$ mass spectrum in Fig. 3c is the distribution that could be due to the $f_2(1500)$. As can be seen the $f_2(1500)$ can describe all the $K^{*0} K^{*0}$ mass spectrum. Assuming that the $f_2(1500)$ has a $K^{*0} K^{*0}$ decay mode then correcting for the unseen decay modes the branching ratio of the $f_2(1500)$ to $f_2(1270)\pi\pi/K^{*0} K^{*0}$ was found to be 33 ± 4. In addition, the branching ratio of the $f_2(1500)$ to $\phi\phi/K^{*0} K^{*0}$ above the $\phi\phi$ threshold is 0.8 ± 0.14.

We have previously published a paper describing the decays of the $f_0(1370)$ and $f_0(1500)$ to $\pi\pi$ and $KK$ [9]. In Ref. [1] a fit has been performed to the $pp$ and $\sigma\sigma$ final states and the contributions of the $f_0(1370)$ and $f_0(1500)$ have been determined. After correcting for the unseen decay modes and the $\sigma\sigma$ decay mode the branching ratio of the $f_0(1500)$ to $4\pi/\pi\pi$ is found to be 1.37 ± 0.16. In the initial Crystal Barrel publication this value was 3.4 ± 0.8 [13]. In the latest preliminary analysis [14] of the Crystal Barrel data the value is 1.54 ± 0.6. Hence although the experiments disagree about the relative amount of $\rho\rho$ and $\sigma\sigma$ in the $4\pi$ decay mode [1], the overall measured branching ratio is consistent.

After correcting for the unseen decay modes and taking into account the above uncertainties the branching ratio of the $f_0(1370)$ to $4\pi/\pi\pi$ is found to be $34\pm 22^\circ$. The large error is due to the fact that there is considerable uncertainty in the amount of $f_0(1370)$ in the $\pi\pi$ final state due to the possible contribution from the high mass side of the $f_0(1000)$. In the latest preliminary analysis [14] of the Crystal Barrel data the value is 12.2 ± 5.4. A coupled channel fit of the $\pi\pi$, $KK$, $4\pi$, $\eta\eta$ and $\eta'\eta'$ final states is in progress and will be reported in a future publication.

In summary, the $dP_T$, $\phi$ and $|t|$ distributions for the $f_0(1370)$, $f_0(1500)$, $f_0(2000)$ and $f_2(1500)$ have been presented. For the $J = 0$ states the $f_0(1370)$ and $f_0(2000)$ have similar $dP_T$ and $\phi$ dependences. These are different to the $dP_T$ and $\phi$ dependences of the $f_0(980)$, $f_0(1500)$ and $f_0(1710)$. For the $J = 2$ states the $f_2(1500)$ has different dependences to the $f_2(1270)$ and $f_2(1520)$. This shows that the $dP_T$ and $\phi$ dependences are not just $J$ phenomena.

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References

High precision mass measurements in $\Psi$ and $\Upsilon$ families revisited


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Abstract

High precision mass measurements in $\Psi$ and $\Upsilon$ families performed in 1980–1984 at the VEPP-4 collider with OLYA and MD-1 detectors are revisited. The corrections for the new value of the electron mass are presented. The effect of the updated radiative corrections has been calculated for the $J/\Psi(1S)$ and $\Psi(2S)$ mass measurements.

Development of the resonant depolarization method (RDM) suggested in Novosibirsk [1,2] opened unique opportunities in the high precision determination of the elementary particle masses. Pioneer experiments in Novosibirsk (see [3] and references therein) were followed by those at Cornell [4], DESY [5] and CERN [6]. In this paper we reconsider our measurements performed at the $e^+e^-$ collider VEPP-4 in Novosibirsk in the $\Psi$ [7,8] and $\Upsilon$ meson families [9–13] with the goal to take into account the change of the electron mass value [14,15] as well as the updated radiative corrections [16] in case of $J/\Psi(1S)$ and $\Psi(2S)$.

$J/\Psi(1S)$ and $\Psi(2S)$ mass measurements [7,8] were performed in 1980 with the OLYA detector [17] while the MD-1 group [18] carried out three independent measurements of the $\Upsilon(1S)$ mass in 1982 [9], in 1983 [10] and in 1984 [11,12] as well as determined the masses of $\Upsilon(2S)$ and $\Upsilon(3S)$ in 1983 [10,13]. The masses of the $\Psi$ and $\Upsilon$ mesons were obtained from a fit of the energy dependence of $\sigma(e^+e^-\rightarrow hadrons)$ followed by relating the value...
of the resonance mass to the beam energy. The absolute calibration of the beam energy was performed using the RDM.

The resonant depolarization method is based upon the fact that in a storage ring with a planar orbit the spin precession frequency \( \Omega_s \) depends on the beam energy \( E \) as

\[
\Omega_s = \omega \left( 1 + \frac{\mu'}{\mu_0} \gamma \right),
\]

where \( \omega \) is the beam revolution frequency, \( \mu'/\mu_0 \) is the ratio of the anomalous and normal parts of the electron magnetic moment, \( \gamma = E/mc^2 \) is the Lorentz factor of electrons. The frequency \( \Omega_s \) is measured at the polarized electron beam using a depolarizer with the frequency \( \Omega_d \) adjusted as \( \Omega_s = \Omega_d + n \omega \), where \( n \) is an arbitrary integer number.

A typical accuracy of the method is about \( 10^{-5} \). However, the measured quantity is a \( \gamma \) factor of electrons rather than their energy. Thus, the beam energy and the resonance mass determined by the RDM depend on the electron mass assumed. In 1986 when the results of the \( T(1S) \) mass measurement were published [11], its accuracy was about five times worse than the claimed accuracy of the electron mass in the MeV scale (2.8 ppm) [14]. However, in “The 1986 adjustment of the fundamental physical constants” [15] the value of the electron mass was decreased by 8.5 ppm while its error was reduced to 0.3 ppm.

The decrease of the electron mass [15] was caused mainly by the 7.8 ppm (about three “old” standard deviations) increase of the \( e/h \) ratio. Taking into account that two other fundamental constants which depend on \( e \), \( h \) and \( m_e \), i.e. the fine-structure constant \( \alpha \) and Rydberg constant \( R_n \), remained almost unchanged, the increase of \( e/h \) propagates to the abovementioned 8.5 ppm decrease of \( m_e \) in the MeV scale. Since resonance masses determined from RDM are based upon the value of the electron mass and are quoted in MeV, they should be also decreased by 8.5 ppm. The corresponding corrections to the values of the \( T(1S), T(2S) \) and \( T(3S) \) meson masses measured by MD-1 were already reported at the Chicago Conference [19].

An additional correction should be applied to the values of the \( J/\Psi(1S) \) and \( \Psi(2S) \) mass obtained in [7,8]. Similarly to most early measurements, a fit of \( \sigma(e^+e^- \rightarrow \text{hadrons}) \) in these papers included the radiative corrections calculated according to the classic work of Jackson and Scharre [20]. Later, in Ref. [16] it was shown that the approach of Ref. [20] is not quite accurate and, in particular, violates the Bloch-Nordsieck theorem. Correspondingly, the analysis of the \( T \) resonances was performed [11–13] using the improved radiative corrections suggested in [16]. In Ref. [11] it was shown that the corresponding shift of the mass was about 0.1 MeV. Somewhat later the paper [21] was published entirely dedicated to the correction of the old measurements of \( \Psi \) and \( T \) parameters using the updated radiative corrections. However, the \( J/\Psi(1S) \) and \( \Psi(2S) \) masses were neither refit by the authors of Ref. [7,8] nor quoted in Ref. [21].

The details of \( J/\Psi(1S) \) and \( \Psi(2S) \) mass measurements [7,8] are not available now. Therefore, the \( J/\Psi(1S) \) and \( \Psi(2S) \) mass corrections were estimated by us as in Ref. [21] from the difference of the fits with the radiative corrections from Ref. [20] and Ref. [16]. Similarly to Ref. [20], only the electron loop was taken into account in the photon vacuum

### Table 1
Revision of mass measurements in \( \Psi \) and \( T \) families

<table>
<thead>
<tr>
<th>Particle</th>
<th>Previous mass (MeV)</th>
<th>( \Delta M ) (MeV)</th>
<th>( \Delta M ) (rad.) (MeV)</th>
<th>Updated mass (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J/\Psi(1S)) [7,8]</td>
<td>3096.93 ± 0.09</td>
<td>−0.026</td>
<td>−0.016</td>
<td>3096.89 ± 0.09</td>
</tr>
<tr>
<td>( \Psi(2S)) [7,8]</td>
<td>3686.00 ± 0.10</td>
<td>−0.031</td>
<td>−0.023</td>
<td>3685.95 ± 0.10</td>
</tr>
<tr>
<td>( T(1S)) [12]</td>
<td>9460.59 ± 0.09 ± 0.05</td>
<td>−0.080</td>
<td>−</td>
<td>9460.51 ± 0.09 ± 0.05</td>
</tr>
<tr>
<td>( T(2S)) [10,13]</td>
<td>10023.6 ± 0.5</td>
<td>−0.085</td>
<td>−</td>
<td>10023.5 ± 0.5</td>
</tr>
<tr>
<td>( T(3S)) [10,13]</td>
<td>10355.3 ± 0.5</td>
<td>−0.088</td>
<td>−</td>
<td>10355.2 ± 0.5</td>
</tr>
</tbody>
</table>
polarization term in Ref. [7,8]. The resulting mass correction for radiative effects equals \(-0.023 \pm 0.003\)\(\sigma_e\), where \(\sigma_e\) is the rms spread of the \(e^+e^-\) center of mass energy and the error accounts for dependence of the correction on the luminosity distribution around the resonance. The correction is somewhat lower than that which can be obtained from Figure 6 of Ref. [21]. At \(\sigma_e = 0.7(1.0)\) MeV in \(J/\psi(1S)\) and \(\psi(2S)\) runs it equals \(-0.016\) MeV and \(-0.023\) MeV respectively.

Table 1 presents a list of the resonance masses measured at the VEPP-4 collider with the corresponding corrections, where \(\Delta M(m)\) and \(\Delta M(rad.)\) stand for the correction for the electron mass and radiative effects respectively.

Let us briefly discuss how the change of the resonance masses above can affect other measurements. The new value of the \(\psi(2S)\) mass should be taken into account during the interpretation of the Fermilab studies of the charmonium family in \(p\bar{p}\) annihilation [22] which used the value of the \(\psi(2S)\) mass from [7,8] as a basic calibration in their determination of the \(J/\psi(1S)\) mass. It is obvious that the obtained values of the \(m_e\) correction for \(T(1S)\) and \(T(2S)\) can also be applied to the Cornell [4] and DESY [5] measurements respectively. Since in these experiments the radiative corrections were calculated according to Ref. [20], their results should be also corrected for the radiative effects. We remind that our value of the \(T(1S)\) mass differs by more than 3.5 standard deviations from that at Cornell while for \(T(2S)\) it is consistent with the one in DESY. Our measurement of the \(T(3S)\) mass has not been repeated by any other group.

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