Cosmological evolution of the rolling tachyon

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Abstract

The cosmological effects of the tachyon rolling down to its ground state are discussed by coupling a simple effective field theory for the tachyon field to Einstein gravity. As the tachyon rolls down to the minimum of its potential the universe expands. Depending upon initial conditions, the scale factor may or may not start off accelerating, but ultimately it ceases to do so and the final flat spacetime is either static in the rest frame of the tachyon (if $k = 0$) or (if $k = -1$) given by the Milne model.

1. Introduction

The reconciliation of fundamental theories such as M/String theory with the basic facts of cosmology continues to present great challenges. The most straightforward approach is to pass to an effective field theory in which gravity is coupled to matter, for instance in a supergravity theory. Because the theories are formulated in higher dimensions, one must either construct a spontaneous compactification scenario or imagine a scheme in which the universe is sort of 3-brane [3]. It seems to be very difficult to construct models with $N = 1$ supersymmetry on the 3-brane [4]. The trouble with compactifications is that they come with associated massless scalars and degenerate vacua: one must address the question of how these evolve with time [1]. One must also ensure that the resultant time-variability of coupling constants is compatible with observations. One possibility is to give the potentials a mass “by hand” such a way that Minkowski-space times a Calabi–Yau is an attractor at late times, [2] but this is ad hoc and ugly.

There is fairly good evidence from the BOOMERANG observations of the Cosmic Microwave Background that the scale factor of the universe underwent a period of acceleration (so-called Primordial Inflation) at early times and from Type Ia super-novae that it may also have been accelerating very recently if not today. (For a recent review see [8].) It is quite difficult to get accelerating universe out of pure supergravity theories [9–14] although with super-matter, providing one gauges a suitable axial current this is possible [5]. The problem is that the axial gauging gives rise to anomalies [6]. It is possible that these anomalies can be cancelled in staring models with D-branes [7].

In recent years there has been great progress, particularly due to Sen, in our understanding of the role of the tachyon in String Theory (see [15] for a recent account with references to earlier work). The basic idea is that the usual open string vacuum is unstable but
there exist a stable vacuum with zero energy density which is stable. There is evidence that this state is associated with the condensation of electric flux tubes of closed strings (see [15,16]). These flux tubes described successfully using an effective Born–Infeld action (see [15,17,18] and references therein). This success of effective action methods, together with the difficulties of other approaches described the encourages one to pursue this further and to attempt a description of the cosmology of tachyon rolling. Moreover not to take into account the effects gravity during the process is inconsistent, since it involves a spatially uniform distribution of energy. It is the purpose of this note to rectify this omission and initiate a study of tachyon cosmology.

2. The rolling tachyon

The tachyon of string theory may be described by effective field theory describing some sort of tachyon condensate which in flat space has a Lagrangian density

\[ \mathcal{L} = -V(T)\sqrt{1 + \eta^{\mu\nu}\partial_\mu T \partial_\nu T}, \] (1)

where \( T \) is the tachyon field, \( V(T) \) is the tachyon potential and

\[ \eta_{\mu\nu} = \text{diag}(-, +, +, \ldots) \]

is the metric of Minkowski spacetime (see [16] for a discussion with references to earlier work). The tachyon potential \( V(T) \) has a positive maximum at the origin and has a minimum at \( T = T_0 \) where the potential vanishes. In [16], \( T_0 \) is taken to lie at infinity. In Minkowski spacetime the rolling down of the towards its minimum value is described by a spatially homogeneous but time-dependent solution obtained from the Lagrangian density

\[ \mathcal{L} = -V(\sqrt{1 - T^2}). \] (2)

During rolling the Hamiltonian density

\[ \mathcal{H} = \frac{V(T)}{\sqrt{1 - T^2}}, \] (3)

has a constant value \( E \). Thus

\[ \dot{T} = \sqrt{1 - \frac{V_0^2(T)}{E^2}}, \] (4)

As \( T \) increases \( V(T) \) decreases and \( \dot{T} \) increases to attain its maximum value of 1 in infinite time as \( T \) tends to infinity. Note that as explained in [16] the tachyon field behaves like a fluid of positive energy density

\[ \rho = \frac{V(T)}{\sqrt{1 - T^2}} \] (5)

and negative pressure

\[ P = -V(T)\sqrt{1 - T^2}. \] (6)

Thus

\[ P\rho = -V^2(T) \] (7)

and

\[ \frac{P}{\rho} = w = -(1 - \frac{\dot{T}^2}{2}), \] (8)

and therefore, \(-1 \leq w \leq 0\). Note that both the Weak Energy Condition, \( \rho > 0 \) and Dominant Energy Condition, \( \rho > |P| \) hold. However, because

\[ \rho + 3P = -\frac{2V(T)}{\sqrt{1 - T^2}} \left( 1 - \frac{3}{2} \frac{T^2}{2} \right) \] (9)

the Strong Energy Condition fails to hold for small \(|\dot{T}|\) but does hold for large \(|\dot{T}|\).

The discussion above has neglected the gravitational field generated by the tachyon condensate. To take it into account we use the Lagrangian density

\[ \sqrt{-g}\left( \frac{R}{16\pi G} - V(T)\sqrt{1 + g^{\mu\nu}\partial_\mu T \partial_\nu T} \right), \] (10)

where \( g_{\mu\nu} \) is the metric and \( R \) its scalar curvature. We shall work in 3 + 1 spacetime dimensions and assume that the metric has Friedman–Lemaître–Robertson–Walker form

\[ ds^2 = -dt^2 + a(t)^2 d\Omega^2_k, \] (11)

where \( a(t) \) is the scale factor and \( d\Omega^2_k \) is, locally at least, the metric on \( S^3, \mathbb{R}^3 \) or \( H^3 \) according as \( k = 1, 0, -1 \), respectively. Note that in this model we have assumed that the cosmological constant \( \Lambda \) vanishes in the tachyon ground state. The expressions (5) and (6) for the density and pressure remain valid and thus the Friedman and Raychaudhuri equations governing the evolution of the scale factor are

\[ \frac{a^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3} \frac{V(T)}{\sqrt{1 - T^2}}. \] (12)
and
\[ \frac{\ddot{a}}{a} = \frac{8\pi G}{3\sqrt{1 - T^2}} \left( 1 - \frac{3}{2} \dot{T}^2 \right). \tag{13} \]

The Hamiltonian density of the tachyon field is no longer constant because the tachyon Lagrangian density is now explicitly time dependent. Eqs. (2) and (3) must be replaced by
\[ \mathcal{L} = -a^3 V\sqrt{1 - T^2} \tag{14} \]
and
\[ \mathcal{H} = a^3 \frac{V(T)}{\sqrt{1 - T^2}}. \tag{15} \]
The equation
\[ \frac{d\mathcal{H}}{dt} = -\frac{\partial \mathcal{L}}{\partial t} \tag{16} \]
is formally equivalent to the conservation of entropy of the fluid which reads
\[ \dot{\rho} = -\frac{\dot{a}}{a} (\rho + P). \tag{17} \]

Because
\[ \rho + P = V(T) \frac{\dot{T}^2}{\sqrt{1 - T^2}}. \tag{18} \]

we have
\[ \frac{d}{dt} \left( \frac{V}{\sqrt{1 - T^2}} \right) = -\frac{3\dot{a}}{a} \left( \frac{V\dot{T}^2}{\sqrt{1 - T^2}} \right). \tag{19} \]

Thus the evolution equation (4) remains valid but the quantity \( E \) is no longer constant but rather decreases in time.

The course of cosmic evolution is now rather clear from these equations. The tachyon field rolls down hill with an accelerated motion and the universe expands. It still follows from (4) that as \( T \) increases \( V(T) \) decreases but since \( E \) decreases, in principal \( \dot{T} \) could decrease but in any case \( \dot{T} \) remains positive and so \( T \) increases monotonically to attain its maximum value of 1. From the Friedman equation (12), it follows that if \( k \leq 0 \), then \( \dot{a} \) will always be positive. This is because the Weak Energy Condition holds, \( \rho > 0 \). From the Raychaudhuri equation (13) one deduces that initially if \( |\dot{T}| < 2/3 \) the scale factor initially accelerates, \( \dot{a} > 0 \) but eventually, once \( \dot{T} \) exceeds \( \sqrt{2/3} \) the acceleration will cease and deceleration will set in. If the universe is flat, i.e., if \( k = 0 \), then ultimately the scale factor may halt \( a(t) \to \text{constant} \). If the universe has hyperbolic sections, that is if \( k = -1 \), then ultimately the scale factor increases linearly with time, \( a \to t \). In both cases the final state of the universe is flat, the case \( k = -1 \) being the Milne model. In the case of spherical sections, \( k = +1 \) re-collapse will take place. The possibility of cosmic acceleration arises from the positive potential \( V(T) \) and should be contrasted with the situation in pure supergravity theories for which the Strong Energy Condition holds and cosmic acceleration is not possible [9,10], unless the internal space is non-compact [12,13]. The inclusion of supermatter matter may allow acceleration [14].

Despite the violation of the Strong Energy condition one sees from the Friedman equation (12) that in the cases \( k \leq 0 \) the positivity of energy precludes the avoidance of singularities in the past. If however \( k = 1 \), it is conceivable that, for special initial conditions, the scale factor might pass through a finite sequence of minima and maxima or even that periodic or quasi-periodic solutions exist with an infinite sequence of maxima and minima.

Acknowledgements

After submitting this Letter to the archive I was told of an earlier paper of Stephon Alexander [19] of which I was unaware and which anticipated some of the ideas discussed here, in terms of \( D-D \) annihilation. The action adopted for the tachyon field is different from that used in this Letter. Another relevant pre-cursor brought to my attention by Anupam Mazumdar and to which the same remarks apply is [20], see also Burgess et al. [21]. I would like to thank them and also Andrew Chamblin, Neil Lambert, Mohammad Garousi, thanu Padmanabhan, Nakoi Saskura and Arkady Tseytlin for helpful comments and pointing out typos and small inaccuracies in the wording.

References

Search for scalar quarks in $e^+e^-$ collisions at $\sqrt{s}$ up to 209 GeV

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Abstract

Searches for scalar top, scalar bottom and mass-degenerate scalar quarks are performed in the data collected by the ALEPH detector at LEP, at centre-of-mass energies up to 209 GeV, corresponding to an integrated luminosity of 675 pb$^{-1}$. No evidence for the production of such particles is found in the decay channels $\tilde{t} \rightarrow c/\bar{u} \chi$, $\tilde{t} \rightarrow b \ell \tilde{\nu}$, $\tilde{b} \rightarrow b \chi$, $\tilde{q} \rightarrow q \chi$ or in the stop four-body decay channel $\tilde{t} \rightarrow b \chi f \bar{f}'$ studied for the first time at LEP. The results of these searches yield improved mass lower limits. In particular, an absolute lower limit of 63 GeV/$c^2$ is obtained for the stop mass, at 95% confidence level, irrespective of the stop lifetime and decay branching ratios. © 2002 Elsevier Science B.V. All rights reserved.
1. Introduction

The results of searches for scalar quarks with the data collected in the year 2000 by the ALEPH detector at LEP are presented in this Letter. The energies and integrated luminosities of the analysed data samples are given in Table 1. Previous results obtained with lower energy data have been reported by ALEPH in Refs. [1–5] and by the other LEP Collaborations in Refs. [6–8].

The theoretical framework for these studies is the supersymmetric extension of the Standard Model [9], with R-parity conservation. The lightest supersymmetric particle (LSP) is assumed to be the lightest neutralino $\chi$ or the sneutrino $\tilde{\nu}$. Such an LSP is stable and weakly interacting. Each chirality state of the Standard Model fermions has a scalar supersymmetric partner. The scalar quarks (squarks) $\tilde{q}_L$ and $\tilde{q}_R$ are the supersymmetric partners of the left-handed and right-handed quarks, respectively. The mass eigenstates are orthogonal combinations of the weak interaction eigenstates $\tilde{q}_L$ and $\tilde{q}_R$. The mixing angle $\theta_{\tilde{q}}$ is defined in such a way that $\tilde{q} = \tilde{q}_L \cos \theta_{\tilde{q}} + \tilde{q}_R \sin \theta_{\tilde{q}}$ is the lighter squark. The off-diagonal terms of the mass matrix, responsible for mixing, read, with standard notation: $m_{\tilde{q}}(A_q - \mu \kappa)$, with $\kappa = \tan \beta$ for down-type and $\kappa = \frac{1}{\tan \beta}$ for up-type quarks. Since the size of this mixing term is proportional to the mass of the Standard Model partner, it could well be that the lightest supersymmetric charged particle is the lighter scalar top (stop, $\tilde{t}$) or, in particular for large $\tan \beta$ values, the lighter scalar bottom (sbottom, $\tilde{b}$). Squarks could be produced at LEP in pairs, $e^+ e^- \rightarrow \tilde{q} \bar{\tilde{q}}$, via s-channel exchange of a virtual photon or Z. The production cross section [10] depends on $\theta_{\tilde{q}}$ when mixing is relevant, i.e., for stops and sbottoms.

The searches for stops described here assume that all supersymmetric particles except the lightest neutralino $\chi$ and possibly the sneutrino $\tilde{\nu}$ are heavier than the stop. Under these assumptions, the allowed decay channels are $\tilde{t} \rightarrow c/\bar{u} \chi$, $\tilde{t} \rightarrow b \chi f \bar{f}'$ and $\tilde{t} \rightarrow b \bar{\nu}$ [10,11]. The corresponding diagrams are shown in Fig. 1. The decay $\tilde{t} \rightarrow c \chi$ (Fig. 1(a)) proceeds only via loops and has a very small width, of the order of $0.01–1$ eV [10], depending on the mass difference $\Delta M$ between the stop and the neutralino, and on the masses and field content of the particles involved in the loops. For low enough $\Delta M$ values ($\Delta M \lesssim 6 \text{ GeV} / c^2$), the stop lifetime becomes sizeable, and must be taken into account in the searches for stop production. If $\Delta M$ is so small that the $\tilde{t} \rightarrow c \chi$ channel is kinematically closed, the dominant decay mode becomes $\tilde{t} \rightarrow u \chi$, and the stop can then be considered as stable for practical purposes.

Table 1

<table>
<thead>
<tr>
<th>Luminosity (pb$^{-1}$)</th>
<th>Energy range (GeV)</th>
<th>$\sqrt{s}$ (GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.4</td>
<td>207–209</td>
<td>208.0</td>
</tr>
<tr>
<td>122.6</td>
<td>206–207</td>
<td>206.6</td>
</tr>
<tr>
<td>75.3</td>
<td>204–206</td>
<td>205.2</td>
</tr>
</tbody>
</table>

Fig. 1. Squark decay diagrams considered in this Letter: (a) $\tilde{t} \rightarrow c/\bar{u} \chi$; (b) $\tilde{t} \rightarrow b \chi f \bar{f}'$ via W exchange and (c) $\tilde{t} \rightarrow b \chi f \bar{f}'$ via sfermion exchange; (d) $\tilde{t} \rightarrow b \bar{\nu}$; (e) $b \rightarrow b \chi$. 
Topologies studied in the different scenarios

<table>
<thead>
<tr>
<th>Production</th>
<th>Decay mode</th>
<th>Topology/Analysis</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{t}\tilde{t}$</td>
<td>$\tilde{t} \rightarrow cX (\Delta M \gtrsim 6 \text{ GeV}/c^2)$</td>
<td>Acoplanar jets (AJ)</td>
<td>[1–3,5]</td>
</tr>
<tr>
<td>$\tilde{t}\tilde{t}$</td>
<td>$\tilde{t} \rightarrow c/\mu X (\Delta M \lesssim 6 \text{ GeV}/c^2)$</td>
<td>Long-lived hadrons</td>
<td>[4]</td>
</tr>
<tr>
<td>$\tilde{t}\tilde{t}$</td>
<td>$\tilde{t} \rightarrow b\chi f f'$</td>
<td>Multi-jets (MJ)</td>
<td>This Letter</td>
</tr>
<tr>
<td>$\tilde{t}\tilde{t}$</td>
<td>$\tilde{t} \rightarrow b\tilde{\nu}$</td>
<td>AJ plus leptons</td>
<td>[1–3,5]</td>
</tr>
<tr>
<td>$\tilde{t}\tilde{t}$</td>
<td>$\tilde{t} \rightarrow b\chi$</td>
<td>AJ plus b tagging</td>
<td>[1–3,5]</td>
</tr>
<tr>
<td>$\tilde{q}\tilde{q}$</td>
<td>$\tilde{q} \rightarrow q\chi$</td>
<td>AJ</td>
<td>[2,3,5]</td>
</tr>
</tbody>
</table>

For stop masses accessible at LEP, i.e., $\lesssim 100 \text{ GeV}/c^2$, the decay mode $\tilde{t} \rightarrow b\chi f f'$ is mediated by virtual chargino and $W$ (Fig. 1(b)) or sfermion (Fig. 1(c)) exchange. It is therefore of the same order in perturbation theory as the loop-induced $\tilde{t} \rightarrow cX$ decay, and can be substantially enhanced if charginos have masses not much larger than their present experimental bounds, and could even dominate for light sfermions [11]. The four-body decay channel yields topologies with $b$-jets, additional jets and/or leptons, and with missing mass and missing energy. A new multi-jet analysis, hereafter called MJ, has been designed to cope with these final states.

The $t \rightarrow b\tilde{\nu}$ channel proceeds via virtual chargino exchange (Fig. 1(d)) and has a width of the order of 0.1–10 keV [10]. This decay channel dominates when it is kinematically allowed, i.e., if the lightest $\tilde{\nu}$ is lighter than the stop. If the lightest neutralino is the LSP, the sneutrino decays invisibly into $\chi\nu$ without any change in the experimental topology.

Under the assumption that the $b$ is lighter than all supersymmetric particles except the $\chi$, the $b$ will decay as $b \rightarrow b\chi$ (Fig. 1(e)). Compared to the $t$, the $b$ decay width is larger, of the order of 10–100 MeV.

The supersymmetric partners of the light quarks are generally expected to be quite heavy. If they are light enough to be within the reach of LEP, their dominant decay mode is expected to be $\tilde{q} \rightarrow q\chi$.

The final state topologies addressed by the searches presented in this Letter are summarised in Table 2, together with the related signal processes and with the references where analysis details can be found.

This Letter is organised as follows. In Section 2, the ALEPH detector and the simulated samples used for the analyses are described. Section 3 is dedicated to the selection algorithms with emphasis on the new search for four-body stop decays. In Section 4 the results of the searches are given, along with their interpretation in the theoretical framework. The conclusions of the Letter are given in Section 5.

2. ALEPH detector and event simulation

A thorough description of the ALEPH detector and its performance, as well as of the standard reconstruction and analysis algorithms, can be found in Refs. [12,13]. Only a brief summary is given here.

The trajectories of charged particles are measured by a silicon vertex detector (VDET), a cylindrical multi-wire drift chamber (ITC) and a large time projection chamber (TPC). These detectors are immersed in an axial magnetic field of 1.5 T provided by a superconducting solenoidal coil. The VDET consists of two cylindrical layers of silicon microstrip detectors; it performs precise measurements of the impact parameter in space, yielding powerful short-lived particle tags, as described in Ref. [14].

The electromagnetic calorimeter (ECAL), placed between the TPC and the coil, is a highly-segmented sandwich of lead planes and proportional wire chambers. It consists of a barrel and two endcaps. The hadron calorimeter (HCAL) consists of the iron return yoke of the magnet instrumented with streamer tubes. It is surrounded by two double layers of streamer tubes, the muon chambers. The luminosity monitors (LCAL and SiCAL) extend the calorimeter hermeticity down to 34 mrad from the beam axis.

The energy flow algorithm described in Ref. [13] combines the measurements of the tracking detectors and of the calorimeters into “objects” classified as charged particles, photons, and neutral hadrons. The energy resolution achieved with this algorithm is $(0.6\sqrt{E} + 0.6) \text{ GeV} (E \text{ in GeV})$. Electrons are identi-
fied by comparing the energy deposit in ECAL to the momentum measured in the tracking system, by using the shower profile in the electromagnetic calorimeter, and by the measurement of the specific ionization energy loss in the TPC. The identification of muons makes use of the hit pattern in HCAL and of the muon chambers. Signal event samples were simulated with the generator described in Ref. [1] for $\tilde{t} \to c\chi$, $b \to b\chi$, $q \to q\chi$ and $t \to b\tilde{b}$. A modified version of this generator was designed to simulate the channel $\tilde{t} \to b\chi f\bar{f}$, where the final state is modelled according to phase space and including parton shower development. The generation of $\tilde{t} \to c/\nu\chi$ with lifetime follows the procedure described in Ref. [4].

To simulate the relevant Standard Model background processes, several Monte Carlo generators were used: BHWISE [15] for Bhabha scattering, KORALZ [16] for $\mu^+\mu^-$ and $\tau^+\tau^-$ production, PHOT02 [17] for $\gamma\gamma$ interactions, KORALW [18] for WW production, and PYTHIA [19] for the other processes ($e^+e^- \to g\bar{g}(\gamma)$, Wev, Zee, ZZ, Zv$\nu$). The sizes of the simulated samples typically correspond to ten times the integrated luminosity of the data.

All background and signal samples were processed through the full detector simulation.

3. Event selections

Several selection algorithms have been developed to search for the topologies given in Table 2. All these channels are characterised by missing energy. The event properties depend significantly on $\Delta M$, the mass difference between the decaying squark and the $\chi$ (or the $\tilde{t}$ in the case of $\tilde{t} \to b\tilde{b}$). When $\Delta M$ is large, there is a substantial amount of visible energy, and the signal events tend to look like WW, Wev, ZZ, and $q\bar{q}(\gamma)$ events. When $\Delta M$ is small, the visible energy is small, and the signal events are therefore similar to $\gamma\gamma$ events. In order to cope with the different signal topologies and background situations, each analysis employs selections dependent on the $\Delta M$ range. The stop lifetime may become sizeable at small $\Delta M$, in which case the signal final state topology depends strongly on the $\tilde{t}$ decay length $\lambda_{\tilde{t}}$; three different selections are used, each designed to cope with a specific $\lambda_{\tilde{t}}$ range [4].

The optimisation of the selection criteria as well as the best combination of selections as a function of $\Delta M$ and $\lambda_{\tilde{t}}$ were obtained according to the $N_{\text{ch}}$ prescription [20], i.e., by minimisation of the 95% C.L. cross section upper limit expected in the absence of a signal. The selections are mostly independent of the centre-of-mass energy except for an appropriate rescaling of the cuts with $\sqrt{s}$ when relevant. The selections applied to the year 2000 data follow closely those described in Refs. [1–5] except for the new analysis developed to address the $\tilde{t} \to b\chi f\bar{f}$ decay, hereafter described in some detail.

3.1. Search for $\tilde{t} \to b\chi f\bar{f}$

The MJ analysis consists of a small, a large and a very large $\Delta M$ selection. These selections are designed to address simultaneously all $b\chi q\bar{q}'$ and $b\chi \ell\nu$ final states, independently of the decay branching ratios. The selections use several anti-$\gamma\gamma$ criteria, reported in Table 3. The cuts are derived from the AJ selection, described in Ref. [1] as well as the variables used. Only the relevant differences are discussed in the following.

In the $\tilde{t} \to b\chi f\bar{f}$ channel, the $b$ quark in the final state produces a visible mass higher than in the $\tilde{t} \to c\chi$ channel. Therefore, for the small $\Delta M$ selection, the cut on the number of charged particle tracks $N_{\text{ch}}$ is reinforced by requiring $N_{\text{ch}} > 10$, and both the visible mass, $M_{\text{vis}}$, and the visible mass computed excluding the leading lepton, $M_{\text{vis}}^{\ell_{\text{vis}}}$, are required to be greater than 10 GeV/$c^2$. These tighter cuts allow others to be loosened: the transverse momentum $p_t$ and that calculated excluding the neutral hadrons, $p_{\text{ex}}^{\text{NH}}$, must be greater than 0.005, $\sqrt{s}$ and 0.01, $\sqrt{s}$, respectively. The remaining background is reduced in the small $\Delta M$ selection by requiring the thrust to be smaller than 0.875, and by the cut $E_{\text{vis}} < 0.26\sqrt{s}$.

For the large $\Delta M$ selections, the multi-jet signature is addressed by requiring $\gamma s$, as calculated with the DURHAM algorithm [21], to be greater than 0.001. The level of the WW, ZZ and Wev background is reduced by taking advantage of the b-quark content in the $\tilde{t} \to b\chi f\bar{f}$ final state. The value of $-\log_{10} P_{\text{uds}}$ is required to be greater than 0.5, where $P_{\text{uds}}$ is the b-tag event probability introduced in Ref. [14]. This background is further suppressed
by a missing mass cut, the location of which is a function of the $\Delta M$ of the signal considered. For example, for $\Delta M = 20, 30$ and $40 \text{ GeV}/c^2$ the optimal cuts are $M_{\text{miss}}/\sqrt{s} > 0.75, 0.70$ and $0.65$, respectively.

The region where the very large $\Delta M$ selection applies is characterised by a higher visible mass. The sliding cut on the missing mass is looser than that in the large $\Delta M$ selection. For $\Delta M = 20, 40$ and $60 \text{ GeV}/c^2$ the optimal cuts are $M_{\text{miss}}/\sqrt{s} > 0.58, 0.34$ and $0.10$. Other cuts are then necessary to reduce the background mainly due to WW events. Similarly to the large $\Delta M$ case, $-\log_{10} P_{\text{vis}}$ and $y_{35}$ are required to be greater than $0.5$ and $0.003$, respectively. The mean momentum of all reconstructed charged particle tracks must be less than $0.007\sqrt{s}$. To reduce background from semileptonic W decays, the fraction of visible energy due to charged objects excluding the leading lepton is required to be greater than $0.5$, and the leading lepton, if present, must not be isolated, i.e., the additional energy deposited in a $30^\circ$ cone around its direction must be at least 50% of its energy. At this level, the remaining background consists of WW events with energy lost in the beam pipe, responsible for the missing mass. These events are rejected by requiring $|p_{z}| < 0.1\sqrt{s}$ and the energy $E_{12}$ deposited at polar angle smaller than $12^\circ$ to be less than $0.015\sqrt{s}$.

The efficiencies of the three selections were parametrised as a function of $\Delta M$ for each stop pair final state that may result from the decay channels considered ($\tilde{t} \rightarrow c\chi, \tilde{t} \rightarrow b\chi \ell\nu, \tilde{t} \rightarrow b\chi q\bar{q}^\prime$). This allows the signal efficiency to be parametrised as a function of the branching ratios. The efficiencies were checked to be practically independent of the lepton flavour (e, $\mu$, $\tau$) in the $\tilde{t} \rightarrow b\chi \ell\nu$ decay. The small, large and very large $\Delta M$ selections are combined using the $N_{95}$ procedure as a function of $\Delta M$ and of the branching ratios. The background to the small $\Delta M$ selection is dominated by $\gamma\gamma \rightarrow q\bar{q}$ events and has a total expectation of 5.0 fb, while the backgrounds of the large and very large $\Delta M$ selections, dominated by WW and other four-fermion processes amount to 3.5 and 4.4 fb, respectively.
3.2. Systematic uncertainties

The efficiencies of the MJ analysis may be affected by uncertainties regarding the assumptions on the stop hadron physics and by uncertainties related to the detector response. The results of the systematic studies are summarised in Table 4 for the three selections.

The systematic effects from the assumptions on the stop hadron physics were assessed by varying the parameters of the model implemented in the generator as in Ref. [1]. The uncertainties from the stop hadron mass were evaluated by varying the effective spectator mass \( M_{\text{eff}} \) set to 0.5 GeV/c\(^2\) in the analysis, in the range between 0.3 and 1.0 GeV/c\(^2\). The efficiencies of the large and very large \( \Delta M \) selections are almost insensitive to this change. The 9% effect found for the small \( \Delta M \) selection reflects the variation in the invariant mass available for the hadronic system.

The systematic error due to the uncertainty on the stop fragmentation was evaluated by varying \( \epsilon_{\tilde{t}} \) by an order of magnitude, where \( \epsilon_{\tilde{t}} \) is the parameter of the Peterson fragmentation function [22]. The effect on the efficiency is very small (\( \sim 2\% \)).

The amount of initial state radiation in stop pair production depends on the value of the stop coupling to the Z boson, which is controlled by the stop mixing angle. A variation of \( \theta_{\tilde{t}} \) from 56\(^\circ\) to 0\(^\circ\), i.e., from minimal to maximal coupling, was applied. The effect was found to be small in all selections, at the level of 1 to 3%.

Detector effects have been studied for the variables used in the selections. The distributions of all relevant variables show good agreement with the simulation. In particular, the b-tagging performance was checked on hadronic events collected at the Z resonance. The systematic errors associated to detector effects and to the reconstruction procedure were found to be negligible.

Beam-related background, not included in the event simulation, may affect the \( E_{12} \) variable. Its effect on the selection efficiency was determined from data collected at random beam crossings. The net effect is a relative decrease of the signal efficiency by about 5%. The uncertainty on this correction is negligible.

Finally, an additional uncertainty of 3% due to the limited Monte Carlo statistics was added. The total systematic uncertainty is at the level of 10% for the small \( \Delta M \) selection. It is dominated by the limited knowledge of the stop hadron physics, and results from rather extreme changes in the model parameters. The systematic uncertainties for the large and very large \( \Delta M \) selections are at the level of 4–5%.

The systematic uncertainties in the selections other than for the \( \tilde{t} \to b \chi f \bar{f}' \) channel are essentially identical to those reported in Refs. [4,5].

4. Results and interpretation

The numbers of candidate events selected and background events expected are reported in Table 5 for all the data samples used to derive the results below. An overall agreement is observed. In particular, a total of six candidate events is selected by the new MJ analysis, with 8.5 events expected from background processes; two events are found by each of the selections, in agreement with predictions of 3.3, 2.3 and 2.9 background events at small, large and very large \( \Delta M \), respectively.

In the framework of the supersymmetric extension of the Standard Model [9], the outcome of these searches can be translated into constraints in the

<table>
<thead>
<tr>
<th>Table 4</th>
<th>Summary of the relative systematic uncertainties (%) on the efficiencies of the MJ analysis</th>
</tr>
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<tbody>
<tr>
<td>MJ selections</td>
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<tr>
<td>( M_{\text{eff}} ) (0.3–1.0 GeV)</td>
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<tr>
<td>( \epsilon_{\tilde{t}} ) (10(^{-5})–10(^{-4}))</td>
<td>2</td>
</tr>
<tr>
<td>( \theta_{\tilde{t}} ) (0(^\circ)–56(^\circ))</td>
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</tr>
<tr>
<td>Detector and reconstruction</td>
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</tr>
<tr>
<td>Monte Carlo statistics</td>
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</tr>
<tr>
<td>Total</td>
<td>10</td>
</tr>
</tbody>
</table>
space of the relevant parameters. In this process the systematic uncertainties on the selection efficiencies were included according to the method described in Ref. [23], and no background subtraction was applied. The constraints discussed below, derived from the Ref. [23], and no background subtraction was applied. 

The very small \( \tilde{M} \) lower limit on \( \sqrt{T} \) shown in Fig. 2(a) for two values of \( \beta \) and \( \theta \), corresponding to the maximal and \( \lambda \) values of \( \tilde{t} \). For any \( \tan \beta \), the stop mass limit is 65 GeV/c^2, reached for \( \tan \beta \sim 2.7 \).

Under the hypothesis that the decay \( \tilde{t} \rightarrow b \chi f^0 \) is dominant, the regions excluded in the plane \( (M_f, M_\chi) \) are shown in Fig. 3(a), for relative proportions of the possible \( f^0 \) final states as in \( W^* \) decays. In Fig. 3(b) the leptonic modes \( b \ell \nu \) (with equal branching ratios for \( \ell = e, \mu \) and \( \tau \)) are assumed to be dominant. The excluded regions are given for \( \theta_1 = 0^\circ \) and \( \theta_1 = 56^\circ \). For \( \Delta M > 8 \) GeV/c^2, the \( \theta_1 \) independent lower limits on \( M_f \) are 78 GeV/c^2 and 80 GeV/c^2, for the two cases of \( W^* \) and leptonic final state dominance, respectively.

The combination of the AJ and MJ analyses allows constraints to be set under the more general hypothesis that both the \( \tilde{t} \rightarrow c \chi \) and \( \tilde{t} \rightarrow b \chi f^0 \) decay channels contribute to stop decays. The excluded regions in the plane \( (M_f, M_\chi) \) are shown in Fig. 4(a) for \( \theta_1 = 0^\circ \) and \( \theta_1 = 56^\circ \). This result was obtained by arbitrarily varying the \( \tilde{t} \rightarrow c \chi \) branching ratio and the leptonic frac-

---

**Table 5**

Numbers of candidate events observed \( (N_{\text{obs}}) \) and expected from background \( (N_{\text{exp}}) \) for the different selections. Also given are the sizes \( (\mathcal{L} dt) \) and the average centre-of-mass energies \( (\sqrt{T}) \) of the samples analysed.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Year</th>
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<td>( \mathcal{L} dt ) (pb^{-1})</td>
<td>( \mathcal{L} dt ) (pb^{-1})</td>
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<td></td>
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<tr>
<td></td>
<td>Large ( \lambda_t )</td>
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<td>0.4</td>
<td>1</td>
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<td>0.7</td>
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<td>0</td>
<td>0.6</td>
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<td>0</td>
<td>0.8</td>
</tr>
<tr>
<td>AJ Plus leptons</td>
<td>Small ( \Delta M )</td>
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<td>0.8</td>
<td>0</td>
<td>1.9</td>
</tr>
<tr>
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<td>Large ( \Delta M )</td>
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<td>0.1</td>
<td>2</td>
<td>0.4</td>
</tr>
<tr>
<td>AJ plus b-tagging</td>
<td>Small ( \Delta M )</td>
<td>0</td>
<td>1.1</td>
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<tr>
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<td>1</td>
<td>0.6</td>
<td>0</td>
<td>0.9</td>
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Fig. 2. (a) Excluded regions at 95% C.L. in the $M_{\tilde{\chi}}$ vs. $M_{\tilde{t}}$ plane from $\tilde{t}\rightarrow c/\mu\chi$ searches; the excluded regions are given for $\theta_{\tilde{t}} = 0^\circ$, corresponding to maximum $\tilde{t}Z$ coupling, and for $\theta_{\tilde{t}} = 56^\circ$, corresponding to vanishing $\tilde{t}Z$ coupling. The dark region in the small $\Delta M$ corridor is excluded by the “long-lived hadrons” analysis. The CDF experiment result is also indicated. (b) Stop mass lower limit at 95% C.L. from the “long-lived hadrons” as a function of $\log(c\tau_{\tilde{t}}/\text{cm})$ for several $\Delta M$ values, without any assumption on the relation between $\Delta M$ and the stop lifetime $\tau_{\tilde{t}}$. (c) Stop mass lower limit at 95% C.L. from the “long-lived hadrons” analysis as a function of $\tan \beta$, independent of the other MSSM parameters.

The exclusion in the $\tilde{t}\rightarrow b\chi f\bar{f}'$ decay, and by using the $\nabla_{95}$ prescription to determine the appropriate combination of selections. The stop mass limit is shown in Fig. 4(b) as a function of the branching ratio $BR(\tilde{t}\rightarrow c\chi)$ for several fixed $\Delta M$ values and for $\theta_{\tilde{t}} = 56^\circ$. The smallest $\Delta M$ value considered is 5 GeV/$c^2$, corresponding to the threshold for the production of a $b$ quark in the final state. The lowest limit obtained is 63 GeV/$c^2$; it is reached for $\Delta M = 5$ GeV/$c^2$, $BR(\tilde{t}\rightarrow c\chi) = 0.22$, and $BR(\tilde{t}\rightarrow b\chi \ell\nu) = 0.55$.

Under the assumption that the $\tilde{t}\rightarrow b\ell\bar{\nu}$ decay mode is dominant, with equal branching ratios for $\ell = e, \mu$ and $\tau$, the excluded region in the plane $(M_{\tilde{t}}, M_{\tilde{\nu}})$ is shown in Fig. 5(a). If $\Delta M > 8$ GeV/$c^2$, and using the LEP1 limit on the sneutrino mass and D0 results [25], the lower limit on $M_{\tilde{t}}$ is 97 GeV/$c^2$, independent of $\theta_{\tilde{t}}$.
Fig. 3. Excluded regions at 95% C.L. in the $M_{\chi}$ vs. $M_{\tilde{t}}$ plane from $\tilde{t} \rightarrow b \chi f \bar{f}'$ searches: (a) the $W^*$ modes or (b) the leptonic modes are assumed to be dominant for the $\tilde{f}'$ final states. The excluded regions are given for $\Theta_{\tilde{t}} = 0^\circ$, corresponding to maximum $\tilde{t} \tilde{t} Z$ coupling, and for $\Theta_{\tilde{t}} = 56^\circ$, corresponding to vanishing $\tilde{t} \tilde{t} Z$ coupling.

Fig. 4. (a) Branching ratio independent excluded regions at 95% C.L. in the $M_{\chi}$ vs. $M_{\tilde{t}}$ plane, from $\tilde{t} \rightarrow b \chi f \bar{f}'$ and $\tilde{t} \rightarrow c \chi$ searches. The excluded regions are given for $\Theta_{\tilde{t}} = 0^\circ$, corresponding to maximum $\tilde{t} \tilde{t} Z$ coupling, and for $\Theta_{\tilde{t}} = 56^\circ$, corresponding to vanishing $\tilde{t} \tilde{t} Z$ coupling. (b) Limit on the stop mass at 95% C.L. as a function of $\text{BR}(\tilde{t} \rightarrow c \chi)$ for various $\Delta M$ values. The limits are given for $\Theta_{\tilde{t}} = 56^\circ$. 
Fig. 5. (a) Excluded regions at 95% C.L. in the $M_{\tilde{t}}$ vs. $M_{\tilde{t}}$ plane from $\tilde{t} \to b \tilde{\nu}$ searches (equal branching fractions for the $\tilde{t}$ decay to $e$, $\mu$, and $\tau$ are assumed). The excluded regions are given for $\theta_{\tilde{t}} = 0^\circ$, corresponding to maximum $\tilde{t}\tilde{t}Z$ coupling, and for $\theta_{\tilde{t}} = 56^\circ$, corresponding to vanishing $\tilde{t}\tilde{t}Z$ coupling. The regions excluded at LEP 1 and by the D0 experiment are also indicated. (b) Excluded regions at 95% C.L. in the $M_{\chi}$ vs. $M_{\tilde{b}}$ plane from $\tilde{b} \to b \chi$ searches. The excluded regions are given for $\theta_{\tilde{b}} = 0^\circ$, corresponding to maximum $\tilde{b}\tilde{b}Z$ coupling, and for $\theta_{\tilde{b}} = 68^\circ$, corresponding to vanishing $\tilde{b}\tilde{b}Z$ coupling. The region excluded by the CDF experiment is also indicated.

The lower limit is 82 GeV/$c^2$ if the $\tilde{t} \to b \tilde{\nu}$ decay mode is dominant and $\Delta M > 8$ GeV/$c^2$, independent of $\theta_{\tilde{t}}$.

The excluded region in the plane ($M_{\tilde{b}}$, $M_{\chi}$) is shown in Fig. 5(b) under the assumption of a dominant $\tilde{b} \to b \chi$ decay. Taking also the CDF exclusion [24] into account, a lower limit of 89 GeV/$c^2$ is set on $M_{\tilde{b}}$, for any $\tilde{b}$ mixing angle and $\Delta M > 8$ GeV/$c^2$. The region excluded for $\theta_{\tilde{b}} = 0^\circ$, for which the $\tilde{b}\tilde{b}Z$ coupling is maximal, is also shown.

As discussed in detail in Ref. [2], the results of the search for acoplanar jets, with or without $b$-tagging, can also be translated into constraints on the mass of degenerate squarks. In order to compare these results with those obtained at the Tevatron [24,25], limits have been evaluated within the MSSM [9] under the following assumptions: a degenerate mass $M_{\tilde{q}}$ for all left-handed and right-handed $\tilde{u}$, $\tilde{d}$, $\tilde{c}$, $\tilde{s}$, $\tilde{b}$ squarks; lowest order GUT relation between the soft supersymmetry breaking gaugino mass terms, allowing the gluino and neutralino masses to be related; $\tan \beta = 4$ and $\mu = -400$ GeV. The results in the plane ($M_{\tilde{g}}$, $M_{\tilde{q}}$) are shown in Fig. 6. Improved constraints are obtained in the region of small $\tilde{q}$ to $\chi$ mass differences.

Fig. 6. Excluded regions at 95% C.L. from the search for generic $\tilde{q}$ pairs, assuming five mass-degenerate $\tilde{q}$ flavours. The results are shown in the gluino–squark mass plane for $\tan \beta = 4$ and $\mu = -400$ GeV, together with results from experiments at pp colliders.
Table 6

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<th>95% C.L. Mass limit (GeV/c²)</th>
<th>ΔM range (GeV/c²)</th>
<th>Dominant decay channel(s)</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>˜t²</td>
<td>92</td>
<td>&gt; 8</td>
<td>˜t → cχ</td>
<td>CDF result [24] used</td>
</tr>
<tr>
<td></td>
<td>78</td>
<td>&gt; 8</td>
<td>˜t → bνW⁺</td>
<td></td>
</tr>
<tr>
<td>97</td>
<td>&gt; 8</td>
<td>˜t → bν</td>
<td>LEP 1, D0 result [25] used</td>
<td></td>
</tr>
<tr>
<td>63</td>
<td>Any</td>
<td>Any</td>
<td>Any branching ratios, any lifetime</td>
<td></td>
</tr>
<tr>
<td>˜b⁸</td>
<td>89</td>
<td>&gt; 8</td>
<td>˜b → bχ</td>
<td>CDF result [24] used</td>
</tr>
</tbody>
</table>

5. Conclusions

Searches for signals of pair-produced scalar partners of quarks have been performed in the data sample of 207 pb⁻¹ collected in the year 2000 with the ALEPH detector at LEP, at centre-of-mass energies ranging from 204 to 209 GeV. The final state topologies studied arise from the decays ˜t → c/ufχ, ˜t → b/ufχ, ˜b → bχ, and ˜q → qχ. The four-body stop decay channel was analysed for the first time at LEP, and the corresponding selections were extended to the 675 pb⁻¹ of data collected by ALEPH at centre-of-mass energies of 183 GeV and above. All numbers of candidate events observed are consistent with the backgrounds expected from Standard Model processes. The results of these searches, combined with earlier ones obtained with data collected from 1997 to 1999, have been translated into improved mass lower limits, of which relevant examples are given in Table 6. In particular, a 95% C.L. lower limit of 63 GeV/c² has been set on the stop mass, irrespective of its lifetime and decay branching ratios.

Acknowledgements

We wish to congratulate our colleagues from the accelerator divisions for the successful operation of LEP at high energies. We would also like to express our gratitude to the engineers and support people at our home institutes without whom this work would not have been possible. Those of us from non-member states wish to thank CERN for its hospitality and support.

References

Study of the decay $\phi \rightarrow \pi^0 \pi^0 \gamma$ with the KLOE detector

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Abstract

We have measured the branching ratio $\text{BR}(\phi \to \pi^0\pi^0\gamma)$ with the KLOE detector using a sample of $\sim 5 \times 10^7$ $\phi$ decays. $\phi$ mesons are produced at DAΦNE, the Frascati $\phi$-factory. We find $\text{BR}(\phi \to \pi^0\pi^0\gamma) = (1.09 \pm 0.03_{\text{stat}} \pm 0.05_{\text{syst}}) \times 10^{-4}$. We fit the two-pion mass spectrum to models to disentangle contributions from various sources. © 2002 Elsevier Science B.V. All rights reserved.

PACS: 13.65.+i; 14.40.-n

Keywords: $e^+e^-$ collisions; $\phi$ radiative decays; Scalar mesons

The decay $\phi \to \pi^0\pi^0\gamma$ was first observed in 1998 [1]. Only two experiments have measured its rate [2,3]. The measured rate is too large if $\phi \to f_0(980)\gamma$, with $f_0 \to \pi^0\pi^0$, were the dominating contribution and $f_0(980)$ is interpreted as a $q\bar{q}$ scalar state [4,5]. Possible explanations for the $f_0$ are: ordinary $q\bar{q}$ meson, $q\bar{q}q\bar{q}$ state, $K\bar{K}$ molecule [4,6–8]. Similar considerations apply also to the $a_0(980)$ meson. The decay $\phi \to \pi^0\pi^0\gamma$ can clarify this situation since both the branching ratio and the line shape depend on the structure of the $f_0$. We present in the following a study of the decay $\phi \to \pi^0\pi^0\gamma$ performed with the KLOE detector [9] at DAΦNE [10], an $e^+e^-$ collider which operates at a center of mass energy $W = M_\phi \sim 1020$ MeV. Data were collected in the year 2000 for an integrated luminosity $L_{\text{int}} \sim 16$ pb$^{-1}$, corresponding to around $5 \times 10^7$ $\phi$-meson decays.

The KLOE detector consists of a large cylindrical drift chamber, DC, surrounded by a lead-scintillating fiber electromagnetic calorimeter, EMC. A superconducting coil around the EMC provides a 0.52 T field. The drift chamber [11], 4 m in diameter and 3.3 m long, has 12 582 all-stereo tungsten sense wires and 37 746 aluminum field wires. The chamber shell is made of carbon fiber-epoxy composite and the gas used is a 90% helium, 10% isobutane mixture. These features maximize transparency to photons and reduce $K_L \to K_S$ regeneration and multiple scattering. The position resolutions are $\sigma_x, \sigma_y \sim 150$ $\mu$m and $\sigma_z \sim 2$ mm. The momentum resolution is $\sigma(p)/p \approx 0.4\%$. Vertices are reconstructed with a spatial resolution of $\sim 3$ mm. The calorimeter [12] is divided into a barrel and two endcaps, for a total of 88 modules, and covers 98% of the solid angle. The modules are read out at both ends by photomultipliers; the readout granularity is $4.4 \times 4.4$ cm$^2$, for a total of 2440 cells. The arrival times of particles and the positions in three dimensions of the energy deposits are obtained from the signals collected at the two ends. Cells close in time and space are grouped into a calorimeter cluster. The cluster energy $E$ is the sum of the cell energies. The cluster time $T$ and position $R$ are energy weighted averages. Energy and time resolutions are $\sigma_E/E = 5.7\%/\sqrt{E}$ (GeV) and $\sigma_T = (57 \text{ ps})/\sqrt{E}$ (GeV) + (50 ps), respectively. The KLOE trigger [13] uses calorimeter and chamber information. For this analysis only the calorimeter signals are relevant. Two energy deposits with $E > 50$ MeV for the barrel and $E > 150$ MeV for the endcaps are required.

Prompt photons are identified as neutral particles with $\beta = 1$ originated at the interaction point requiring $|T - R/c| < \min(5\sigma_T, 2 \text{ ns})$, where $T$ is the photon flight time and $R$ the path length; $\sigma_T$ includes also the contribution of the bunch length jitter. The photon detection efficiency is $\sim 90\%$ for $E_\gamma = 20$ MeV, and reaches 100% above 70 MeV. The sample selected...
by the timing requirement contains a $< 1.8\%$ contamination due to accidental clusters from machine background.

Two amplitudes contribute to $\phi \to \pi^0\pi^0\gamma$: $\phi \to S\gamma$, $S \to \pi^0\pi^0$ ($S\gamma$) and $\phi \to \rho^0\pi^0$, $\rho^0 \to \pi^0\gamma$ ($\rho\pi\gamma$) where $S$ is a scalar meson. The event selection criteria of the $\phi \to \pi^0\pi^0\gamma$ decays ($\pi\pi\gamma$) have been designed to give similar efficiencies for both processes. The first step, requiring five prompt photons with $E_\gamma \geq 7$ MeV and $\theta \geq \theta_{\text{min}} = 23^{\circ}$, reduces the sample to 124 575 events. The background due to $\phi \to K\bar{K}L$ is removed requiring that $E_{\text{tot}} = \sum E_{\gamma,i}$ and $p_{\text{tot}} = \sum p_{\gamma,i}$ satisfy $E_{\text{tot}} > 800$ MeV and $|p_{\text{tot}}| < 200$ MeV/$c$. We are left with 15 825 events. Other reactions which give rise to background are: $e^+ e^- \to \omega\pi^0 \to \pi^0\pi^0\gamma$ ($\omega\pi\gamma$), $\phi \to \eta\pi^0\gamma \to S\gamma$ ($\eta\pi\gamma$) and $\phi \to \eta\gamma \to 3\pi^0\gamma$ ($\eta\gamma\gamma$) with 2 undetected photons.

A kinematic fit (Fit1) requiring overall energy and momentum conservation improves the energy resolution to 3%. Photons are assigned to $\pi^0$'s by minimizing a test $\chi^2$-function for both the $\pi\pi\gamma$ and $\omega\pi\gamma$ cases. For the $\omega\pi\gamma$ case we also require $M_{\pi\gamma}$ to be consistent with $M_\omega$. The correct combination is found by this procedure 89%, 96% of the time for the $\pi\pi\gamma$, $\omega\pi\gamma$ case, respectively. Good agreement is found with the Monte Carlo simulation, MC, for the distributions of the $\chi^2$ and of the invariant masses. A second fit (Fit2) requires the mass of $\gamma\gamma$ pairs to equal $M_\pi$.

The $e^+ e^- \to \omega\pi^0 \to \pi^0\pi^0\gamma$ background is reduced rejecting the events satisfying $\chi^2/\text{ndf} \leq 3$ and $\Delta M_{\pi\gamma} = |M_{\pi\gamma} - M_\omega| \leq 3\sigma_\omega$ using Fit2 in the $\omega\pi\gamma$ hypothesis. Data and MC are in good agreement (Fig. 1). The $\phi \to \pi^0\pi^0\gamma$ events must then satisfy $\chi^2/\text{ndf} \leq 3$ for Fit2 in the $\pi\pi\gamma$ hypothesis. We also require $\Delta M_{\pi\gamma} = |M_{\pi\gamma} - M_\pi| \leq 5\sigma_\pi$ using the photon momenta of Fit1. The efficiency for the identification of the signal is evaluated applying the whole analysis chain to a sample of simulated $\phi \to S\gamma$, $S \to \pi^0\pi^0$ events with a $\pi^0\pi^0$ mass ($m$) spectrum consistent with the data. We use the symbol $M_{\pi\pi}$ to denote the reconstructed value of $m$. The selection efficiency as a function of $M_{\pi\pi}$ is shown in Fig. 2. The average over the whole mass spectrum is $\epsilon_{\pi\pi\gamma} = 41.6\%$. A similar efficiency function is obtained for the process $\phi \to \rho^0\pi^0$ with $\rho^0 \to \pi^0\gamma$. Fig. 3 shows various distributions for the 3102 events surviving the selection together with MC predictions. The angular distributions prove that $S\gamma$ is the dominant process. The rejection factors

![Image 310x487 to 525x704](image)

Fig. 1. Data–MC comparison for $\omega\pi$ events: (a) $\chi^2/\text{ndf}$ and (b) $\Delta M_{\pi\gamma}/\sigma_\omega$.

![Image 424x182 to 460x182](image)

Fig. 2. Efficiency vs. $\pi^0\pi^0$ invariant mass for $\phi \to \pi^0\pi^0\gamma$ events.

and the expected number of events for the background processes are listed in Table 1 [14–16]. After subtracting the background $2438 \pm 61$ $\phi \to \pi^0\pi^0\gamma$ events remain. Their $M_{\pi\pi}$ spectrum is shown in Fig. 4.

The $\phi \to \pi^0\pi^0\gamma$ branching ratio, BR, is obtained normalizing the number of events after background subtraction, $N - B$, to the $\phi$ cross section, $\sigma(\phi)$, to the selection efficiency and to $L_{\text{int}}$:

$$\text{BR}(\phi \to \pi^0\pi^0\gamma) = \frac{N - B}{\epsilon_{\pi\pi\gamma} \sigma(\phi) L_{\text{int}}}$$

The luminosity is measured using large angle Bhabha scattering events. The measurement of $\sigma(\phi)$ is obtained from the $\phi \to \eta\gamma \to \gamma\gamma\gamma$ decay in the same


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Fig. 3. Data–MC comparison for \( \phi \rightarrow \pi^0\pi^0\gamma \) events after \( \omega \pi \) rejection: (a) \( \chi^2/\text{ndf} \); (b) \( (M_{\gamma\gamma} - M_{\pi})/\sigma_{\pi} \) with \( \chi^2/\text{ndf} \leq 3 \); (c), (d) angular distributions with all analysis cuts applied. \( \theta \) is the polar angle of the radiative photon, \( \psi \) is the angle between the radiative photon and \( \pi^0 \) in the \( \pi^0\pi^0 \) rest frame.

Table 1
Background channels for \( \phi \rightarrow \pi^0\pi^0\gamma \)

<table>
<thead>
<tr>
<th>Process Rejection factor</th>
<th>Expected events</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^+e^- \rightarrow \omega\pi^0 \rightarrow \pi^0\pi^0\gamma )</td>
<td>8.7</td>
</tr>
<tr>
<td>( \phi \rightarrow \eta\pi^0 \gamma \rightarrow \gamma\gamma\pi^0\gamma )</td>
<td>4.0</td>
</tr>
<tr>
<td>( \phi \rightarrow \eta\gamma \rightarrow \pi^0\pi^0\pi^0\gamma )</td>
<td>5.9 ( \times ) 10^3</td>
</tr>
</tbody>
</table>

is [17]:

\[
\text{BR}(\phi \rightarrow \pi^0\pi^0\gamma) = (1.08 \pm 0.03_{\text{stat}} \pm 0.03_{\text{syst}} \pm 0.04_{\text{norm}}) \times 10^{-4}.
\]

The contributions to the uncertainties are listed in Table 2. Details can be found in Ref. [16].

In order to disentangle the contributions of the various processes and to determine the normalized differential decay rate, \( \text{dBR}/\text{dm} = (1/\Gamma)\text{d}\Gamma/\text{dm} \), we fit the data to a mass spectrum \( f(m) \). This spectrum is taken as the sum of \( S \gamma, \rho \pi \) and interference term, \( f(m) = f_{S\gamma}(m) + f_{\rho\pi}(m) + f_{\text{int}}(m) \). The scalar term sample [15]. We obtain

\[
\text{BR}(\phi \rightarrow \pi^0\pi^0\gamma) = (1.08 \pm 0.03_{\text{stat}} \pm 0.03_{\text{syst}} \pm 0.04_{\text{norm}}) \times 10^{-4}.
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\[
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\[
\text{BR}(\phi \rightarrow \pi^0\pi^0\gamma) = (1.08 \pm 0.03_{\text{stat}} \pm 0.03_{\text{syst}} \pm 0.04_{\text{norm}}) \times 10^{-4}.
\]
this meson, its decay rate is given by [19]:
\[
\Gamma_{\phi\gamma}(m) = \frac{e^2 g_{\phi\gamma}^2}{12\pi m^2} \left( \frac{m^2 - m_\phi^2}{2M_\phi} \right)^3.
\]
(5)

where \( g_{\phi\gamma} \) is a point-like \( \phi \gamma \) coupling.

Finally, \( \Gamma_{S\pi\gamma} \) is related to \( \Gamma_{S\pi^+\pi^-} \) by:
\[
\Gamma_{S\pi\gamma}(m) = \frac{1}{2} \Gamma_{S\pi^+\pi^-}(m) = \frac{g_{S\pi\gamma}}{32\pi m} \sqrt{1 - \frac{4m^2}{m_\phi^2}}.
\]
(6)

For the inverse propagator, \( D_S \), we use the formula with finite width corrections [17] for the \( f_0 \) and a Breit–Wigner for the \( \sigma \). The parametrization of Ref. [20] has been used for the \( \rho \pi \) and the interference term.

The observed mass spectrum \( S_{\text{obs}}(M_{\pi\pi}) \) is fit folding into the theoretical shape experimental efficiency and resolution after proper normalization for \( \sigma(\phi) \) and \( L_{\text{int}} \). Two different fits have been performed varying \( f_{S\gamma}(m) \): in Fit (A) only the \( f_0 \) contribution is considered while in Fit (B) we also include the contribution of the \( \sigma \) meson. The mass and width of the \( \sigma \) were fixed to their central values. If the normalization of the \( \rho \pi \) term is left free during fitting, its contribution and the associated interference terms turn out to be negligibly small. When \( BR(\phi \rightarrow \rho^0\pi^0 \rightarrow \pi^0\pi^0\gamma) \) is fixed at \( 1.8 \times 10^{-5} \) as in Ref. [20], the \( \chi^2/\text{ndf} \) increases by more than a factor of 2. The fits without the \( \rho \pi \) contribution are shown superimposed over the raw spectrum in Fig. 4(b).

The result of the fits are listed in Table 3. In Fit (A) we use as free parameters \( M_{f_0}, g_{f_0K^+K^-}^2 \) and the ratio \( g_{f_0K^+K^-}^2/g_{f_0\pi^+\pi^-}^2 \). The fit gives a large \( \chi^2/\text{ndf}; integrated the theoretical spectrum a value \( BR(\phi \rightarrow f_0\gamma \rightarrow \pi^0\pi^0\gamma) = (1.11 \pm 0.06_{\text{stat+sys}}) \times 10^{-4} \) is obtained.

A much better agreement with data is given by Fit (B), where we add as a free parameter also the coupling \( g_{\phi\gamma} \). The negative interference between the \( f_0 \) and \( \sigma \) amplitudes results in the observed decrease of the \( \pi^0\pi^0\gamma \) yield below 700 MeV. Integrating over the theoretical \( \sigma \) and \( f_0 \) curves we obtain \( BR(\phi \rightarrow \sigma \gamma \rightarrow \pi^0\pi^0\gamma) = (0.28 \pm 0.04_{\text{stat+sys}}) \times 10^{-4} \) and \( BR(\phi \rightarrow f_0\gamma \rightarrow \pi^0\pi^0\gamma) = (1.49 \pm 0.07_{\text{stat+sys}}) \times 10^{-4} \). Multiplying the latter \( BR \) by a factor of 3 to account for \( f_0 \rightarrow \pi^+\pi^- \) decay, the \( BR(\phi \rightarrow f_0\gamma) \) is determined to be
\[
BR(\phi \rightarrow f_0\gamma) = (4.47 \pm 0.21_{\text{stat+sys}}) \times 10^{-4}.
\]
(7)

The values of the coupling constants from Fit (B) are in agreement with those reported by the SND and CMD-2 experiments [2,3]. The coupling constants differ from the WA102 result on \( f_0 \) production in central \( pp \) collisions \( (g_{f_0K^+K^-}^2/3g_{f_0\pi^+\pi^-}^2 = g_K/1.33g_\pi = 1.63 \pm 0.46) \) [21] and from those obtained when the \( f_0 \) is produced in \( D^+ \rightarrow \pi^+\pi^- \) decays [22], where \( g_K \) is consistent with zero.

In order to allow a detailed comparison with other experiments and theoretical models, we have unfolded \( S_{\text{obs}}(M_{\pi\pi}) \). For each reconstructed mass bin, the ratio between the theoretical and the smeared function, \( SF(M_{\pi\pi}) \), is calculated. The d\( BR/\text{d}m \) is then given by
\[
\frac{\text{d}BR}{\text{d}m} = \frac{S_{\text{obs}}(M_{\pi\pi})}{SF(M_{\pi\pi})} \frac{1}{L_{\text{int}}\sigma(\phi)\Delta M_{\pi\pi}}.
\]
(8)

The value of d\( BR/\text{d}m \) as a function of \( m \) is given in Table 4 and shown in Fig. 5. Integrating over the whole mass range we obtain:
\[
BR(\phi \rightarrow \pi^0\pi^0\gamma) = (1.09 \pm 0.03_{\text{stat}} \pm 0.03_{\text{sys}} \pm 0.04_{\text{norm}}) \times 10^{-4}
\]
(9)

which well compares with the result obtained correcting for the average selection efficiency (Eq. (2)). If we limit the integration to the \( f_0 \) dominated region, above 700 MeV, we get:
\[
BR(\phi \rightarrow \pi^0\pi^0\gamma; m > 700 \text{ MeV}) = (0.96 \pm 0.02_{\text{stat}} \pm 0.02_{\text{sys}} \pm 0.04_{\text{norm}}) \times 10^{-4}
\]

which is in agreement with our previous measurement in the same mass range [23].

---

**Table 3**

| Fit results using \( f_0 \) only, Fit (A), and including the \( \sigma \), Fit (B) |
|-----------------|-----------------|-------------------|
| \( \chi^2/\text{ndf} \) | 109.53/34 | 43.15/33 |
| \( M_{f_0} \) (MeV) | 962 ± 4 | 973 ± 1 |
| \( g_{f_0K^+K^-}^2/(4\pi) \) (GeV²) | 1.29 ± 0.14 | 2.79 ± 0.12 |
| \( g_{f_0\pi^+\pi^-}^2 \) | 3.22 ± 0.29 | 4.00 ± 0.14 |
| \( g_{\phi\gamma} \) | - | 0.060 ± 0.008 |
dBR/\text{d}m is in units of 10^8 \text{ MeV}^{-1}. The errors listed are the total uncertainties.

<table>
<thead>
<tr>
<th>m</th>
<th>dBR/\text{d}m</th>
<th>m</th>
<th>dBR/\text{d}m</th>
</tr>
</thead>
<tbody>
<tr>
<td>290</td>
<td>2.0 ± 2.9</td>
<td>670</td>
<td>11.2 ± 1.9</td>
</tr>
<tr>
<td>310</td>
<td>2.2 ± 1.4</td>
<td>690</td>
<td>11.0 ± 1.9</td>
</tr>
<tr>
<td>330</td>
<td>3.0 ± 1.5</td>
<td>710</td>
<td>12.5 ± 1.9</td>
</tr>
<tr>
<td>350</td>
<td>0.9 ± 1.3</td>
<td>730</td>
<td>14.0 ± 2.0</td>
</tr>
<tr>
<td>370</td>
<td>2.9 ± 1.4</td>
<td>750</td>
<td>17.3 ± 2.3</td>
</tr>
<tr>
<td>390</td>
<td>2.2 ± 1.3</td>
<td>770</td>
<td>17.0 ± 2.4</td>
</tr>
<tr>
<td>410</td>
<td>1.4 ± 1.1</td>
<td>790</td>
<td>19.4 ± 2.5</td>
</tr>
<tr>
<td>430</td>
<td>1.8 ± 1.0</td>
<td>810</td>
<td>27.4 ± 3.1</td>
</tr>
<tr>
<td>450</td>
<td>1.9 ± 0.8</td>
<td>830</td>
<td>29.2 ± 3.2</td>
</tr>
<tr>
<td>470</td>
<td>1.1 ± 0.5</td>
<td>850</td>
<td>30.6 ± 3.2</td>
</tr>
<tr>
<td>490</td>
<td>0.5 ± 0.2</td>
<td>870</td>
<td>41.7 ± 3.8</td>
</tr>
<tr>
<td>510</td>
<td>0.2 ± 0.1</td>
<td>890</td>
<td>39.6 ± 3.6</td>
</tr>
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<td>530</td>
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<tr>
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<td>47.2 ± 4.3</td>
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<td>970</td>
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<td>0.2 ± 0.1</td>
</tr>
<tr>
<td>650</td>
<td>7.0 ± 1.7</td>
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</tr>
</tbody>
</table>

In a separate paper [14], we present a measurement of BR(\phi \rightarrow a_0\gamma), together with a discussion of the implications of f_0 and a_0 results.

Acknowledgements

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References


A measurement of the $K_S$ lifetime

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Abstract

A measurement of the $K_S$ lifetime is presented using data recorded by the NA48 experiment at the CERN-SPS during 1998 and 1999. The $K_S$ lifetime is derived from the ratio of decay time distributions in simultaneous, collinear $K_S$ and $K_L$ beams, giving a result which is approximately independent of the detector acceptance and with reduced systematic errors. The result obtained is $\tau_S = (0.89598 \pm 0.00048 \pm 0.00051) \times 10^{-10}$ s, where the first error is statistical and the second systematic.

1. Introduction

Precise measurements of the basic physics parameters defining the neutral kaon system, such as the masses and mean lifetimes of the $K_S$ and $K_L$ states, are important not only in their own right but also as essential inputs to many kaon physics analyses such as studies of indirect and direct CP violation or precision tests of CPT invariance. The $K_S$ lifetime is presently known with a relative precision of about 0.1% [1], dominated by measurements from the NA31 experiment at CERN [2] and from the E731 and E773 experiments at Fermilab [3,4]. Here we present a measurement of the $K_S$ lifetime from the NA48 experiment at the CERN-SPS, based on the same $K^0 \rightarrow \pi\pi$ data samples as used for the precise determination of the direct CP violation parameter $Re(\epsilon'/\epsilon)$ [5]. The $K_S$ lifetime is measured using an analysis technique introduced by...

2. The method

A defining principle of the NA48 experiment is the simultaneous recording of decays occurring within a common decay region traversed by two almost collinear beams of neutral kaons. The relative target positions for each beam, one far upstream of the decay region and the other much closer, ensure that kaon decays in the two beams are due dominantly to the K_S or to the K_L component, respectively. Assuming equal detection efficiencies for decays from the K_S and K_L beams, the ratio R = N_S/N_L of decay rates observed in each beam as a function of the longitudinal position z can be expressed as

\[ R(E_K, z) = A(E_K) \frac{f_{S,L}(t_S)}{f_{L}(t_L)}. \tag{1} \]

where \( E_K \) is the kaon energy, \( A(E_K) \) is a normalisation function which depends on the relative beam intensities and \( t_{S,L} = (z - z_{\text{beam},S,L})(m_K/p_K) \) are the proper lifetimes for kaon decays in the K_S or K_L beams. The functions \( f(t) \) are given by

\[ f_{S,L}(t) = e^{-\tau_S t} + |\eta|^2 e^{-\tau_L t}, \]

\[ + 2D_{S,L}(E_K)|\eta|e^{-\left((\tau_S + \tau_L)/2\right) t}, \]

\[ \times \cos(\Delta m \cdot t - \phi), \tag{2} \]

where \( \tau_S \) and \( \tau_L \) are the K_S and K_L mean lifetimes, \( \Delta m = m_L - m_S \) is the mass difference between the K_S and K_L states, \( |\eta| \) and \( \phi \) are the modulus and phase of the ratio \( A(K_L \rightarrow \pi\pi)/A(K_S \rightarrow \pi\pi) \) of decay amplitudes, and the dilutions \( D_{S,L}(E_K) = \frac{[N(K^0) - N(\overline{K}^0)]/[N(K^0) + N(\overline{K}^0)]} \) reflect the initial admixture of \( K^0 \) and \( \overline{K}^0 \) in each beam. Except for the K_L beam at higher energies, where the interference term contributes appreciably, the functions \( f_{S}(t_S) \) and \( f_{L}(t_L) \) are dominated by the exponential terms \( e^{-\tau_S t} \) and \( |\eta|^2 e^{-\tau_L t} \), respectively. Hence the K_S/K_L ratio is approximately of the form

\[ R \propto e^{-\tau_S t} / e^{-\tau_L t} \propto e^{-\tau(1/\tau_S - 1/\tau_L)}. \]

Since \( \tau_L \gg \tau_S \), the ratio R is primarily sensitive to the K_S lifetime \( \tau_S \). The K_S lifetime is determined by fitting a function \( R(E_K, \tau) \) of the form given in Eqs. (1) and (2) to the ratio \( N_S/N_L \) of K^0 \( \rightarrow \pi\pi \) decays reconstructed in the two beams. Besides the K_S lifetime itself, the normalisation \( A(E_K) \) and the dilutions \( D_{S,L}(E_K) \) are also taken as free parameters in the fit, while the remaining physics parameters in Eq. (2) are taken from published measurements. Small acceptance differences between K_S and K_L decays are corrected using Monte Carlo, and background to the K_L samples is subtracted using the data. The \( \pi^+\pi^- \) and \( \pi^0\pi^0 \) decay modes are analysed separately and the results subsequently combined.

The K_S/K_L ratio is reconstructed in 20 bins of energy with width \( \Delta E_K = 5 \) GeV covering the range \( 70 < E_K < 170 \) GeV and in lifetime bins of width \( \Delta \tau = 0.1 \tau_S \). The fit to determine the K_S lifetime is carried out in the lifetime range \( 0.5\tau_S < \tau < 3.5\tau_S \). The lower lifetime limit of 0.5\tau_S largely avoids detector resolution effects associated with the start of the decay region at \( \tau = 0 \) while the upper limit at 3.5\tau_S is dictated by the trigger requirements. The choice of lifetime range approximately minimises the total error on the measured K_S lifetime.

3. The NA48 experiment

The K_L and K_S beams in the NA48 experiment are derived from 450 GeV protons incident on separate

\[ \Delta \tau \]
The range 5–100 GeV. The photon time resolution is 0.8%. The energy response is linear to about 0.1% in photon showers is 1 mm and the energy resolution is energy of 25 GeV, the transverse spatial resolution for and depth 27 radiation lengths. For the average photon ∼

K_S target.

at the centre of the detector, 120 m downstream of the K_S target. The K_S target is targets positioned 126 m and 6 m upstream of the decay region, respectively, [6]. The K_S target is located 72 mm above the axis of the K_L beam. The K_S target and collimator system is aligned along an axis directed slightly downwards such that the K_S and K_L beams subtend an angle of 0.6 mrad and intersect at the centre of the detector, 120 m downstream of the K_S target.

The beginning of the decay volume is accurately defined by a K_S anticounter (AKS) located at the exit of the K_S collimator. The AKS is composed of a photon convertor followed by three scintillator counters and is used to veto all upstream decays in the K_S beam. To minimise interactions of beam particles with air and material, the decay region itself is contained in a 90 m long evacuated tank terminated by a thin composite polyamide (Kevlar) window of $3 \times 10^{-3}X_0$ thickness. The tank is followed by the NA48 detector, the principal components of which are a magnetic spectrometer for charged particle detection, a liquid krypton calorimeter for photon and electron detection, an iron-scintillator hadron calorimeter, and muon counters consisting of three planes of scintillator shielded by 80 cm thick iron walls.

The charged particle spectrometer [7] consists of two drift chambers (DCH1 and DCH2) located before, and two drift chambers (DCH3 and DCH4) located after, a central dipole magnet. Each chamber has an area of 4.5 $m^2$ and is made up of four sets of two staggered sense wire planes oriented along four directions (horizontal, vertical, ±45°). Track positions are reconstructed with a precision of 100 µm per view, while the momentum resolution is $\sigma(p)/p = 0.48% \oplus 0.009 \times p(\text{GeV}/c)%$.

The liquid krypton calorimeter (LKr) [8] consists of ~13000 cells, each of cross section about 2 cm × 2 cm and depth 27 radiation lengths. For the average photon energy of 25 GeV, the transverse spatial resolution for photon showers is 1 mm and the energy resolution is 0.8%. The energy response is linear to about 0.1% in the range 5–100 GeV. The photon time resolution is ~500 ps and the $\pi^0\pi^0$ event time is known with a precision of ~220 ps.

An evacuated beam pipe of radius 8 cm running along the full length of the detector on its central axis transports undecayed beam particles through each of the detector components. The beam tube and DCH drift chambers are aligned along the bisector of the converging K_S and K_L beams in order to equalise the acceptance for K_S and K_L decays. Undecayed neutral kaons in the K_L beam are largely confined within a transverse beam profile of radius ~3.5 cm at the position of the NA48 detector. The K_S beam has a larger beam divergence and a correspondingly larger transverse profile of about 5 cm radius. Due to scattering in the beam collimators, both the K_S and K_L beams have an associated halo of particles extending to larger radii from the beam axis.

The trigger for $\pi^0\pi^0$ decays [9] requires that the total energy deposited in the LKr calorimeter be greater than 50 GeV, that the kaon impact point at the calorimeter (had it not decayed) be within 15 cm of the beam axis, that the decay vertex be less than 5 K_S lifetimes from the beginning of the decay volume, and that each of the horizontal and vertical projections of the LKr energy distribution contain at most five peaks. The $\pi^0\pi^0$ trigger operates with negligible deadtime and high efficiency, (99.920 ± 0.009)% with no significant difference between K_S and K_L decays. The first level of the $\pi^+\pi^−$ trigger [10] is based on signals from a scintillator hodoscope positioned in front of the LKr calorimeter and on the hit multiplicity in DCH1, and also requires a total energy in the LKr and hadron calorimeters of at least 35 GeV. The second level trigger is based on tracks reconstructed using the information from DCH1, 2 and 4, and includes requirements that the decay vertex be less than 4.5 K_S lifetimes from the beginning of the decay volume and that the reconstructed mass be larger than 0.95$m_K$. The efficiency of the first and second level triggers is about 99.5% and 98.3%, respectively, again with no significant difference between K_S and K_L decays. The $\pi^+\pi^−$ trigger introduces a deadtime of about 1.1%.

4. Event selection

$K^0 \rightarrow \pi\pi$ decays are reconstructed and selected using the same procedures and selection requirements as for the Re($\epsilon'/\epsilon$) analysis [5]. The level of background remaining in the selected $K^0 \rightarrow \pi\pi$ samples, and the correction for acceptance differences between K_S and K_L decays, are also evaluated using similar techniques to those in [5]. For the determination of the K_S lifetime, it is the lifetime dependence of these corrections which is of importance, rather than their energy dependence or their overall normalisation.
4.1. The $\pi^0\pi^0$ sample

The reconstruction of $\pi^0\pi^0$ events is based entirely on data from the LKr calorimeter. Any group of four showers, each reconstructed within 5 ns of their average time, is considered. The energy of each shower is required to lie between 3 and 100 GeV, and showers close to the edges of the calorimeter (within 11 cm of the outer edge or within 15 cm of the central axis) or within 2 cm of a defective cell are excluded. The transverse distance between any pair of showers must be greater than 10 cm. The position of the centre of gravity of the event is defined as the energy-weighted average of the four shower positions. The radial distance, $C_R$, of the centre of gravity from the detector axis is required to be less than 10 cm to suppress events due to the decay of particles in the beam halo.

The kaon energy is estimated simply as the sum of the four shower energies. The longitudinal decay vertex position is reconstructed from the energies and positions of the four showers, under the assumption that they come from the decay of a particle with the kaon mass, $m_K$, moving along the beam axis. A resolution of $\sim$40–60 cm is achieved on the vertex position, depending on the kaon energy and decay point. The shower pairing which best represents two $\pi^0$ decay is inferred using a $\chi^2$ variable constructed from the sum, $m_1 + m_2$, and difference, $m_1 - m_2$, of the two candidate $m_{\gamma\gamma}$ masses and a parameterisation of the resolutions on $m_1 \pm m_2$.

Background from $K_L \rightarrow \pi^0\pi^0\pi^0$ decays is suppressed by requiring no additional showers with energy greater than 1.5 GeV within $\pm$3 ns around the event time, and by requiring $\chi^2 < 13.5$ (which corresponds to 3.7$\sigma$ on the $m_{\gamma\gamma}$ resolution). All other potential sources of background to the $\pi^0\pi^0$ sample are negligibly small.

The level of $K_L \rightarrow \pi^0\pi^0\pi^0$ background remaining in the $K_L \rightarrow \pi^0\pi^0$ sample is estimated using a control region at large values of $\chi^2$, $36 < \chi^2 < 135$, dominated by background. Genuine $K_L \rightarrow \pi^0\pi^0$ signal events populating this control region are first subtracted using the $K_S \rightarrow \pi^0\pi^0$ data, for which background is negligible. Small differences in the shape of the $\chi^2$ distribution for $K_S \rightarrow \pi^0\pi^0$ and $K_L \rightarrow \pi^0\pi^0$ decays are taken into account by applying a correction derived from Monte Carlo. The $K_L \rightarrow \pi^0\pi^0\pi^0$ background in the signal region, $\chi^2 < 13.5$, is estimated by extrapolating uniformly from the control region, with an additional factor $\lambda_{ext} = 1.2 \pm 0.2$ derived from Monte Carlo simulation. The background estimation is carried out separately for each bin of kaon energy and proper lifetime. At low energy and at low lifetime, the fraction of $\pi^0\pi^0\pi^0$ background in the $\pi^0\pi^0$ sample is negligibly small, but rises with both energy and lifetime, reaching about 3% for $E_K = 170$ GeV and $\tau = 3.5\tau_S$. An increase of the background fraction with lifetime is to be expected since, when only four photons are detected in the LKr calorimeter, the reconstructed vertex position for a $K_L \rightarrow \pi^0\pi^0\pi^0$ decay is shifted downstream from its true position, while $K_L \rightarrow \pi^0\pi^0\pi^0$ decays occurring upstream of the decay region are largely removed by the $K_L$ beam collimators.

4.2. The $\pi^+\pi^-$ sample

$K^0 \rightarrow \pi^+\pi^-$ events are reconstructed from oppositely charged pairs of tracks found in the magnetic spectrometer. Each track is required to have momentum greater than 10 GeV and to lie at least 12 cm from the centre of each DCH. Each track is also required to lie within the acceptance of the LKr calorimeter and the muon counters after extrapolation downstream. The momentum-weighted average of the track positions after extrapolation to the LKr calorimeter is used to define the position of the centre of gravity of the event. Its radial distance, $C_R$, from the detector axis is required to be less than 10 cm.

The separation between the two tracks at their point of closest approach after extrapolation upstream from the spectrometer is required to be less than 3 cm. The point midway between the tracks at their closest approach is used to define the decay vertex position. The resolution on the longitudinal position of the decay vertex is in the range $\sim$30–50 cm, while the transverse resolution is about 2 mm. The kaon momentum is computed from the opening angle $\theta$ of the two tracks and from the ratio $p_+/p_-$ of their momenta, assuming that the event corresponds to a $K^0 \rightarrow \pi^+\pi^-$ decay. Thus the reconstruction of both the decay vertex position and the kaon momentum, and hence also of the proper time for the decay, depend only on the geometry of the detector.

Background to the $\pi^+\pi^-$ sample from $\Lambda \rightarrow p\pi\pi$ decays is reduced to a negligible level by applying an
energy dependent upper cut on the track momentum asymmetry $|p_+ - p_-|/(p_+ + p_-)$. This cut also serves to remove events with tracks at low radius to the beam axis in a way which depends only on the momentum ratio of the two tracks. Background from $K_L \to \pi e\nu$ ($K_{e3}$) decays is suppressed by requiring $E/p$ to be less than 0.8 for each track, while $K_L \to \pi \mu\nu$ ($K_{\mu3}$) decays are suppressed by rejecting events where one or both of the tracks is associated with a signal in the muon counters within ±4 ns. Additional suppression of both $K_{e3}$ and $K_{\mu3}$ decays is obtained by requiring that the $\pi^+\pi^-$ invariant mass, $m_{\pi\pi}$, be compatible with the kaon mass to within 3σ of the energy-dependent mass resolution, and by requiring a small missing transverse momentum, $p_T^m < 200$ MeV/c^2, where $p_T^m$ is the component of the kaon momentum perpendicular to the line joining the production target (identified from the vertical position of the decay vertex, as described below) and the point where the kaon trajectory crosses the plane of the first drift chamber. The quantity $p_T^m$ has approximately the same resolution for $\pi^+\pi^-$ decays in both the $K_S$ and $K_L$ beams.

The level of background remaining in the $\pi^+\pi^-$ sample is estimated by extrapolating an exponential fit to the $p_T^2$ distribution in a control region, 800 < $p_T^2$ < 2000 MeV^2/c^2, dominated by background, into the signal region $p_T^2 < 200$ MeV^2/c^2. Due to limited statistics, the lifetime bin width was increased from 0.1τ_{K_S} to 0.5τ_{K_S} for this procedure. The fraction of background in the $K_L \to \pi^+\pi^-$ sample is found to be about $2 \times 10^{-3}$, and no significant dependence of the background fraction on lifetime is observed in any bin of energy. An appreciable lifetime dependence is not expected in this case since both the signal and background are observed in the detector as two-track final states with configurations which vary in similar fashion with the decay vertex longitudinal position.

### 4.3. $K_S$ tagging

For $\pi^+\pi^-$ decays, the good resolution on the transverse position of the decay vertex allows a clean separation of decays from the $K_S$ and $K_L$ beams; decays with a vertex position more than 4 cm above the $K_L$ beam axis after extrapolation of the reconstructed parent kaon trajectory back to the position of the AKS counter were classed as belonging to the $K_S$ beam ("vertex tagging"). For $\pi^0\pi^0$ decays, only the longitudinal position of the decay vertex can be reconstructed. In this case, the identification of decays from the $K_S$ beam is accomplished using a tagging station (Tagger) traversed by the proton beam during transport to the $K_S$ target and consisting of two scintillator ladders (one horizontal, one vertical) [11]. Each scintillator counter has a time resolution of ~140 ps, and a proton crosses at least two counters. A $\pi^0\pi^0$ decay is classed as belonging to the $K_S$ beam if a signal is observed in the Tagger within ±2 ns of the reconstructed event time. A small fraction, $\alpha_{SL}^{00} = (1.6 \pm 0.5) \times 10^{-4}$, of $K_S \to \pi^0\pi^0$ decays are mistagged as belonging to the $K_L$ beam due to Tagger detection inefficiencies. A larger fraction, $\alpha_{LS}^{00} = (10.692 \pm 0.020)\%$, of $K_L \to \pi^0\pi^0$ decays are mistagged as belonging to the $K_S$ beam due to accidental proton signals in the Tagger in time with the event. The values of $\alpha_{SL}^{00}$ and $\alpha_{LS}^{00}$ were inferred from studies of the mistagging fractions $\alpha_{SL}^{\pi\pi}$ and $\alpha_{LS}^{\pi\pi}$ for vertex tagged $\pi^+\pi^-$ decays, combined with studies of $K_L \to \pi^0\pi^0$ and $K_L \to \pi^0\pi^0\pi^0$ decays to determine the small differences in mistagging rates between charged and neutral modes.

### 4.4. Event samples

Applying the selection cuts above to the data recorded by NA48 during 1998 and 1999 yielded samples of 13.2M $K_S \to \pi^+\pi^-$, 12.2M $K_L \to \pi^+\pi^-$, 3.1M $K_S \to \pi^0\pi^0$ and 2.8M $K_L \to \pi^0\pi^0$ events with reconstructed lifetimes in the range 0.5 < $\tau/\tau_{K_S}$ < 3.5, where the number of $\pi^0\pi^0$ events has been corrected for $K_{S-L}$ mistagging.

Samples of simulated $K^0 \to \pi\pi$ events corresponding to about 3–5 times the data statistics were also available for analysis. The Monte Carlo included a detailed modelling of the $K_S$ and $K_L$ beams (including the $K_S$ beam halo) and used the GEANT package [12] for particle tracking and the simulation of processes such as multiple scattering, photon conversion, and secondary interactions. Simulation of the LKr calorimeter response was based on a library of electromagnetic and hadronic showers generated using GEANT. The simulated events were passed through the same reconstruction and selection code as the data. Asymmetric non-Gaussian tails in photon shower energies due to hadron photoproduction in the liquid
krypton, which arise in about $3 \times 10^{-3}$ of cases, were modelled using a parameterisation which was applied to the reconstructed shower energies in the Monte Carlo events.

4.5. Acceptance correction

Since the $K_S$ and $K_L$ beams are almost collinear, and since $K_S$ and $K_L$ decays are recorded simultaneously using a common trigger, the acceptances for $K_S$ and $K_L$ decays as a function of energy and position are equal to good approximation, and the detector acceptance essentially cancels in the $K_S/K_L$ ratio of lifetime distributions from the two beams. For the $\pi^+\pi^-$ decay mode, effects due to decay in flight of the charged pions also cancel in the $K_S/K_L$ ratio.

Small acceptance differences arise due to the different divergences and transverse profiles of the $K_S$ and $K_L$ beams and the consequent differences in illumination of the detector. Also, the different definition of the upstream edge of the decay region for the two beams introduces large acceptance differences at low values of the reconstructed lifetime. The AKS veto requirement effectively represents a cut on the reconstructed lifetime. The $\Delta_{1}\pi$ veto rejection of simulated events passing the selection cuts in a bin defined by-bin for residual background in the $K_L$ acceptance with increasing radius is due dominantly to the geometrical cuts and to the cut on the track momentum asymmetry, and can be reliably modelled and checked by comparing the relevant reconstructed distribution in data and Monte Carlo. Similarly, the modelling of the transverse beam profiles is checked by comparing the reconstructed $C_g$ distributions in data and Monte Carlo.

5. Results

The $K_S/K_L$ ratio $R = N_S/N_L$ is computed in two-dimensional bins of energy and lifetime of width $\Delta E_K = 5 \text{ GeV}$ and $\Delta \tau = 0.1 \tau_S$, and corrected bin-by-bin for residual background in the $K_L \rightarrow \pi^+\pi^-$ and $K_L \rightarrow \pi^0\pi^0$ samples and for small differences in the $K_S$ and $K_L$ acceptances, as described above. For the $\pi^0\pi^0$ mode, the $K_S/K_L$ ratio is also corrected for $K_S$-$K_L$ mistagging. Examples of the dependence of the corrected $K_S/K_L$ ratio on the reconstructed proper lifetime are shown in Fig. 1. The shapes of the separate $K_S$ and $K_L$ lifetime distributions change appreciably with energy, due largely to changes in the lifetime dependences of the $\pi^+\pi^-$ and $\pi^0\pi^0$ acceptances, but the corrected $K_S/K_L$ ratio is approximately independent of energy.

The $K_S$ lifetime is determined by fitting a function of the form given in Eqs. (1) and (2) to the $K_S/K_L$ ratio over the lifetime range $0.5 < \tau/\tau_S < 3.5$ and energy range $70 < E_K < 170 \text{ GeV}$. The following $\chi^2$ quantity is minimised in the fit:

$$\chi^2 = \sum_{i=1}^{20} \sum_{j=1}^{30} \left( \frac{R_{ij} - R(E_i, \tau_j)}{\sigma_{ij}} \right)^2,$$

where $E_i$ is the central energy of the $i$th bin of energy, $\tau_j$ is the central value of the $j$th bin of proper lifetime, $R_{ij}$ is the corrected $K_S/K_L$ ratio from the data, $\sigma_{ij}$ is the statistical error on $R_{ij}$, and $R(E_i, \tau_j) = A_i f_S(t_S)/f_L(t_L)$ is the expected $K_S/K_L$ ratio. The nor-
Fig. 1. The points with error bars in the upper plot of each pair show examples of the corrected \( K_S/K_L \) ratio as a function of the reconstructed proper lifetime, \( \tau \), expressed in nominal \( K_S \) lifetime units of \( 0.8927 \times 10^{-10} \) s. The dotted and dashed histograms show the corresponding uncorrected lifetime distributions for the \( K_S \) and \( K_L \) beams, arbitrarily normalised. The curves show the results of the fit for the \( K_S \) lifetime, while the lower plot of each pair shows the normalised fit residuals.

Table 1

<table>
<thead>
<tr>
<th>Year</th>
<th>( \tau_S^{0.99} ) (( 10^{-10} ) s)</th>
<th>( \chi^2/\text{dof} )</th>
<th>( \tau_S^{0.99} ) (( 10^{-10} ) s)</th>
<th>( \chi^2/\text{dof} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1998</td>
<td>0.89578 ± 0.000109</td>
<td>628.2/573</td>
<td>0.89608 ± 0.000247</td>
<td>551.0/573</td>
</tr>
<tr>
<td>1999</td>
<td>0.89598 ± 0.000072</td>
<td>601.2/573</td>
<td>0.89635 ± 0.000167</td>
<td>543.5/573</td>
</tr>
</tbody>
</table>

malisation parameters \( A_i = A(E_i) \) are allowed to vary in the fit. At low energies, \( 70 < E_K < 140 \, \text{GeV} \), the values of the dilutions \( D_S(E_K) \) and \( D_L(E_K) \) appearing in the interference terms in the functions \( f_S(t_S) \) and \( f_L(t_L) \) are fixed using measurements of the dilution by the NA31 experiment [13]. The dilutions \( D_S(E_K) \) and \( D_L(E_K) \) rise from about 0.2 at \( E_K = 70 \, \text{GeV} \) to about 0.35 at \( E_K = 140 \, \text{GeV} \) and differ slightly because of the different kaon production angles (4.2 mrad and 2.4 mrad) used to define the \( K_S \) and \( K_L \) beams; small corrections are applied to the NA31 measurements to account for the different production angles used in this experiment. At high energies, \( 140 < E_K < 170 \, \text{GeV} \), where the interference term contributes significantly to the ratio \( R(E_K, \tau) \), the values \( D_i = D_S(E_i) = D_L(E_i) \) of the dilution are allowed to vary in the fit, neglecting the small differences between the \( K_S \) and \( K_L \) dilutions. The remaining physics parameters appearing in the functions \( f(t) \) are taken from the PDG averages of existing published measurements, except the \( K_S \) lifetime \( \tau_S \) which is a free parameter in the fit.

The fit was carried out separately for the \( \pi^+\pi^- \) and \( \pi^0\pi^0 \) samples from each year of data taking. The values of the \( K_S \) lifetime from each fit are summarised in Table 1. The errors are statistical only and include the contribution from finite Monte Carlo statistics. The minimum value of the fit \( \chi^2 \) is also given in Table 1 and corresponds in all cases to an acceptable fit probability. Examples of the fit results and the fit quality for the \( \pi^+\pi^- \) and \( \pi^0\pi^0 \) data samples from the 1999 run are shown in Fig. 1. The effect of the interference terms in the fit function is visible in the plots for the high energy bin as a slight curvature of the lines representing the fit result.

Consistent values of the fitted \( K_S \) lifetime \( \tau_S \), and of the fitted normalisation and dilution parameters \( A_i \) and \( D_i \), were found for the charged and neutral modes and for the samples from the different years of data taking. The fitted values of the dilution \( D_i \) within the energy range \( 140 < E_K < 170 \, \text{GeV} \) were also consistent with the published NA31 measurements.

6. Systematic errors

Various sources of systematic error on the measured \( K_S \) lifetime were considered, including uncertainties in the reconstruction of the energy and distance scales for the charged and neutral decay modes, and uncertainties due to the background subtraction, the tagging correction, the Monte Carlo acceptance correction, and the external physics parameters used in the fitting function.

As noted in Section 4.2, the energy and distance scales for reconstructed \( \pi^+\pi^- \) events are determined largely by the detector geometry, especially of DCH1 and DCH2. The systematic error due to uncertainties in detector geometry was estimated by considering a variation of \( \pm 2 \, \text{mm} \) in the longitudinal separation of DCH1 and DCH2, and a variation of \( \pm 20 \, \text{\mu m/m} \) in the relative transverse scale of the two chambers. The position of the upstream edge of the \( K_S \to \pi^+\pi^- \) decay vertex distribution was found to be consistent within these tolerances with the nominal position of the AKS counter. In addition, the momentum scale for reconstructed tracks was varied by \( \pm 0.1\% \), as determined from the consistency of the reconstructed \( \pi^+\pi^- \) invariant mass with the kaon mass, \( m_K \). In each case, the kaon decay reconstruction was redone for all events in the data using the modified track positions or momenta, and the lifetime analysis repeated. The resulting change in the fitted \( K_S \) lifetime was taken as the estimate of the corresponding systematic error.

The systematic error due to uncertainties in the reconstruction of \( \pi^0\pi^0 \) events was estimated by varying the energy scale of reconstructed showers by \( \pm 0.03\% \) and by varying the linearity and uniformity of the calorimeter response by modifying the reconstructed shower energy by an amount

\[
\Delta E = \alpha + \beta E^2 + \gamma r E,
\]

where \( r \) is the radial distance of the shower from the central detector axis. The allowed variation in the neutral energy scale is determined from a comparison of the position of the edge of the \( K_S \to \pi^0\pi^0 \) decay vertex distribution with the nominal position of the AKS counter. The allowed ranges of the constants \( \alpha \), \( \beta \) and \( \gamma \) are determined from studies of \( K_{S,0} \) to \( \pi^0\pi^0 \) and \( K_L \to \pi^0\pi^0 \) decays, and of \( \pi^0 \to \gamma\gamma \) and \( \eta \to \gamma\gamma \) decays in special runs with \( \pi^- \) beams.

The uncertainty due to the \( K_L \to \pi^+\pi^- \) background subtraction was assessed by conservatively assigning a systematic error equal to \( \pm 100\% \) of the change in the fitted \( K_S \) lifetime when the background subtraction was removed. The systematic error associated with the \( K_L \to \pi^0\pi^0 \) background subtraction
The transverse beam profile was studied by switching their nominal positions. In addition, the sensitivity to the KL gave the above fraction of collimator scattered events in the acceptance correction was estimated by varying collimator scattering component in the fit. This contribution is already included in the charged background estimate described in Section 4.2, and no additional systematic error is warranted. Studies of events with large transverse momentum \( p_T^2 \) showed that the lifetime distribution of collimator scattered events corresponds approximately to an exponential decay characterised by the K_S, rather than K_L, lifetime, and that their energy distribution is similar to that of unscattered K_L decays. The high \( p_T^2 \) events were also used to estimate the fraction, \((4.2 \pm 1.0) \times 10^{-4}\), of collimator scattered events in the K_L \( \rightarrow \pi^0 \pi^0 \) sample, where no transverse momentum cut is possible. For the \( \pi^0 \pi^0 \) mode, the effect of collimator scattering on the K_S lifetime measurement was studied by modifying the fit function to be of the form

\[
R(E_i, \tau_j) = A_i \frac{f_{\text{LS}}(t_S)}{[f_{\text{LS}}(t_L) + A_{\text{coll}} e^{-t_{\text{coll}}/\tau_S}]}.
\]

where \( t_{\text{coll}} = (z - z_{\text{coll}})(m_K/p_K) \) is the proper lifetime of the decay relative to the position of the final K_L collimator and \( A_{\text{coll}} \) is a constant which is adjusted to give the above fraction of collimator scattered events in the K_L \( \rightarrow \pi^0 \pi^0 \) sample. This fraction was assumed to be independent of energy. The resulting systematic error was conservatively estimated as \( \pm 100\% \) of the change in the K_S lifetime due to inclusion of the collimator scattering component in the fit.

The systematic error associated with uncertainties in the acceptance correction was estimated by varying the assumed vertical position of the K_S and K_L beams in the Monte Carlo within a range \( \pm 2\text{ mm} \) around their nominal positions. In addition, the sensitivity to the transverse beam profile was studied by switching off the simulation of the K_S beam halo in the Monte Carlo. In each case, the resulting change in the fitted K_S lifetime was taken as the systematic error. The sensitivity of the acceptance correction to the modelling of the detector resolution was assessed by removing the simulation of non-Gaussian tails from the reconstructed photon shower energies, and a systematic error of \( \pm 50\% \) of the effect on the fitted K_S lifetime was assigned. The effect of possible non-Gaussian tails in the reconstructed drift chamber hit positions was also studied in the Monte Carlo, but found to have a negligible effect on the fitted K_S lifetime. Finally, the component of the overall statistical error on the fitted lifetime arising from the finite Monte Carlo statistics was extracted, and classified as a separate systematic error.

The external physics parameters: \( t_\pi, \eta_{+\pi}, \eta_{00}, \phi_{+\pi}, \phi_0 \) and \( \Delta m \) appearing in the functions \( f(t) \) were each varied in turn within a range given by the error on the PDG average value. The mistagging fractions \( \alpha_{00} \) and \( \alpha_{00} \) were varied within the uncertainties given in Section 4.3. The value of the dilutions \( D_{\text{SL}}(E_K) \) in the energy range \( 70 < E_K < 140 \text{ GeV} \) was varied by twice the error on the NA31 measurements [13]. The extra factor of two conservatively takes into account uncertainties in correcting the NA31 measurements to the NA48 experiment.

The fitting procedure itself was tested by applying the fit to the generated inclusive energy and lifetime distributions of parent kaons from the Monte Carlo samples. No significant bias was observed on the fitted K_S lifetime, and the statistical precision of the test was assigned as a systematic error.

The systematic errors on the K_S lifetime from each of the above sources for the 1998 and 1999 data combined are summarised in Table 2. For comparison, the effect of the background subtraction is to increase the fitted K_S lifetime for the \( \pi^0 \pi^0 \) mode by \( 12.8 \times 10^{-14} \text{ s} \), while the acceptance correction from the Monte Carlo changes the K_S lifetime by \( +22.5 \times 10^{-14} \text{ s} \) and \( -9.1 \times 10^{-14} \text{ s} \) for the \( \pi^+ \pi^- \) and \( \pi^0 \pi^0 \) modes, respectively.

The analyses carried out for each year of data taking and for each decay mode are combined taking into account any correlations between the separate analyses. The total systematic error is obtained by summing the individual errors in quadrature. The measured values of the K_S lifetime for each decay...
Table 2
Summary of systematic errors on the measured K_S lifetime, in units of 10^{-14} s. The final column corresponds to a combination of the \( \pi^+ \pi^- \) and \( \pi^0\pi^0 \) results.

<table>
<thead>
<tr>
<th>Source</th>
<th>Variation</th>
<th>( \pi^+ \pi^- )</th>
<th>( \pi^0\pi^0 )</th>
<th>( \pi \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>DCH1 radial scale</td>
<td>±20 ( \mu )m/m</td>
<td>±1.8</td>
<td>±1.5</td>
<td></td>
</tr>
<tr>
<td>DCH1 ( z ) position</td>
<td>±2 mm</td>
<td>±0.2</td>
<td>±0.1</td>
<td></td>
</tr>
<tr>
<td>Momentum scale</td>
<td>±0.001</td>
<td>±0.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>LKr energy scale</td>
<td>±0.0003</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Energy loss (( \alpha ))</td>
<td>±10 MeV</td>
<td>±1.7</td>
<td>±0.3</td>
<td></td>
</tr>
<tr>
<td>Non-linearity (( \beta ))</td>
<td>±0.00002 GeV^{-1}</td>
<td>±1.6</td>
<td>±0.6</td>
<td></td>
</tr>
<tr>
<td>Non-uniformity (( \gamma ))</td>
<td>±0.00001 cm^{-1}</td>
<td>±1.0</td>
<td>±0.2</td>
<td></td>
</tr>
<tr>
<td>LKr radial scale</td>
<td>±0.0003</td>
<td>±1.2</td>
<td>±0.5</td>
<td></td>
</tr>
<tr>
<td>Charged background</td>
<td>±100%</td>
<td>±1.4</td>
<td>±1.2</td>
<td></td>
</tr>
<tr>
<td>Neutral background</td>
<td>( \lambda_{\text{ext}} = 1.2 \pm 0.2 )</td>
<td>±2.1</td>
<td>±0.3</td>
<td></td>
</tr>
<tr>
<td>Collimator scattering</td>
<td>±100%</td>
<td>±3.2</td>
<td>±0.5</td>
<td></td>
</tr>
<tr>
<td>MC: K_S beam y position</td>
<td>±2 mm</td>
<td>±0.2</td>
<td>±1.3</td>
<td></td>
</tr>
<tr>
<td>MC: K_S beam x position</td>
<td>±2 mm</td>
<td>±1.4</td>
<td>±1.2</td>
<td></td>
</tr>
<tr>
<td>MC: K_S beam halo</td>
<td>±100%</td>
<td>±1.3</td>
<td>±1.4</td>
<td></td>
</tr>
<tr>
<td>MC: non-Gaussian tails</td>
<td>±50%</td>
<td>±4.4</td>
<td>±0.7</td>
<td></td>
</tr>
<tr>
<td>MC: statistics</td>
<td></td>
<td>±3.0</td>
<td>±4.9</td>
<td>±2.7</td>
</tr>
<tr>
<td>( \tau_L )</td>
<td>5.17 ± 0.04 ( \times 10^{-8} ) s</td>
<td>±0.1</td>
<td>±0.1</td>
<td>±0.1</td>
</tr>
<tr>
<td>(</td>
<td>\eta_+</td>
<td>-</td>
<td>\eta_0</td>
<td>)</td>
</tr>
<tr>
<td>( \phi_+ - \phi_0 )</td>
<td>43.3 ± 0.5°, 43.2 ± 1.0°</td>
<td>±0.8</td>
<td>±0.4</td>
<td>±0.8</td>
</tr>
<tr>
<td>( \Delta m )</td>
<td>0.5300 ± 0.0012 ( \times 10^{10} ) s^{-1}</td>
<td>±0.5</td>
<td>±0.7</td>
<td>±0.5</td>
</tr>
<tr>
<td>( D(E_K) )</td>
<td>±2( \sigma )</td>
<td>±0.3</td>
<td>±1.0</td>
<td>±0.4</td>
</tr>
<tr>
<td>( \alpha_{\text{LS}}^0 )</td>
<td>0.10692 ± 0.00020</td>
<td>±2.1</td>
<td>±0.4</td>
<td></td>
</tr>
<tr>
<td>( \alpha_{\text{SL}}^0 )</td>
<td>(1.6 ± 0.5) ( \times 10^{-4} )</td>
<td>±0.4</td>
<td>±0.1</td>
<td></td>
</tr>
<tr>
<td>Fit method</td>
<td></td>
<td>±2.7</td>
<td>±2.7</td>
<td>±2.7</td>
</tr>
<tr>
<td>Total systematic error</td>
<td></td>
<td>±5.4</td>
<td>±10.9</td>
<td>±5.1</td>
</tr>
<tr>
<td>Statistical error</td>
<td></td>
<td>±5.2</td>
<td>±12.9</td>
<td>±4.8</td>
</tr>
</tbody>
</table>

where the first error is statistical and the second systematic.

Various cross-checks were performed to verify the integrity and stability of the result. No significant dependence of the fitted K_S lifetime was found on the lifetime or energy range used in the fit, or on the energy range within which \( D(E_K) \) was allowed to vary in the fit. The effect of varying the main selection cuts used in the analysis was also studied, namely the cuts on \( p_T^2, C_g \), the momentum asymmetry \( |p_+ - p_-|/(p_+ + p_-) \), and the minimum radii of tracks and clusters in DCH1 and the LKr calorimeter. In each case, either no statistically significant variation in the fitted lifetime was found, or the observed variation was found to be within the systematic errors assigned. The stability of the analysis was also tested by dividing the \( \pi^+ \pi^- \) data sample into topologies for which the positive and negative tracks initially curve towards or away from each other in the spectrometer magnetic field; consistent values of the fitted lifetime were found for the two topologies. Similarly, selecting either of the two polarities of the spectrometer magnetic field setting gave consistent results. Other tests involved dividing the data samples according to the primary data-taking periods, the event time within the SPS spill, and the azimuthal orientation of the decay.
For the charged mode, separating $K_S$ and $K_L$ decays using the Tagger in place of vertex tagging gave no significant change in the result.

7. Summary

The $K_S$ lifetime has been measured using $K_S \to \pi\pi$ and $K_L \to \pi\pi$ decays recorded by the NA48 experiment in 1998 and 1999. The combined result for the $\pi^+\pi^-$ and $\pi^0\pi^0$ decay modes, $\tau_S = (0.89598 \pm 0.00048 \pm 0.00051) \times 10^{-10}$ s, has a precision better than that of the current PDG average [1] of existing measurements, and lies about 1.7 standard deviations above it.

References

The ratio, \( \rho \), of the real to the imaginary part of the \( \bar{p}p \) forward elastic scattering amplitude at \( \sqrt{s} = 1.8 \) TeV

E-811 Collaboration

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Abstract

We have measured \( \rho \), the ratio of the real to the imaginary part of the \( \bar{p}p \) forward elastic scattering amplitude, at \( \sqrt{s} = 1.8 \) TeV. Our result is \( \rho = 0.132 \pm 0.056 \); this can be combined with a previous measurement at the same energy to give \( \rho = 0.135 \pm 0.044 \). © 2002 Elsevier Science B.V. All rights reserved.

Measurements of \( \rho \), the ratio of the real to the imaginary part of the forward \( \bar{p}p \) elastic scattering amplitude, together with measurements of \( pp \) and \( \bar{p}p \) total cross sections and some very general assumptions, allow prediction of total cross section behavior at considerably higher energies than are presently available [1,2]. We report here a measurement of \( \rho \) at \( \sqrt{s} = 1.8 \) TeV, made at the same time as our Fermilab Tevatron measurement [3] of the total cross section, \( \sigma_T \). Previous measurements of \( \rho \) at \( \bar{p}p \) colliders have been made at the ISR [4], the SPS [5], and the Tevatron [6].

The experimental apparatus and method have been described in Ref. [3]; the major difference in the measurements reported here is analysis of data at smaller \( |t| \) values, in order to study the \( |t| \) region where Coulomb effects become important.

After event selection for elastic candidates as described in Ref. [3], the data for a typical run are illustrated in Fig. 1: (a) shows the correlation for each event between the vertical (\( Y \)) coordinates of particles in proton and antiproton conjugate detectors, while (b) shows the corresponding horizontal (\( X \)) correlation. Elastic events are in the diagonal bands in each figure, with background increasing towards the region closest to the beams, as can be seen most clearly in Fig. 1a. We would expect that the background is due to un-
correlated particles in the detectors caused by, for example, beam halo, which is known to increase sharply close to the beams. We used two methods to remove the remaining background under the elastic signal.

A. Events in the elastic region were removed from the $X_{\text{proton}}-X_{\text{antiproton}}$ distribution; the $Y_{\text{proton}}-Y_{\text{antiproton}}$ correlation of the remaining events allowed determination of the background under the elastic region of the $Y_{\text{proton}}-Y_{\text{antiproton}}$ distribution.

B. We used in the $Y_{\text{proton}}-Y_{\text{antiproton}}$ distribution only those events in the elastic region of the $X_{\text{proton}}-X_{\text{antiproton}}$ distribution. We then determined an analytic expression for the combined signal and background shapes perpendicular to the elastic correlation of Fig. 1a in the region with lower background. This analytic form was extrapolated into the small $Y_{\text{proton}}-Y_{\text{antiproton}}$ region to determine the background in that region.

Methods A and B for background subtraction gave identical results, within statistical uncertainties, for $\rho$; results are quoted using method A. Our analysis used data in the range $0.002 < |t| < 0.035$ (GeV/c)$^2$; at the smallest $|t|$, up to 70% of the events were background, but this dropped rapidly with increasing $|t|$; however, the background could be determined with sufficient statistical precision, and our final statistical uncertainties include the contribution from the background determination. The number of elastic events used in this analysis was about 40,000.

We use the following expression for the elastic differential cross section:

$$\frac{1}{L} \frac{dN_{\text{el}}}{dt} = \frac{d\sigma}{dt} = \frac{4\pi\alpha^2(hc)^2G^4(t)}{|t|^2}$$

$$+ \frac{\alpha(\rho - \alpha\phi)\sigma_T G^2(t)}{|t|} \exp\left(-\frac{B|t|}{2}\right)$$

$$+ \frac{\sigma_t^2(1 + \rho^2)}{16\pi(hc)^2} \exp\left(-\frac{B|t|}{2}\right). \tag{1}$$

The three terms in Eq. (1) are due to, respectively, Coulomb scattering, Coulomb-nuclear interference, and nuclear scattering. $L$ is the integrated accelerator luminosity, $dN_{\text{el}}/dt$ is the observed elastic differential distribution, $\alpha$ is the fine structure constant, $\phi$ is the relative Coulomb-nuclear phase, given by [7] $\ln(0.08/|t|^{-1}) - 0.577$, and $G(t)$ is the nucleon electromagnetic form factor, which we parameterize in the usual way as $(1 + |t|/0.71)^{-2}$ [t is in (GeV/c)$^2$].
We also use the following two equations:

\[ \sigma_T^2 = \frac{1}{L} \frac{16\pi (hc)^2}{(1 + \rho^2)} \frac{dN_{el}^n}{dt} \bigg|_{t=0}, \]

(2)

\[ \sigma_T = \frac{1}{L} (N_{el} + N_{inel}). \]

(3)

Eq. (2) is obtained from the optical theorem. \( N_{el}^n \) is the total number of nuclear elastic events, obtained from the observed \( dN_{el}/dt \) distribution in the \( t \) region where nuclear scattering dominates, and extrapolated to \( t = 0 \) and \( t = \infty \) using the form \( \exp(-B|t|) \). \( dN_{el}^n/dt|_{t=0} \) is the observed differential number of nuclear elastic events extrapolated to \( t = 0 \) using the same form. \( N_{inel} \) is the total number of inelastic events; our method for obtaining this, using detectors close to the interaction point, has been described earlier [3]. Note that Eqs. (2) and (3) allow us to express \( L \) in terms of \( \sigma_T \) and \( \rho \). Then \( dN_{el}/dt \) in Eq. (1) can be expressed in terms of just two unknowns, \( \sigma_T \) and \( \rho \). Our input data are our measurements of \( dN_{el}/dt \) together with the total number of inelastic events \( N_{inel} \) for the same runs as the elastic data, and the value of \( B = 16.98 \pm 0.22 \text{ (GeV/c)}^{-2} \) (the mean from Refs. [6] and [8]). We carry out a least-squares analysis for \( \sigma_T \) and \( \rho \) in Eq. (1) using all of our input data.

The result of the fit is

\[ \sigma_T = 71.42 \pm 1.55 \text{ mb}; \quad \rho = 0.132 \pm 0.049, \]

where the uncertainties quoted are statistical only. In this fit the \( \chi^2 \) per degree of freedom is 1.05. We show in Fig. 2 the fit to our \( dN_{el}/dt \) data, together with two other fits for illustration, fixing \( \rho = 0 \) and 0.24 as noted, and allowing only \( \sigma_T \) to vary; these latter two fits both give \( \chi^2 \) per degree of freedom of 1.4.

We considered 11 sources of systematic uncertainties, with the major ones including the uncertainties in \( N_{inel} \) and \( B \), cuts on the elastic event sample, detector calibrations and efficiencies, and detector positions with respect to the beam center. Almost all of these were obtained from our own data, with uncertainties that were statistical. These systematic uncertainties total \( \pm 1.85 \text{ mb} \) in \( \sigma_T \) and \( \pm 0.028 \text{ in } \rho \). Combining statistical and systematic uncertainties in quadrature leads to our final result of \( \sigma_T = 71.42 \pm 2.41 \text{ mb}; \quad \rho = 0.132 \pm 0.056 \).

The value of \( \sigma_T \) given here [9] supersedes that of Ref. [3].

Our value of \( \rho \) is identical within uncertainties to that of E710 [6]. We can combine the two results in quadrature (since there is little common systematic uncertainty) to give a value at \( \sqrt{s} = 1.8 \text{ TeV} \) of \( \rho = 0.135 \pm 0.044 \).

This value is shown in Fig. 3, together with results at lower energies [4,5,10], and a prediction [11] based on existing \( \rho \) and \( pp \) and \( \bar{p}p \) \( \sigma_T \) data. Using the approximate asymptotic dispersion relation [2],

\[ \rho \approx \frac{\pi}{2 \sigma_T} \frac{d \sigma_T}{d \log s}. \]
we find consistency between our values of $\rho$ and $\sigma_T$ and previous lower energy $\sigma_T$ data.

In summary, we have made a new measurement of $\rho$ at $\sqrt{s} = 1.8$ TeV, which confirms previous data with higher accuracy, and provides a consistent picture, via an approximate dispersion relation, with our measurement and lower energy $\sigma_T$ data.

Acknowledgements

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References

[9] Note that this value of $\sigma_T$ has a larger uncertainty than that of Ref. [3]. In some previous measurements (e.g., Ref. [3] and F. Abe et al., Phys. Rev. D 50 (1994) 5550) the quantity actually measured is $\sigma_T (1 + \rho^2)$, and a value of $\rho$ is assumed from dispersion relation fits, with the uncertainty in $\rho$ set to zero. In the current measurement, we do a two-parameter fit to $\sigma_T$ and $\rho$, and the uncertainty in $\rho$ affects the final uncertainty in $\sigma_T$.
[11] The curve shown is from dispersion relations, and is similar to and consistent with that of C. Augier et al. (Ref. [1]). An essentially identical curve is also obtained by M.M. Block et al. (Ref. [1]).
Spin and magnetic moment of $^{31}$Al ground state

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Abstract

The $g$-factor of the ground state of $^{31}$Al ($T_{1/2} = 640$ ms) has been measured by the $\beta$-NMR technique on a polarized secondary fragment beam at GANIL, Caen, France. The measured value of $|g| = 1.517(20)$ is compared with $sd$-shell model calculations, which allows the assignment of spin and parity $I^\pi = 5/2^+$ to this state. The spectroscopy of neutron-rich Al isotopes on the border of the “island of inversion” is discussed in terms of the persistence of the $N = 20$ shell closure for $Z = 13$ nuclei. © 2002 Elsevier Science B.V. All rights reserved.

PACS: 21.10.Ky; 21.10.Hw; 21.60.Cs; 23.20.En; 25.70.Mn; 27.30.+t

Keywords: $\beta$-NMR; Spin-polarization from projectile-fragmentation; $^{31}$Al ground state $g$-factor; Spin and parity

Recent developments in the production of radioactive beams have provided intensive investigations of nuclei with large neutron or proton excess, enriching our knowledge about the structure of nuclear matter at the extremes and on the properties of the strong nuclear interaction. A number of fundamental questions, such as evolution of nuclear masses, sizes and shapes, as well as interplay between single particle and collective degrees of freedom, have initiated different experimental studies.

The investigation of neutron-rich Na and Mg isotopes, often referred to as the “island of inversion”, started already in late seventies after the pioneering experiments on mass measurements [1–3] and $\beta$-decay studies [4] in this region which revealed an unusual
mass excess and unexpected spin and parity assignments as compared to the standard \(sd\)-shell model calculations [5]. These abnormalities were assigned to a large deformation in those nuclei, and this was supported later by the observation of a low-lying first excited state with large \(B(E2)\) transition probabilities to their ground states in \(^{32}\text{Mg}\) [6–8], \(^{28}\text{Ne}\) [7], and \(^{31}\text{Na}\) [9], as well as by the recent measurements of the quadrupole moments in neutron-rich Na isotopes [10]. The need for explanations of the observed phenomena gave rise to the development of various theoretical approaches, from the earlier Hartree–Fock [11] and shell-model calculations [12–14] towards the more elaborate large-scale shell model theories with realistic interactions [15–18]. It is by now well established that the ground states of the \(^{31}\text{Na}\) and \(^{32}\text{Mg}\) have an intruder nature with two neutrons excited from the \(sd\) to the \(pf\) shell (in the spherical shell model language), thus defined as nuclei inside the island of inversion. Such an inversion of normal and intruder states takes place due to the lowering of an effective single-particle gap between the \(sd\) and \(pf\) shells for these nuclei and an essential gain in deformation energy caused by the proton–neutron interaction. At the same time most of the self-consistent mean-field calculations (e.g., Refs. [18–20]), as well as the relativistic mean-field approaches [21–23] favor the spherical “normal” ground state, e.g., for \(^{32}\text{Mg}\), and the techniques beyond the mean field are required in order to generate the quadrupole collectivity of the \(^{31}\text{Na}\) and \(^{32}\text{Mg}\) ground states [24].

Up to now, the unusual properties of \(^{31}\text{Na}\) and \(^{32}\text{Mg}\) have been rather well established by the good agreement of the data with the majority of recent theoretical calculations. The nuclei around them have been less investigated experimentally, while the models give different predictions for their ground states. The exact borders of the island of inversion are not determined yet. Recent \(\beta\)-decay studies supported by the state-of-the-art shell model calculations revealed that the ground state of \(^{34,35}\text{Si}\) can be classified as normal ones [25]. On the other hand, very little is known experimentally on the properties of the neutron-rich Al isotopes, which occur in between the intruder Mg and normal Si isotopes. Recent shell model calculations [15,17] indicate that the ground state of \(^{31}\text{Al}\) \((N = 20)\) could be influenced by particle–hole excitations to the \(pf\) shell. Apart from decay and spectroscopy studies, very important characteristics of a nuclear state are its static moments, namely: the electric quadrupole \((Q)\) and the magnetic dipole \((\mu)\) moment. They are extremely sensitive to the single-particle structure of the nuclear state, as well as to its collective properties and are directly related to the nuclear spin. They provide a test ground for the nuclear models.

Since \(^{31}\text{Al}\) is expected to be on the border of the deformation region, a firm spin assignment of its ground state and detailed understanding of its level structure would permit to understand how the transition from the normal \(sd\)-shell nuclei to the island of inversion happens. In the present study we exploit the experimentally determined \(g\)-factor of the \(^{31}\text{Al}\) ground state for the determination of its ground state spin and for a discussion on the structure of neutron-rich Al-isotopes, including \(^{32}\text{Al}\).

\(^{31}\text{Al}\) was produced using the fragmentation of a \(^{36}\text{Si}\)^{16+} primary beam (77.5 MeV/\(\alpha\)) on a rotating \(^{9}\text{Be}\)-target (177 mg/cm\(^2\)) located at the entrance of the LISE spectrometer [26,27]. A 99% pure secondary beam of \(^{31}\text{Al}\) was selected using the LISE spectrometer and identified by standard energy loss versus time-of-flight techniques. Spin polarization of the \(^{31}\text{Al}\) fragments was produced by deflecting the primary beam by \(^2\) with respect to the spectrometer entrance where the fragmentation target was mounted. This target was tilted \(56^\circ\) with respect to the vertical direction to increase the effective target thickness to its optimal value (317 mg/cm\(^2\)). The production of spin-polarization in the secondary beam was verified by selecting first a pure \(^{27}\text{Na}\) beam and measuring its known magnetic moment via \(\beta\)-NMR [28]. For \(^{27}\text{Na}\) \((I^\pi = 5/2^+,\ t_{1/2} = 301\ ms, \ Q_\beta = 9.010\ MeV)\) implanted in a NaCl single crystal, an experimental polarization \(|P_{\text{exp}}| = 4.3(6)\%\) was measured.

The \(^{31}\text{Al}\) fragments were implanted in a MgO single crystal (20 mm \(\times\) 20 mm \(\times\) 1 mm) at room temperature (Fig. 1). The ionic MgO has a face-centered cubic (fcc) lattice structure with a two atom basis. The \(^{31}\text{Al}\) nuclei maintain most of their polarization by applying a static magnetic field \(B_0\) up to 2000 G. The field was induced by two coils and an iron yoke placed around the vacuum chamber in which the crystal was mounted. A radio frequency (rf) field \(B_{rf}\), was induced by an adjustable LCR circuit with a rf-coil fixed around the MgO crystal. This LCR circuit was optimized to have maximum power of the rf-field at the
Fig. 1. Schematic layout of the NMR setup at GANIL. The static magnetic field is oriented vertically, perpendicular to the beam. A rf coil inducing a radio-frequency field $B_{\text{rf}}$ is fixed around the implantation crystal inside a vacuum chamber. Two scintillator detector telescopes are placed above and below the crystal, in the vacuum chamber as well.

chosen frequencies $\nu_{\text{rf}} = 450$ kHz and $\nu_{\text{rf}} = 1000$ kHz. The rf-field destroys resonantly the spin-polarization when the applied rf-frequency $\nu_{\text{rf}}$ matches the Larmor frequency $\nu_L$ [26,29,30]:

$$\nu_L = -\frac{g\mu_N B_0}{h} = \nu_{\text{rf}},$$ (1)

The resonant destruction of the polarization is measured through detection of the resonant change of the $\beta$-decay asymmetry, $N_{\text{up}}/N_{\text{down}}$, where $N_{\text{up}}$ and $N_{\text{down}}$ are the coincident count rates in the up and down detector telescope, respectively. The two detectors telescopes each consist of two plastic scintillators of 1 mm and 2 cm thickness, respectively, packed in an aluminum sheet of 0.1 mm. The scintillators are placed above and below the crystal (Fig. 1) and optically coupled via plexiglass light guides to the photomultiplier tubes which are placed outside the static magnetic field and shielded for any stray fields. The $N_{\text{up}}/N_{\text{down}}$ ratio was measured as a function of the static magnetic field strength $B_0$. The field strength was varied in regular time intervals of a few minutes and monitored with a Hall probe. The $\beta$-decay energy and time signals, as well as the Hall probe output, were registered event by event. The rf-frequency was swept continuously around a fixed value $\nu_{\text{rf}}$ over a modulation range $\nu_{\text{rf}} - \Delta\nu_{\text{rf}}$ to $\nu_{\text{rf}} + \Delta\nu_{\text{rf}}$ [31]. According to Eq. (1), this means that at one particular field $B_0$ a $g$-factor range $[g - \Delta g, g + \Delta g]$ is scanned, inducing a field-dependent uncertainty $\Delta g$ on the measured $g$-factor (Fig. 2). The modulation range $2\Delta\nu_{\text{rf}}$ is scanned with a modulation frequency of 50 Hz, using a ‘ramp’ scan profile. Because of the ‘ramp’ shape of the frequency modulation, each $g$-factor value in the range $[g - \Delta g, g + \Delta g]$ is scanned equally in time. Therefore, the amount of $\beta$-asymmetry being destroyed by a resonant rf-signal is expected to be the same all over the modulation interval.

In a first rough scan a relative large frequency modulation $\Delta\nu_{\text{rf}} = 35$ kHz at $\nu_{\text{rf}} = 450$ kHz is used. Like this a wide $g$-factor range could be covered in eight data points at fields between $B_0 = 340$ G and $B_0 = 920$ G, corresponding to a $g$-factor range from $g = 1.74$ to 0.64. At $B_0 = 393$ G a clear change in the $\beta$-asymmetry is observed (Fig. 3(a)), corresponding to a $g$-factor range of $g = 1.51(1)$. The systematic error of 0.3%, caused by the readout of the magnetic field, is negligible compared to the uncertainty of 7%, induced by the frequency modulation. In the range $g = 1.387$ to 1.621 a fine scan was made using a higher frequency $\nu_{\text{rf}} = (1000 \pm 12)$ kHz which allowed to measure the $g$-factor more precisely, $g = 1.517(20)$ (Fig. 3(b)). Note that a change in the frequency requires a change in the field range (as indicated in Fig. 3(b)) in order to locate the resonance signal. The uncertainty on the $g$-factor is determined by the modulation range, which is 1.2%, and by the systematic uncertainty of 0.1% on the static field. As demonstrated in Table 1, the measured $g$-factor of the $^{31}$Al ground state is similar to the known $g$-factors for two other Al-isotopes [32].
For the interpretation of this result, we have performed sd-shell model calculations using the USD interaction [33] for $^{31}$Al and systematically for other less-exotic odd-A Al-isotopes. In Fig. 4 we compare theoretical and experimental spectra for $^{25-31}$Al. The calculation yields a $5/2^+$ ground state spin and parity assignment for all the isotopes. The experimental position and ordering of the lowest levels in $^{25-29}$Al is well reproduced, reflecting the filling of the $s_1/2$ orbital for $^{29}$Al. The first two excited states of $^{31}$Al are predicted to be $1/2^+$ and $3/2^+$. In general, we point out that the structure of all low-lying states is rather far from any pure configuration, and represents a complex mixture of many sd-shell configurations. E.g., the ground state of $^{31}$Al with 57% of $\pi(d_{5/2})^2 \nu(d_{3/2})^3 (s_{1/2})^2 (d_{3/2})^2$ configuration has the purest ground state among the mentioned isotopes, and the largest contribution to its first excited $3/2^+$ state is 35% of $\pi(d_{5/2})^4 s_{1/2} v(d_{3/2})^3 (s_{1/2})^2 (d_{3/2})^2$, while the other 75% of the $3/2^+$ wave functions is spread among many other configurations.

In Table 1, we compare the calculated values for the ground state $g$-factors ($g = \mu/I$) and spectroscopic quadrupole moments $Q$ to available experimental values. The theoretical values are obtained with free-nucleon $g$-factors, i.e., $g^e_\nu = 5.58$, $g^f_\nu = 1.0$, $g^e_\pi = -3.81$, $g^f_\pi = 0.0$ and standard effective electric charges $e_\pi = 1.35 e$, $e_\nu = 0.5 e$. The detailed empirical study [36,37] concerning the necessity of introducing effective values for $g^e$ revealed that there is less quenching of the M1 spin operator as compared to the Gamow–Teller one for the sd-shell nuclei. The reason is that the quenching in the isovector spin contribution of the M1 operator is largely compensated by the enhancement in the orbital contribution. As is seen from Table 1, the theoretical $g$-factors obtained using free-nucleon $g$-factors agree within 10% with the measured ones. The calculated $g$-factors for the tentatively assigned $^{31}$Al ground state spins ($5/2^+$ and $3/2^+$) are $g(5/2^+) = 1.524$ and $g(3/2^+) = 0.804$. The value for the $5/2^+$ state agrees very well with our measured value of $g = 1.517(20)$, while the value for the $3/2^+$ state is far off. This allows us to firmly assign the $^{31}$Al ground state spin and parity as $I^\pi = 5/2^+$. The calculated ground state quadrupole moments of odd-A $^{25-31}$Al remain almost constant with increas-

### Table 1

Experimental and calculated $g$-factors and electric quadrupole moments of the $I^\pi = 5/2^+$ ground states of different odd-A Al-isotopes. For $^{33}$Al (S) and (I) refer to the spherical and the lowest intruder states. See text for the details.

<table>
<thead>
<tr>
<th>Isotope</th>
<th>$^{25}$Al</th>
<th>$^{27}$Al</th>
<th>$^{29}$Al</th>
<th>$^{31}$Al</th>
<th>$^{33}$Al (S)</th>
<th>$^{33}$Al (I)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g^{\exp}(5/2^+)$</td>
<td>1.4582(5)</td>
<td>1.4566</td>
<td></td>
<td>1.517(20)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$g^{\th}(5/2^+)$</td>
<td>1.512</td>
<td>1.424</td>
<td>1.476</td>
<td>1.524</td>
<td>1.70</td>
<td>1.34</td>
</tr>
<tr>
<td>$Q^{\exp}(5/2^+)$ ($e\text{fm}^2$)</td>
<td>14.02(10)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Q^{\th}(5/2^+)$ ($e\text{fm}^2$)</td>
<td>15.44</td>
<td>14.44</td>
<td>14.20</td>
<td>14.21</td>
<td>11.3</td>
<td>20.4</td>
</tr>
</tbody>
</table>

Fig. 3. (a) Rough scan with $\nu_{rf} = (450 \pm 35)$ kHz and $B_{rf} = 8$ G. (b) Fine scan with $\nu_{rf} = (1000 \pm 12)$ kHz and $B_{rf} = 10$ G. The difference in the magnetic field range is caused by the choice of a different rf-frequency. The gray bar indicates the mean of the experimental asymmetries in the base line.
ing number of neutrons (Table 1). This suggests that the deformation which is naturally inherent to Al-isotopes, having five valence protons, is predicted to persist and to even slightly decrease towards $N = 18$, provided this ground state is not influenced by the intrusion mechanism. This is supported by the fact that calculations allowing two neutrons being excited from $sd$ to $pf$ shells [15] predict the lowest $5/2^+$ intruder state in $^{31}$Al at about 4.7 MeV, thus its admixture to the ground state is expected to be small. Such a behaviour is opposite to the trend for deformation in neutron-rich Na isotopes, the only available experimental data from that region [10], where a drastic increase of the quadrupole moments already for $N = 18$ is observed, contrary to the predictions of the $sd$-shell model. The reason for the onset of deformation in the ($Z = 11$) Na-isotopes, and possible persistence of a moderate deformation in Al-isotopes, is the stabilizing role of the protons for the Al nuclei ($Z = 13$) against the intrusion mechanism [15,17].

It remains thus to be verified experimentally, whether the trend in the Al-isotopes indeed behaves as expected from the $sd$-shell model. The quadrupole and magnetic moments of $^{33}$Al seem to be the most sensitive probes to investigate whether $N = 20$ is a good shell closure for the Al-isotopes. Calculations allowing for two neutrons to be excited in the $pf$-shell [15] predict a low-lying intruder ($I$) $5/2^+$ state below 1 MeV, in addition to the spherical ($S$) $5/2^+$ ground state, obtained within the $sd$-shell model space. The two configurations have very different values for their nuclear moments, as shown in Table 1. A Monte Carlo Shell Model calculation performed in a $sd-f_{7/2}p_{3/2}$ space [17] predicts a mixed ground state $5/2^+$ with $g = 1.552$ and $Q(5/2^+) = 16.0$ $e$ fm$^2$. Experimental results on these nuclear moments, in particular the quadrupole moments, would permit to determine on the amount of intruder configuration mixing into the ground state wave function.

To summarize, we have measured the $g$-factor of the ground state of neutron-rich $^{31}$Al to be $|g| = 1.517(20)$. Comparison of the experimental $g$-factor to shell model calculations for the lowest lying states in a $sd$ model space using the USD interaction, allowed to unambiguously determine the spin and parity of the $^{31}$Al ground state as $I^π = 5/2^+$. This result confirms the validity of the $sd$-shell model description for the low-lying spectrum of neutron-rich Al-isotopes up to $N = 18$ and places $^{31}$Al well outside the “island of inversion”.

Fig. 4. Experimental low-lying spectra of Al-isotopes [34,35] in comparison to $sd$-shell model calculations with the USD interaction [33].
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Signals for black body limit in coherent ultraperipheral heavy ion collisions

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Abstract

We argue that study of total cross section of photoabsorption and coherent photoproduction of $\rho$, $\rho'$-mesons in ultraperipheral heavy ion collisions (UPC) is effective method to probe onset of black body limit (BBL) in the soft and hard QCD interactions. We illustrate the expected features of the onset of BBL using generalized vector dominance model. We show that this model describes very well $\rho$-meson coherent photoproduction at $6 \leq E_\gamma \leq 10$ GeV. In the case of $\rho$-meson production we find a UPC cross section which is a factor $\sim 1.5$ larger than the one found by Klein and Nystrand. The advantages of the process of coherent dijet production to probe onset of BBL in hard scattering regime where decomposition over the twists becomes inapplicable are explained and relative importance of the $\gamma +$ pomeron and $\gamma + \gamma$ mechanisms is estimated. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

Studies of the coherent interactions of photons with nucleons and nuclei were one of the highlights of the strong interaction studies of the seventies, for the excellent summaries see [1,2].

The fundamental question which one can investigate in the coherent processes is how interactions change for different types of projectiles with increase of the size/thickness of the target. Several regimes appear possible. In the case of a hadronic projectile (proton, pion, etc.) high-energy interactions with the nucleus rather rapidly approach a black body limit (BBL) in which the total cross section of the interaction is equal to $2\pi R^2_A$. Another extreme limit is the interaction of small size projectiles (or wave packages). In this case at sufficiently high energies the system remains frozen during the passage through the nucleus and the regime of color transparency is reached in which the amplitude of interactions is proportional to the gluon density of the nucleus which is somewhat smaller than the sum of the nucleon gluon densities due to the leading twist nuclear shadowing. In this regime the cross section of interaction rapidly grows with energy reflecting the fast increase of the gluon densities at small $x$ and large $Q^2$ and it may reach ultimately the black limit of interaction from the perturbative domain. (This limit can correspond to quite differ-
ent perturbative QCD dynamics, in particular, it could be reached already at $x \approx 10^{-3}$ where ln $x$ effects are a small correction.) The BBL for the interaction of the small size dipoles with heavy nuclei represents a new regime of interactions when the leading twist approximation and therefore the whole notion of the parton distributions becomes inapplicable for the description of hard QCD processes in the small $x$ regime. Obviously there should exist also many cases when the projectile represents a superposition of configurations of different sizes (leading to fluctuations of the strength of interaction).

In this respect interactions of photons with heavy nuclei provide unique opportunities since the photon wave function contains both the hadron-like configurations (vector meson dominance) and the direct photon configurations (small $q\bar{q}$ components). The important advantage of the photon is that at high energies the BBL is manifest in diffraction into a multitude of the hadronic final states (elastic diffraction $\gamma \rightarrow \gamma$ is negligible) while in the hadron case only elastic diffraction survives in the BBL and details of the dynamics leading to this regime remain hidden. Spectacular manifestations of BBL in (virtual) photon diffraction include strong enhancement of the large mass tail of the diffractive spectrum as compared to the expectations of the triple pomeron limit, large cross section of the double Pomeron, and manifestations of BBL in (virtual) photon diffraction.

At the same time even at small $x$ some components are still small enough, so that they interact with a small cross section—for these components the color transparency still survives. Hence one needs smaller $x$ to reach the BBL than allowed by the cutoff in the integral in Eq. (1). This $x$-range was not reached so far experimentally in $ep$ collisions.

It is worth emphasizing that the hypothesis of BBL corresponds to the assumption that at sufficiently small $x$ partons with large virtuality interact with heavy nuclei without any suppression with a cross sections $\approx 2\pi R_A^2$. It is this feature of the BBL which is responsible for the gross violation of the Bjorken scaling and for the above mentioned qualitative difference of the energy dependence of $\sigma_{tot}$ and $\sigma_{tot}^{hA}$.

One of striking features of the BBL regime is the suppression of nondiagonal transitions in the photon interaction with heavy nuclei [14]. In the BBL the dominant contribution to the coherent diffraction originates from “a shadow” of the fully absorptive interactions at impact parameters $b \leq R_A$ and hence the orthogonality condition is applicable.

In Ref. [15] it was assumed that one can neglect interference effects for a nucleon target also. In this case in order to preserve the Bjorken scaling one has to make an assumption that the cross section of the interaction of heavy mass configurations with nucleons decreases $\propto 1/M^2$. 

\[ F_T(x, Q^2) = \int_{m_0^2}^{2q_0/R_A} dM^2 \frac{2\pi R_A^2}{12\pi^5} \frac{Q^2M^2\rho(M^2)}{(M^2 + Q^2)^2}, \] (1)

where $q_0 = \omega_T$ is the photon energy, $m_0^2 \approx m_T^2$. The upper cutoff in the integral in the black body limit formulae (the Gribov approximation) comes from the nucleus form factor

\[ -I_{\min} R_A^3/3 \approx \left( \frac{M^2 + Q^2}{2q_0} \right)^2 R_A^3/3 \ll 1. \] (2)

The distinctive feature of Eq. (1) is that the contribution of large masses in the wave function of projectile photon (a direct photon contribution) is not suppressed. Consequently, Eq. (1) leads to $\sigma_{tot}^{hA} \propto 2\pi R_A^2 \alpha_{em} \ln(2q_0/R_A m_0^2)$ for $A \gg 1$ (this is qualitatively different from the hadron case where $\sigma_{tot}^{hA} \approx 2\pi R_A^2$), and grossly violates expectations of the Bjorken scaling for the $Q^2$ dependence of $\sigma_{tot}^{hA}$.

To overcome this puzzle J. Bjorken suggested a long time ago the aligned jet model in which only $q\bar{q}$ pairs with small $p_t$ can interact while high $p_t$ configurations in the photon wave function remain sterile [9]. Existence of sterile states has been explained later as due to the color transparency phenomenon [10]. More recently it was understood that states which behave as sterile at moderate energies, may interact at high enough energies with a hadron target with cross sections comparable to that for soft QCD phenomena.

Thus the Gribov’s assumptions are justified in QCD for the interaction of a range of hadronic components of the photon wave function with heavy nucleus target. At the same time even at small $x$ some components are still small enough, so that they interact with a small cross section—for these components the color transparency still survives. Hence one needs smaller $x$ to reach the BBL than allowed by the cutoff in the integral in Eq. (1). This $x$-range was not reached so far experimentally in $ep$ collisions.

It is worth emphasizing that the hypothesis of BBL corresponds to the assumption that at sufficiently small $x$ partons with large virtuality interact with heavy nuclei without any suppression with a cross sections $\approx 2\pi R_A^2$. It is this feature of the BBL which is responsible for the gross violation of the Bjorken scaling and for the above mentioned qualitative difference of the energy dependence of $\sigma_{tot}$ and $\sigma_{tot}^{hA}$.
Very little is known experimentally so far about coherent photon induced diffractive phenomena due to the problems of separating events where nucleus remained intact in the fixed target experiments and absence of electron–nucleus colliders. New opportunities for the investigation of photon–nucleus interactions become available in ultraperipheral collisions (UPC) of heavy nuclei at RHIC and LHC. These studies will allow to extract the cross section of the UPC of heavy nuclei at RHIC and LHC. These opportunities for the investigation of photon–nucleus interactions up to \( \sqrt{s} \sim 60(15) \) GeV (LHC/RHIC) due to a possibility to select the events where colliding nuclei remain intact or nearly intact, see, e.g., [4,5], see Refs. [6,7] for the recent reviews and extensive lists of references. Recently we investigated possibilities of studying color transparency and perturbative color opacity related to the leading twist gluon shadowing in \( J/\psi \) UPC and commented on the onset of BBL for \( J/\psi \) production [8].

In this Letter we will continue studies of the UPC phenomena. Our aim is to evaluate pattern of soft QCD phenomena in the proximity to black body limit, disappearance of color transparency phenomenon in the hard processes with increase of energies. We will study photoproduction of \( \rho \)-mesons and the \( I = 1 \) mesonic states with masses \( 1.5 \leq M^2 \leq 4 \) GeV\(^2\) usually generically referred to as a \( \rho^\prime \)-meson in the processes: \( \gamma + A \rightarrow V + A, A + A \rightarrow A + A + V; V = \rho, \rho^\prime \). To visualize expected new phenomena we will use generalized vector dominance model which takes into account fluctuations of the interaction strength to show that relative yield of \( \rho \) and \( \rho^\prime \)-mesons is sensitive to the onset of BBL physics in soft regime. We will argue that the production of two jets in the process \( A + A \rightarrow A + A + 2 \) jets in collisions of heavy nuclei provides a new effective method of probing the onset of BBL for the hard QCD phenomena.

2. Vector meson production off nuclei in the generalized vector dominance model

In this section we will use generalized vector dominance model to describe coherent photoproduction of hadronic states of \( M \leq 2 \) GeV off nuclei.

The vector dominance model (VDM) [11] was first suggested as an explanation of the nuclear shadowing in the interactions of photons with nuclei [12] in a close connection with the Bell discussion of the shadowing in neutrino–nucleus scattering [13]. It was also pointed out in [12] that at sufficiently high energies heavier states may become important. Importance of extending VDM to include the heavy mass states—Generalized VDM (GVDM) was further emphasized and explored in the late sixties [14,15]. In particular, one needs large mass states to explain the slope of \( Q^2 \) dependence of structure functions at small \( Q^2 \leq 1/(1 + Q^2/0.71) \) behavior instead of \( 1/(Q^2 + m^2_\rho)^2 \) predicted by the VDM.

The main ambiguity in such an extension was the issue of nondiagonal transitions where a photon initially converts into one vector state \( V_i \) which through diffractive interactions with a nucleon converts into another state \( V_j \). Such amplitude would interfere with the process of direct production of \( V_j \). Such nondiagonal transitions were introduced in a number of GVDM models [16,17]. Physically the importance of such transitions could be justified on the basis of the interpretation of the early Bjorken scaling for moderately small \( x \sim 10^{-2} \) as due to the color transparency phenomenon—presence in the virtual photon of hadron type and point-like type configurations [10]. Presence of nondiagonal transitions is also crucial for ensuring a quantitative matching with perturbative QCD regime for \( Q^2 \leq \) few GeV\(^2\) [18]. Hence it is reasonable to use GVDM for the modeling of the production of the light states off nuclei.

The amplitude of the vector meson production off a nucleon can be written within the GVDM as

\[
A(\gamma + N \rightarrow V_j + N) = \sum_i \frac{e}{f_{V_i}} A(V_i + N \rightarrow V_j + N),
\]

where \( f_{V_i} \) are connected to the width of decay of the corresponding resonance in the process \( e^+e^- \rightarrow \) hadrons. In the case of nuclei calculation of the amplitude of the Glauber scattering with production of a meson \( V \) requires taking into account both the nondiagonal transitions due to the transition of the photon to a different meson \( V' \) in the vertex \( \gamma \rightarrow V' \) and due to change of the meson in multiple rescatterings like \( \gamma \rightarrow V \rightarrow V' \rightarrow V \). This physics is equivalent to inelastic shadowing phenomenon familiar from hadron–nucleus scattering [19]. The Glauber model for the
description of these processes is well known, so, we present here only the basic formulæ which we will use to calculate the photoproduction cross section\(^2\):

\[
d\sigma_{\gamma A \rightarrow V A(t)} = \pi \int_0^\infty J_0(p_1b) \Gamma(b) db db^2.
\]

(4)

Here \(J_0(z)\) is the Bessel function, \(p_t = \sqrt{m^2 - t}\), 
\(-t_{\text{min}} = \frac{M^2}{4t_0}\) is longitudinal momentum transfer in \(\gamma - V\) transition, and \(\Gamma(b)\) is the nuclear profile function which is obtained in impact parameter space from the solution of the coupled multichannel Glauber equations for production of vector mesons \(\rho, \rho'\) which takes into account the finite coherence length effects due to the longitudinal momentum transfers (see, e.g., [20] for the explicit expressions).

In Ref. [20] the simplest nondiagonal model (which is a truncation of a more general model [16,17]) was considered with two states \(\rho\) and \(\rho'\) which have the same diagonal amplitudes of scattering off a nucleon and the fixed ratio of coupling constants

\[
f_{\rho'}/f_{\rho} = \sqrt{3},
\]

(5)

while the ratio of the nondiagonal and diagonal amplitudes

\[
A(\rho + N \rightarrow \rho' + N) = -\epsilon.
\]

(6)

and the value \(\sigma^{\text{tot}}_{\rho N}\) were found from the fit to the forward \(\gamma + A \rightarrow \rho + A\) cross sections measured at \(\omega_{\gamma} = 6.1, 6.6\) and 8.8 GeV [21]. It was pointed out that this model with reasonable values of \(\sigma^{\text{tot}}_{\rho N}\) and \(\epsilon\) allows to bring the value of \(f_\rho\) determined from the photoproduction of \(\rho\)-mesons off protons assuming approximate equality of the cross sections of \(\rho - N\) and \(\pi - N\) interactions into a good agreement with the \(e^+ e^-\) data thus removing a long standing 20% discrepancy between two determinations. One should emphasize here that in the logic of GVDM \(\rho'\)-meson approximates the hadron production in the interval of hadron masses \(\Delta M^2 \sim 2\text{ GeV}^2\), so the values of the production cross section refer to the corresponding mass interval.

As a first step we shall refine the model and then compare it with more detailed experimental data. First of all we diminish the dependence on the nuclear structure parameters by calculating the nuclear densities in the Hartree–Fock–Skyrme (HFS) approach. This model not only provided an excellent description (with an accuracy \(\approx 2\%\)) of the nuclear root mean square radii and the binding energies of spherical nuclei along the periodical table from carbon to uranium [22] but also was successfully used to describe in the Glauber approximation such detailed characteristics of the nuclear structure as the shell model momentum distributions in the high energy \((p, 2p)\) [23] and \((e, e' p)\) [24] reactions. Next, we fixed the values of the total cross section of the \(\rho N\) interaction and \(\eta_{\rho N} = \frac{\text{Re} A_{\rho N}}{\text{Im} A_{\rho N}}\) using the corresponding parameterizations suggested in the Landshoff–Donnachie model [25]. Accounting for the nondiagonal \(\rho - \rho'\) transitions the value of \(\epsilon\) was looked for to provide a best fit to the differential cross section of the \(\rho\)-meson photoproduction off lead at \(\omega_{\gamma} = 6.2\) GeV and \(p_t^2 = 0.001\) GeV\(^2\). As a result (Fig. 1(a)) we found \(\epsilon = 0.18\) which is indeed very close to the lower end of the range \(\epsilon = 0.2–0.28\) suggested in [20]. Note that this value leads to a suppression of the differential cross section of the \(\rho\)-meson production in \(\gamma + p \rightarrow \rho + p\) by a factor of \((1 - \epsilon/\sqrt{3})^2 \approx 0.80\) practically coinciding with phenomenological renormalization factor \(R = 0.84\) introduced in [25] to achieve the best fit of the elementary \(\rho\)-meson photoproduction forward cross section in the VDM which neglects mixing effects.

With all parameters fixed we calculated the differential cross sections of \(\rho\)-production off nuclei and found a good agreement (Figs. 1(b)–(f)) with available data [21].

A rather small systematic discrepancy with the data at \(p_t^2 \approx 0.01\) GeV\(^2\) appears to be due to the incoherent \(\rho\) photoproduction which is strongly suppressed for the very small \(p_t\) but gives a contribution comparable to the coherent one for \(p_t^2 \approx 0.01\) GeV\(^2\).

We have also checked the description of the \(A\)-dependence for the forward \(\rho\) photoproduction cross section (Fig. 2). In difference from Ref. [25] we

\(\text{\footnotesize \(2\) In this calculation we neglect the triple pomeron contribution which is present at high energies. This contribution though noticeable for the scattering off the lightest nuclei becomes a very small correction for the scattering of heavy nuclei due the strongly absorptive nature of interaction at the central impact parameters.} \)
did not find any evidence for an increase of $\epsilon$ by almost 50% (from 0.2 to 0.28) when the energy of photons is increased from 6.2 GeV up to 8.8 GeV.

As far as we know previously this important check of the Glauber model predictions in the vector meson production off $A > 2$ nuclei has never been performed in such self-consistent way. In view of a good agreement of the model with the data on $\rho$-meson production in the low energy domain we will use this model to consider the $\rho$ meson photoproduction at higher energies of photons. The increase of the coherence length with the photon energy leads to a qualitative difference in the energy dependence of the coherent vector meson production off light and heavy nuclei (Fig. 3) and to a change of the $A$-dependence for the ratio of the forward $\rho'$ and $\rho$-meson production cross sections between $\omega_\gamma \sim 10$ GeV and $\omega_\gamma \sim 50$ GeV (Fig. 4). The observed pattern reflects the difference of the coher-
ence lengths of the $\rho$-meson and a heavier $\rho'$-meson which is important for the intermediate photon energies $\lesssim 30$ GeV.

Unfortunately no experimental data are available at the moment on the coherent $\rho'$ photoproduction and on the $\rho$ photoproduction at energies $\geq 10$ GeV. Such studies may be possible with the HERMES detector at DESY and in the E-160 experiment at SLAC. On the other hand, a very promising way to collect such data would be a study of the coherent light vector meson production in the ultraperipheral ion collisions (UPC) at RHIC and LHC where one can explore the wide range of the quasi-real photon energies.

3. Vector meson production in ultraperipheral collisions

Production of vector mesons in ultraperipheral heavy ion collisions can be expressed in the Weizsacker–Williams (WW) approximation through the cross section of the vector meson production in $\gamma A$ scattering

$$d\sigma_{AA \rightarrow AA V} \over dy = 2 \int d\hat{b} T_{AA}(\hat{b}) n(\hat{b}, y) \sigma_{\gamma A \rightarrow V A(y)}. \quad (7)$$

Here $y$ is rapidity of the produced vector meson, $T_{AA}(\hat{b})$ is the thickness function of colliding nuclei on the impact parameter $\hat{b}$, $n(\hat{b}, y)$ is the flux of photon with energy

$$\omega = m_V e^y \over 2$$

emitted by one of nuclei and $\sigma_{\gamma A \rightarrow V A(y)}$ we calculated integrating the Eq. (4) over the momentum transfer in the range $t_{\text{min}} \leq t \leq \infty$.

As we discussed in Section 2, the GVDM with the value of $f_\rho$ fixed to the value determined from the $e^+e^-$ annihilation gives a better description of the cross section of the coherent $\rho$ production from nucleons. We also demonstrated that it gives a very good description of the absolute cross section and $t$-dependence of the cross section of the $\rho$-meson photoproduction off nuclei. Hence it is natural to expect that it would provide a reliable prediction for production of vector mesons in UPC. In particular, we calculated within this model coherent cross sections of both the $\rho$ and $\rho'$ mesons. The inelastic diffractive contribution is expected to be rejected using the veto from Zero Degree Neutron Calorimeter which is implemented in the RHIC experiments and is planned
for the LHC. This veto is the least effective for the single inelastic diffraction as this process will often result in the events where one nuclear proton is removed and the residual nucleus remains in the ground or a low excitation state. Our calculation of the single inelastic diffraction shown in Fig. 5 by dotted lines demonstrates that this background is very small for a wide range of central rapidities.

The results of our calculations for the total cross sections are given in Table 1. It should be emphasized that we have got the cross sections of the coherent \( \rho \) production considerably larger than estimates in Ref. [26] where the first quantitative study of the coherent \( \rho \)-meson production in kinematics of the peripheral ion collisions at RHIC and LHC was presented. In [26] as well as in [27] the cross section was calculated as:

\[
d\sigma_{\gamma^* A \rightarrow V + A} = \frac{\sigma_{\text{em}}}{f_\rho^2} \sigma_{\text{tot}}^2 (\rho A) \int_{t_{\text{min}}}^{\infty} dt F^2_A (t),
\]

where \( F_A (t) \) is the nuclear form factor. Further it was assumed in [26] that \( \sigma_{\text{tot}} (\rho A) \) is given by the classical mechanics formula:

\[
\sigma_{\text{tot}} (\rho A) = \int d^2 \vec{b} \left[ 1 - \exp\left(-\sigma_{\text{tot}} (\rho N) T (\vec{b})\right) \right],
\]

where \( T (\vec{b}) \) is the usual thickness function. It is easy to estimate that this formula leads to a substantially smaller value of the total cross section than the quantum mechanical Glauber expression

\[
\sigma_{\text{tot}} (\rho A) = 2 \int d^2 \vec{b} \left[ 1 - \exp\left(-\frac{\rho N}{2} T (\vec{b})\right) \right]
\]

a factor of two smaller for heavy enough nuclei: \( \sigma_{\text{tot}} (\rho A) \approx \pi R^2_A \) instead of \( 2 \pi R^2_A \). To show explicitly the difference in results we compare in Fig. 6 the rapidity distributions obtained in the VDM + Glauber model with correct accounting for the longitudinal momentum transfer but without nondiagonal terms (solid line) and result of calculations (dashed line) with the same parameters [25] and the HFS nuclear form factor but in the model based on Eqs. (8), (9) used in [26].

---

**Table 1**

<table>
<thead>
<tr>
<th></th>
<th>Au+Au at RHIC</th>
<th>Pb+Pb at LHC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coherent ( \rho )</td>
<td>934 mb</td>
<td>9538 mb</td>
</tr>
<tr>
<td>Coherent ( \rho' )</td>
<td>133 mb</td>
<td>2216 mb</td>
</tr>
<tr>
<td>Incoherent ( \rho )</td>
<td>201 mb</td>
<td>846 mb</td>
</tr>
</tbody>
</table>

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Fig. 5. Rapidity distributions for the light vector meson production at RHIC and LHC.

Fig. 6. Comparison of the rapidity distributions calculated in VDM + classical mechanics formula for total cross section (dotted line) with calculations within VDM + Glauber model (solid line).
In the follow up paper [27] authors considered $p_t$ distribution of the produced vector mesons and made an interesting observation that the amplitudes of the production of a vector mesons produced when a left moving nucleus emits the photon and when right moving nucleus emits a photon should destructively interfere. Due to the condition that essential impact parameters in $AA$ collisions are larger than $2R_A$ a significant interference occurs only for $p_t \leq 1/2R_A$ corresponding to $p_t \leq 10$ MeV [27]. This $p_t$ range constitutes a small fraction of whole permitted phase volume and hence the interference effects can be neglected for the case of the cross sections integrated over $p_t$ which were required to calculate the rapidity distributions presented here (Fig. 3).

In the case of $\rho$ production corrections due to nondiagonal transitions are relatively small ($\sim 15\%$) for the case of scattering off a nuclei. As a result we find that the GVD cross section is close to the one calculated in the VD model for heavy nuclei as well.

Situation is much more interesting for $\rho'$ production. In this case cross section of production of $\rho'$ off a nucleon is strongly suppressed as compared to the case when the $\rho \leftrightarrow \rho'$ transitions are switched off. The extra suppression factor is $\approx 0.5$.

In accordance with the general argument of Gribov the nondiagonal transitions disappear in the limit of large $A$ (black body limit) due to the condition of orthogonality of hadronic wave functions [14]. Hence we expect that in the limit of $A \to \infty$:

$$\frac{d\sigma(\gamma + A \to V_1 + A) / dt}{d\sigma(\gamma + A \to V_2 + A) / dt} \bigg|_{A \to \infty} = \left( f_2/f_1 \right)^2. \quad (10)$$

In reality the $\rho$-meson is a broad resonance which also interferes with the nonresonance $\pi^+\pi^-$ continuum, and $\rho'$ represents a set of overlapping resonances and continuum. Also the detectors are likely to be able to detect only some of the final states. Hence it is convenient to use a more general relation for the productions of states $h_1, h_2$ of invariant masses $M_1^2, M_2^2$:

$$\frac{d\sigma(\gamma + A \to h_1 + A) / dt}{d\sigma(\gamma + A \to h_2 + A) / dt} \bigg|_{A \to \infty} = \frac{\sigma(\epsilon^+\epsilon^- \to h_1)}{\sigma(\epsilon^+\epsilon^- \to h_2)}. \quad (11)$$

Indeed we have found from calculations that in the case of the coherent photoproduction off lead the nondiagonal transitions becomes strongly suppressed with increase of the photon energy. As a result the $\rho'/\rho$ ratio increases, exceeds the ratio of the $\gamma p \to Vp$ forward cross sections calculated with accounting for $\rho-\rho'$ transitions already at $\omega_\gamma \geq 50$ GeV and becomes close to the value of $f_2^V/f_1^V$ which can be considered as the limit when one can treat the interaction with the heavy nucleus as a black one. The same trend to BBL is seen from $A$-dependence presented for kinematics at LHC corresponding the value of energy $W_{\gamma p} = 60$ GeV (Fig. 7).

It is worth noting here that presence of nondiagonal transitions which in terms of the formalism of the scattering eigenstates [28] corresponds to the fluctuations of the interaction cross section leads to a substantial modification of the pattern of the approach to BBL. For example, if one would neglect nondiagonal transitions one would have to reduce both $\rho - N$ and $\rho' - N$ cross sections in order to keep the values of the production cross sections in $\gamma + p \to \rho + p$ the same as in the considered GDVM. For the $\rho$-meson the reduction effect is a small correction $(1 - \epsilon/\sqrt{3}) \approx 0.9$, while the cross section of $\rho' - N$ interaction is reduced by a substantially larger factor $(1 - \sqrt{3}\epsilon) \approx 0.7$. This would lead to a noticeable reduction of the total cross section of the $\rho' - A$ interaction as compared to the BBL value of $2\pi R_A^2$ and reduces the $\rho'/\rho$ ratio for $A \sim 200$

![Fig. 7.](image)

Fig. 7. (a) Energy dependence of the ratio of $\rho'$ and $\rho$-meson production cross sections, (b) $A$-dependence of the ratio of $\rho'$ and $\rho$-meson production forward cross sections in kinematics at LHC.
by ≈ 10% as compared to reduction by a factor 0.9 in the original model. At the same time in a number of GVD models it is assumed that $\sigma_{\text{tot}}(\gamma N) \propto 1/M^2$. In such a model the $\rho/\rho$ ratio for Pb would be reduced by a factor $\sim 3$.

The general BBL expression for the differential cross section of the production of the invariant mass $M^2$ [3] is

$$\frac{d\sigma_{\gamma A \rightarrow M^2 + A}}{dt \, dM^2} = \frac{\alpha_{\text{em}} (2\pi R_A^2)^2 \rho(M^2)}{16\pi} \frac{4 |J_1(\sqrt{-t} R_A)|^2}{-t R_A^2}. \quad (12)$$

Hence by comparing the extracted cross section of the diffractive production of states with certain masses with the black body limit result—Eq. (12) one would be able to determine up to what masses in the photon wave function interaction remains black. Onset of BBL limit for hard processes should reveal itself also in the faster increase with energy of cross sections of photoproduction of excited states with that for ground state meson. It would be especially advantageous for these studies to use a set of nuclei—one medium range like Ca and another heavy one—one could remove the edge effects and use the length of about 10 fm of nuclear matter.

Note in passing that an interesting change of the low-mass dipion spectrum is expected in the discussed limit. It should be strongly suppressed as compared to the case of scattering off proton where nonresonance continuum is much larger than in $e^+e^- \rightarrow \pi^+\pi^-$ process.

### 4. Diffractive dijet production

For the $\gamma A$ energies which will be available at LHC one may expect that the BBL in the scattering off heavy nuclei would be a good approximation for the masses $M$ in the photon wave function up to few GeV. This is the domain which is described by perturbative QCD for $x \sim 10^{-3}$ for the proton targets and larger $x$ for scattering off nuclei. The condition of large longitudinal distances—small longitudinal transfer will be applicable in this case up to quite large values of the produced diffractive mass (though it will not hold for masses above 3 GeV or so at RHIC).

Really $x_{\text{eff}} = M^2/s_{\gamma N} = M/2E_N$ will be $\sim 10^{-3}$ for $M = 4$ GeV for $y = 0$. So that the condition $l_{\text{coh}} = 1/m_N x_{\text{eff}} \gg 2R_A$ is satisfied.

In the BBL the dominant channel of diffraction for large masses is production of two jets with the total cross section given by Eq. (12) and with a characteristic angular distribution $(1 + \cos^2 \theta)$, where $\theta$ is the c.m. angle [3]. On the contrary in the perturbative QCD limit the diffractive dijet production except charmed jet production is strongly suppressed [29,30]. The suppression is due to the structure of the coupling of the wave function of the real photon wave to two gluons when calculated in the lowest order in $\alpha_s$. As a result in the real photon case hard diffraction involving light quarks is connected to production of $q\bar{q}g$ and higher states. Thus the dijet photoproduction should be very sensitive to the onset of BBL regime.

Note that in the case of photon nucleon scattering at $\alpha_{\gamma} \sim 100$ GeV [31] the normalized differential

$$\frac{1}{\sigma_{\text{tot}}^N} \frac{1}{d^4q} \frac{d\sigma_{\gamma^2}}{dM^2}$$

for diffraction into large masses ($\gtrsim 2$ GeV) is very similar to that for the pion–nucleon scattering and appears to be dominated by the triple Reggeon limit corresponding to the process where a photon first converts to a $\rho$-meson and next a large mass is produced in the $\rho - N$ diffractive scattering. Since the triple pomeron coupling constant is quite small this process should be a small correction in the BBL. Besides in this limit the triple pomeron process is screened by the multiple pomeron exchanges and originates solely from the scattering off the rim of the nucleus. Hence it is suppressed at least by a factor $\sim A^{1/3}$ as compared to the process of direct diffraction into heavy masses.

A competing process for photoproduction of dijets off heavy nuclei is production of dijets in $\gamma - \gamma$ collisions where the second photon is provided by the Coulomb field of the nucleus. Note that the dijets produced in this process have positive $C$-parity and hence this amplitude does not interfere with the amplitude of the dijet production in the $\gamma p$ interaction which have negative $C$-parity.

For the calculation of the cross section of dijet production in $\gamma - \gamma$ collisions we use the lowest order perturbative QCD result which coincides up to the number of colors factor and summation over the quark
flavors with the well known QED result for the lepton pair production in $\gamma\gamma$ collisions:

$$\frac{d\sigma(\gamma + \gamma \rightarrow \text{jet} + \text{jet})}{d\Omega} = 3 \sum_i e_i^4 \alpha_{em}^2 \frac{1}{M^2} \left[ \frac{2}{\sin^2 \theta} - 1 \right]. \quad (13)$$

Here the sum over the quark flavors goes over quarks with $m_q \ll M/2$ and $p_t^\text{jet}$ is sufficiently large to suppress nonperturbative contribution. Using the Weizsacker–Williams approximation we evaluate the ratio of the $\gamma\gamma$ and $\gamma^\prime\rho$ contributions to the dijet production in AA collisions in the BBL with the logarithmic accuracy:

$$R = \frac{d\sigma_{\gamma\gamma}(A + A \rightarrow \text{dijet} + A + A)}{d\sigma_{\gamma\prime\rho}(A + A \rightarrow \text{dijet} + A + A)} = \frac{\sum_i e_i^4}{\sum_i e_i^2} \frac{16Z^2\alpha_{em}^2}{M^2 R_A^2 \sin^2 \theta} \ln \frac{24\theta}{M^2 R_A}. \quad (14)$$

In the derivation of Eq. (14) we neglected a difference of the energy dependences of the processes. For the kinematics of interest (large $p_t$ of jets and region of produced masses $M \lesssim 3$ GeV) $\theta \approx 90^\circ$ in the center of mass of the produced system and we can account for three lightest flavors, hence

$$\sum_i e_i^4 = \frac{1}{3}. \quad (15)$$

One can easily see that $R \ll 1$ for production of high $p_t$ jets corresponding to $\sin \theta \sim 1$, and hence the $\gamma\gamma$ contribution can be safely neglected.

It is worth emphasizing that at the energies below the BBL, where diffraction of the photon to dijets can be legitimately calculated in the lowest order in $\alpha_s$ (cf. calculation of a similar process of dijet production in the pion–hadron scattering in Ref. [32]) the electromagnetic mechanism is much more prominent. It is enhanced by a factor $1/\alpha_s^2$ and becomes much more prominent with increase of $p_t$ of the jet. Also it enhanced for very small total momentum of the dijet system. Observation of the last effect is hardly feasible, cf. the above discussion of the vector meson production.

5. Conclusions

We demonstrated that ultraperipheral AA collisions is effective method of probing onset of BBL regime in hard processes at small $x$. We have demonstrated that the Glauber model predicts a significantly larger coherent $\rho$-meson production rates than the previous calculations. We predict a significant increase of the ratio of the yields of $\rho$, $\rho'$ mesons in coherent processes off heavy nuclei due to the blackening of the soft QCD interactions in which fluctuations of the interaction strength are present. An account of nondiagonal transitions leads to a prediction of a significant enhancement of production of heavier diffractive states especially production of high $p_t$ dijets. Study of these channels may allow to get an important information on the onset of the black body limit in the diffraction of real photons.

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Ground state properties of many-body systems in the two-body random ensemble and random matrix theory

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Abstract

We explore generic ground-state and low-energy statistical properties of many-body bosonic and fermionic one- and two-body random ensembles (TBRE) in the dense limit, and contrast them with random matrix theory (RMT). Weak differences in distribution tails can be attributed to the regularity or chaoticity of the corresponding Hamiltonians rather than the particle statistics. We finally show the universality of the distribution of the angular momentum gap between the lowest energy levels in consecutive \( J \)-sectors for the four models considered. © 2002 Elsevier Science B.V. All rights reserved.

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Wigner introduced the Gaussian orthogonal ensemble (GOE) to deal with the statistics of high-lying levels of many-body quantum systems [1]. Although the fluctuations of certain observables predicted by the GOE agree well with experimental observations, it represents a system in which all the particles interact simultaneously and this is a priori not appropriate to describe many-body systems in which the two-body interaction is predominant [2,3]. The two-body random ensemble (TBRE) was introduced to improve upon the physical limitations of RMT [4]. The TBRE Hamiltonian includes only up to two-body operators, whose coefficients are real random numbers. This ensemble reproduces a Gaussian level density and the GOE level repulsion as desired, but until recently, only the dilute limit was analytically tractable [5]. This limit corresponds to having a large number of particles \( N_p \gg 1 \) and an even larger number of single-particle states \( K \gg 1 \), so that \( N_p/K \ll 1 \). In such a limit Pauli’s principle has only a marginal effect, so particles statistics are unimportant. Many physical regimes lie outside the dilute limit: for bosonic particles, we may consider a large number of particles with a finite \( K \ll N_p \); similarly, fermionic antisymmetry becomes relevant as one approaches half-filling, \( N_p \approx K/2 \). Since we are interested in those aspects, we do not restrict ourselves to the dilute limit. It has only recently been shown, from the connection between the Lanczos tridiagonalization and random polynomials, that this case can also be treated analytically in some instances [6].
We investigate the statistical properties of the low-lying levels of interacting complex systems, extracting from the edge of the spectra, generic properties that would reflect the basic structure of many-body systems such as molecules, atomic nuclei or quantum dots close to, or in their ground-state. We compute the lowest eigenvalue distribution for two bosonic and two fermionic TBREs and show that, after a proper rescaling, the distributions have a surprisingly weak dependence on the nature of the particles. Comparing to predictions from RMT lead to surprising similarities. The fact that the distribution of the lowest energy state depends only weakly on the nature of the particles and not on the ensemble used to study it points to its universality, but a closer inspection of the tails of these distributions indicate that they depend on the chaoticity of the system analyzed. To understand the role of chaos, we study models which are regular as well as chaotic and find that the tails of the ground state distributions for chaotic systems are closer to that obtained from the large-\(N\) GOE, while for integrable systems it is closer to the \(N = 2\) GOE. The distribution of energy differences between the lowest energy levels in two consecutive angular momentum sector (\(J\) and \(J + 1\)) is found to be more robust, having the same shape in all four cases within numerical accuracy (deviations occur for the single \(j\)-shell model at small gap values). Since this distribution depends on correlations between Hilbert subspaces with different quantum numbers, it is outside the scope of RMT. We expect that this distribution is generic, a conclusion which is borne out by the bosonic nature of the corresponding excitation. Our analysis is based on numerical and analytical results. The analytical treatment is now possible not only for the TBRE [6], as already pointed out, but also for the large \(N\) limit of the GOE, for which an expression for the distribution function of the largest eigenvalue was obtained in terms of a particular Painlevé II function [7].

We start by summarizing analytical results obtained for the GOE. A general matrix \(H\) of this ensemble can be expressed in terms of a rotation matrix \(O\) and the diagonal eigenvalue matrix \(E = \text{diag}(E_1, E_2, \ldots, E_N)\) as \(H = O^T E O\). The distribution of the ground state energy is \(P(E_{\text{gs}}) = \langle \delta(E_{\text{gs}} - E_1) \rangle\), where \(\langle \cdots \rangle\) indicates GOE averaging and the probability of having a matrix element \(H_{ij}\) is

\[
P(H_{ij}) \propto \exp\left(-\frac{1}{2} \text{Tr} H^2\right).
\]

Two limits are considered: \(N = 2\) and \(N \rightarrow \infty\). For \(N = 2\), direct integration gives

\[
P^{\text{GOE}}_{2}(E_{\text{gs}}) = \frac{1}{2\sqrt{\pi}} \left\{ \exp(-E_{\text{gs}}^2) - \frac{\pi}{2} \exp(-E_{\text{gs}}^2/2) E_0\left[ \text{erfc}\left(\frac{E_{\text{gs}}}{\sqrt{2}}\right)\right]\right\}. \tag{1}
\]

This distribution is not readily calculated for arbitrary \(N\), however, in the limit \(N \rightarrow \infty\), Tracy and Widom [7] derived an expression for the distribution of the largest eigenvalue (\(f_1(s)\) [7]), which in terms of the lowest eigenvalue is written as

\[
P_{\infty}^{\text{GOE}}(E_{\text{gs}}) = \frac{1}{2}\left\{ \int_{-E_{\text{gs}}}^{\infty} q(x)^2 dx + q(-E_{\text{gs}}) \right\}
\]

\[
\times \exp\left[\left\{-\frac{1}{2} \left( \int_{-E_{\text{gs}}}^{\infty} (x + E_{\text{gs}})q(x)^2 dx \right) + \int_{-E_{\text{gs}}}^{\infty} q(x) dx \right\}\right]. \tag{2}
\]

Here \(q(E_{\text{gs}})\) satisfies the Painlevé II equation and the boundary condition is \(q(E_{\text{gs}}) \sim \text{Ai}(E_{\text{gs}})\) as \(E_{\text{gs}} \rightarrow \infty\), where \(\text{Ai}\) is the Airy function. The numerical solutions for this expression are plotted on the top panel of Fig. 1, where one sees that it agrees well with the distribution for a GOE of \(N = 200\). This function has the asymptotics

\[
\log P_{\infty}^{\text{GOE}}(E_{\text{gs}}) \sim -|E_{\text{gs}}|^3, \quad E_{\text{gs}} \rightarrow -\infty,
\]

\[
\log P_{\infty}^{\text{GOE}}(E_{\text{gs}}) \sim -E_{\text{gs}}^{3/2}, \quad E_{\text{gs}} \rightarrow \infty. \tag{3}
\]

In contrast to \(N = 2\), the tails deviate from Gaussian behavior. For completeness, we also show the average ground state energy for the GOE, although this quantity is not expected to have physical consequences. We find that

\[
\langle E_{\text{gs}} \rangle = 2.01(2)\sqrt{N} + 1.58(2)N^{-1/6},
\]
with the numerically obtained distribution for the theoretical estimate. The expression fits numerical results for \( N \). On the bottom panel of Fig. 1, we show that this agrees with the predicted functional form \([7, 8]\). In contrast to the GOE, the TBRE is not a purely statistical model, but rather is constructed in a particular bosonic or fermionic space. We take it in the form

\[
H = \sum_{ij} \epsilon_{ij} a_i^+ a_j + \sum_{ijkl} V_{ijkl} a_i^+ a_j^+ a_k a_l.
\]  

(4)

When originally introduced, the Hamiltonian operators of the TBRE had only (interaction) two-body terms, but we include the one-body term. \( a_i^+ (a_i) \) represents boson or fermion creation (annihilation) operator and the coefficients \( \epsilon_{ij} \) and \( V_{ijkl} \) are taken as Gaussian random variables once certain physical constraints are imposed such as rotation invariance, time reversal invariance and conservation of total spin, isospin and parity.

As already mentioned, some analytical results for bosonic \([5, 9]\) and fermionic \([3, 4]\) models in the dilute limit were obtained, but here we consider the non-dilute limit, for which little is known. We want to find out how this limit and some other model dependencies, such as the nature of the particles or the system’s integrability, may affect the statistical properties of many-body systems.

### Bosonic models

We study two bosonic models, the \( U(4) \) vibron and \( U(6) \) interacting boson models, used to explore molecular or nuclear collective excitations. In the \( U(4) \) TBRE model, bosons with \( J^π = 0^+(s, s^+) \) and \( J^π = 1^+(p, p^+) \) are coupled to form a scalar Hamiltonian \([10]\)

\[
H_{U(4)} = \left( \epsilon_s s^+ s + \epsilon_p p^+ \hat{\rho} \right) / N_p
+ \left\{ \frac{C_0}{2} \left[ p^+ p^+ \right]^{(0)} \left[ \hat{\rho} \hat{\rho} \right]^{(0)}
+ \frac{C_2}{2} \left[ p^+ p^+ \right]^{(2)} \left[ \hat{\rho} \hat{\rho} \right]^{(2)}
+ \frac{H_0}{2} \left[ s^+ s^+ \right]^{(0)} \left[ s \hat{s} \right]^{(0)}
+ \frac{H_1}{2} \left[ s^+ p^+ \right]^{(1)} \left[ s \hat{\rho} \right]^{(1)}
+ \frac{H_2}{2} \left[ s^+ s^+ \right]^{(0)} \left[ \hat{s} \hat{\rho} \right]^{(0)} + \text{h.c.} \right\}
\times \left[ N_p (N_p - 1) \right]^{-1}.
\]

(5)

The square brackets denote angular momentum coupling, the dots represent scalar products and \( N_p \) is the total number of bosons. Since the 1- and 2-body matrix elements are proportional to \( N_p \) and \( N_p (N_p - 1) \), respectively, scaling allows all coefficients in (5) to be Gaussian random numbers of unit variance. For the choice of coefficients above, the lowest eigenvalue distribution has been obtained analytically, and in the large \( N_p \) limit \([6]\)

\[
P(E_{gs}) \propto e^{-0.42E_{gs}^2} \text{erfc}(0.663 E_{gs})
+ 0.423 e^{-0.467 E_{gs}^2} \text{erfc}(0.632 E_{gs}).
\]

(6)

The second model we investigate is the nuclear \( U(6) \) model (IBM) which consists of scalar \( J^π = 0^+(s, s^+) \) and quadrupole \( J^π = 2^+(d, d^+) \) bosons coupled by one- and two-body interactions. The Hamiltonian has the form \([11]\)

\[
H_{U(6)} = \left( \epsilon_s s^+ s + \epsilon_d d^+ \hat{\delta} \right) / N_p
+ \left\{ \sum_{L=0,2,4} c_L \left[ d^+ d^+ \right]^{(L)} \left[ \hat{\delta} \hat{\delta} \right]^{(L)} \right\}
\]
We will consider \( N_p = 16 \), which is typical for a heavy collective nucleus, and the distribution for the ground state energy is obtained numerically. An important distinction between the \( U(4) \) and \( U(6) \) models is that the former is integrable for all parameters, while the latter is generally chaotic [12]. Chaos usually applies to fluctuation properties near the middle of the spectrum, rather than at the edges. Never the less, there is a subtle difference between these models which we attribute to the underlying chaos, as we discuss below. The ground states of these models also have well known phase transition behaviors. In spite of this, the ground state distributions will be seen to be surprisingly insensitive to this.

**Fermionic models**

The two fermionic models we investigate are a model for interacting nucleons in a single-\( j \) shell and a model for randomly interacting electrons. The single-\( j \) shell model consists of \( N_p \) identical fermions interacting with scalar, pairwise interactions \( V_{nn'} \) (for particles \( n \) and \( n' \)) in a degenerate multiplet of spin \( j \). We use \( N_p = 6 \) particles in the \( j = 15/2 \) shell. The Hamiltonian is in the form (4) with \( \varepsilon_{ij} = 0 \), since the interactions are two-body. Since the particles are identical, the sum over all pairwise interactions, \( \sum_{n<n'} V_{nn'} \), can be expressed in terms of a single two-body matrix element

\[
v_{j} = \langle j^2(J') J | V_{nn'} | j^2(J') J \rangle
\]

using coefficients of fractional parentage [13]. Note that the particle indices drop from the equation. Consequently, there is a large reduction in the number of distinct matrix elements, and the Hamiltonian (4) reduces to [13]

\[
\langle j^N \alpha J, \alpha' J | \sum_{n<n'} V_{nn'} | j^N \alpha' J, \alpha J \rangle = \frac{N_p(N_p - 1)}{2} \times \sum_{n<n'} \left[ j^N \alpha J \left[ \sum_{n<n''} j^{N_p-2}(\alpha'' J''_n) j^2(J') J \right] \times \left[ j^{N_p-2}(\alpha'' J''_n) j^2(J') J \right] \times j^N \alpha' J \right] v_{j'}.
\]

Here \( J \) is the total angular momentum, \( M \) is the \( z \)-projection, \( \alpha \) denotes all other quantum numbers, the term in brackets corresponds to the coefficients of fractional parentage. The matrix elements \( v_{j'} (J' = 0, 2, \ldots, 2j - 1) \) are Gaussian random numbers.

The second fermionic model is a model for randomly interacting \( j = 1/2 \) fermions (IEM) and is given by [15]

\[
H = \sum_{a} \varepsilon_{a} a \dagger_{a} a_{a} + \sum_{a b l n j} U_{a b}^{l n} \langle a \dagger_{a} b \dagger_{b} a_{l} a_{n} \rangle,
\]

where the one-body spectrum is Wigner–Dyson distributed with \( \varepsilon_{a} \in [-K/2; K/2] \) (\( K/2 \) gives the number of spin-degenerate orbitals, the average level spacing is then \( \Delta = 1 \)), \( U_{a b}^{l n} \in [-U; U] \) are Gaussian random numbers and \( j(j') = \dagger (\downarrow) \) are spin indices. We study up to \( N_p = 7 \) fermions on \( K/2 = 7 \) orbits.

While the single-\( j \)-shell model describes non-chaotic system [14], the spectral properties of the IEM depend on the ratio of the interaction strength to the one-body level spacing \( U/\Delta \). For \( U/\Delta \gg 1/(K N_p^2) \) the model is non-chaotic with a Poissonian spectrum while for \( U/\Delta \gtrsim 1/(K N_p^2) \) it is chaotic with a Wigner–Dyson distributed spectrum [16].

To compare the lowest eigenvalue distribution for these four models with the ones for the GOE, we re-center the distributions and rescale their widths by using the following scaling prescription

\[
\varepsilon_{gs} = \frac{E_{gs} - \langle E_{gs} \rangle}{\sigma_{Egs}}.
\]

This removes model dependencies associated with scales, level densities and so forth. We show in Fig. 2 distributions of the rescaled ground state energy \( \varepsilon_{gs} \) for...
Fig. 2. Distributions of the ground state energy for the vibron model (top left), the single $j$-shell model (top right), the IBM (bottom left) and the IEM (bottom right) for $N_p = 3$ (circles) and 4 (squares). In all panels, dashed and solid lines give the distribution for the $N = 2$ and $N = \infty$ GOE, respectively. The top panels correspond to non-chaotic systems, for which the data are well fitted by a $N = 2$ GOE. The bottom panels correspond to chaotic systems, where data are closer to the $N = \infty$ GOE.

Fig. 3. Crossover to quantum chaoticity for the IEM with $N_p = 4$ and $U/\Delta = 10^{-7}$ (diamonds), 0.05 (triangles), 1 (squares) and $\infty$ (circles). Symbols for data in the integrable regime are filled, while dashed and solid lines give the distribution for the $N = 2$ and $N = \infty$ GOE, respectively.

the four models described above. The figure splits into regular systems on the top two panels and chaotic ones on the two bottom panels. Bosonic models are left and fermionic ones are right. The chaoticity of the IEM depends on $U/\Delta$ and the data shown on the lower right panel of Fig. 2 for the IEM correspond to $U/\Delta = 1$, well into the chaotic regime [16].

One clearly sees that the ground state energies have approximately the same distribution for the four TBREs and the GOE of small and large dimension—this seems to be a robust property, and in particular, these distributions seem not to depend on the nature of the particles. Moreover, the agreement of the distributions with the ones obtained from the GOE demonstrates that the distribution of the ground state energy is also not sensitive to the type of ensemble used. A closer inspection of the presented data does reveal discrepancies between different models in the distribution tails. The lowest energy distribution for regular systems coincides better with the distribution for the two-dimensional GOE, while the distribution
for chaotic systems is in better agreement with the one obtained with a large-$N$ GOE. The GOE dimension necessary to describe a system can be associated with the integrability of the model. This conclusion is corroborated by the data shown on Fig. 3 for the IEM at various $U/\Delta$, both in the integrable and chaotic regimes. Evidently the crossover from Poisson to Wigner–Dyson distributed level spacing (i.e., from integrable to chaotic quantum dynamics) is accompanied by a crossover from two-dimensional to infinite-dimensional GOE fitting of the ground-state distribution. We also found that the ground-state distribution for the IEM exactly at half-filling ($K = 14$ and $N_p = 7$), gets close to a two-dimensional GOE, as shown at the top of Fig. 4. Since the associated spectrum is still Wigner–Dyson distributed, we conclude that this is a manifestation of the particle statistics. The only influence of the latter therefore is to single out half-filled fermionic systems which despite their chaotic character have a distribution of ground-state energies which is closer to the $N = 2$ GOE.

Consider now distributions which cannot be addressed through RMT, namely distributions which involve correlations between Hilbert subspaces with different quantum numbers. One quantity of fundamental interest is the energy gap. In the models we study, this typically involves states of different spin: for the vibron model it is

$$
\epsilon(J^\pi = 1^{-}) - \epsilon(J^\pi = 0^{+}),
$$

for the IBM

$$
\epsilon(J^\pi = 2^{+}) - \epsilon(J^\pi = 0^{+}),
$$

while for the single $j$-shell model and the IEM

$$
\epsilon(J = 1) - \epsilon(J = 0).
$$

We will call this energy difference the spin gap. The ground state is mostly dominated by $0^+$ (this predominance almost reaches 100% for the IEM [15]) and the lowest level with a different spin usually corresponding to the first excited state (this is however
not the case for the IEM), so we are looking at the very edge of the spectrum. Even though we have both bosonic and fermionic models, the distributions are very similar and also agree very well with the analytical expression derived for the vibron model [6] as it is evident from Fig. 5. The distribution of the spin gaps is thus also a generic property which we attribute to the bosonic nature (i.e., $\delta J$ is an integer) of the corresponding excitation in all four models studied here. This explains why the particle statistics have no influence here. Note that in the case of the IEM, universality of the spin gap requires a finite interaction strength $U/\Delta \gtrsim 1$. In the limit $U/\Delta \rightarrow 0$, the spin gap is determined by the one-body spectrum $\epsilon_\alpha$ and thus given by a GOE one- (two-) spacing distribution for $N_p$ even (odd).

In summary, investigating various TBREs, we have found that in spite of their fundamental differences, they have strikingly similar features in both their ground state energy distribution and their spin gap distribution. The ground state energy distributions have common features, and coincide to GOE distributions of either small or large dimensions, for regular or chaotic TBREs, respectively. We conclude that the edge of the spectra exhibit two unexpected generic properties: the distribution of the ground state energy and the distribution of the energy differences between the lowest levels with different spins. This latter point is presumably related to the apparent regular structure found in the angular momentum structure of low-lying levels for several TBREs [17].

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References

Unquenched domain wall quarks with multi-bosons

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Abstract

The numerical simulation of domain wall quarks with the two-step multi-boson (TSMB) algorithm is considered. The inclusion of single quark flavours, as required for strange quarks, is discussed. The usage of computer memory can be kept relatively low, independently of the order of polynomial approximations. Tests are performed with two flavours ($N_f = 2$) of degenerate quarks near the $N_t = 4$ thermodynamical cross over. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

Domain wall fermions [1–4] offer the possibility of improving chiral symmetry of lattice discretizations for fermionic theories by tuning the action parameters in an extra (fifth) dimension. It can be shown [5,6] that in the limit of vanishing lattice spacing in the fifth dimension the domain wall formulation is equivalent to the overlap formulation [7,8] which fulfills the Ginsparg–Wilson relation [9] for lattice chiral symmetry [10]. The price of the chiral symmetry at non-zero lattice spacing is the extra dimension enlarging the number of degrees of freedom and, from a technical point of view, the extensions of the fermion matrix. It is an interesting question how much numerical simulations with light domain wall quarks become slower than, say, with “unimproved” Wilson fermions.

Up to now unquenched domain wall fermions have been treated either by the Hybrid Monte Carlo [11] or, in case of an odd number of fermionic flavours, by the Hybrid Molecular Dynamics R-algorithm [12]. (For a few examples of these simulations see, for instance, [13–16]. For a recent review, see [17].) In the present Letter the application of the two-step multi-boson (TSMB) algorithm [18] for domain wall fermion simulations is considered. This algorithm is applicable for any number of fermion flavours and has a tolerable slowing down towards light fermions.

The plan of this Letter is as follows. In Section 2 the lattice action for two degenerate domain wall quarks is formulated. In Section 3 the TSMB algorithm is briefly recapitulated and the generalization to the case of an odd number of domain wall fermion flavours is discussed. In Section 4 the results of test runs on $8^3 \cdot 4$ lattices near the thermodynamical cross over are given.

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2. Lattice action

In this Letter the domain wall fermion action is constructed according to Shamir’s prescription [3]. Therefore, the fermion field is defined on a five dimensional hypercubic lattice and the light chiral fermion modes are located on two boundaries of the fifth dimension. The gauge field links depend only on the coordinates of the four-dimensional space–time. The bosonic Pauli–Villars fields subtracting the heavy fermion modes are introduced as in [14].

The complete lattice action is given by

\[ S = S_G[U] + S_F[\overline{\Psi}, \Psi, U] + S_{PV}[\Phi^\dagger, \Phi, U]. \] (1)

Here the standard Wilson action \( S_G \) for the \( SU(N_c) \) gauge field is a sum over plaquettes

\[ S_G = \beta \sum_{pl} \left( 1 - \frac{1}{N_c} \text{Re} \text{Tr} U_{pl} \right). \] (2)

with the bare gauge coupling given by \( \beta \equiv 2N_c/g^2 \). In particular, in QCD the number of colours is \( N_c = 3 \). The Pauli–Villars action in (1) \( S_{PV} \) will be discussed later and the fermion action \( S_F \) for a single fermion flavour is given by

\[ S_F = \sum_{x,x',s,s'} \overline{\Psi}(x',s') D_F(x',s';x,s) \Psi(x,s). \] (3)

The four-dimensional space–time coordinates are denoted by \( x, x' \) and the fifth coordinates are \( s, s' \). The domain wall fermion matrix \( D_F \) is constructed from the standard four-dimensional Wilson fermion matrix

\[ D(x',x) = \delta_{x,x'} (4 - am_0) - \frac{1}{2} \sum_{\mu=1}^{4} \left[ \delta_{x',x+\hat{\mu}} (1 + \gamma_\mu) U_{x\mu} + \delta_{x',x} (1 - \gamma_\mu) U_{x\mu}^\dagger \right]. \] (4)

The notations are standard: \( a \) is the (four-dimensional) lattice spacing and \( \hat{\mu} \) denotes the unit vector in direction \( \mu \). The bare mass \( -m_0 \) is chosen to be negative and should be tuned properly for producing the light boundary fermion state. In an \( s \)-block notation the domain wall fermion matrix is

\[ D_F = \begin{pmatrix} \sigma + D & -\sigma P_L & 0 & 0 & \cdots & 0 & 0 & am_f P_R \\ -\sigma P_R & \sigma + D & -\sigma P_L & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\sigma P_R & \sigma + D & -\sigma P_L & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\sigma P_R & \sigma + D & -\sigma P_L \\ am_f P_L & 0 & 0 & 0 & \cdots & 0 & -\sigma P_R & \sigma + D \end{pmatrix}. \] (5)

Here \( m_f \) denotes the bare fermion mass of the light boundary fermion, \( P_R = \frac{1}{2}(1 + \gamma_5) \), \( P_L = \frac{1}{2}(1 - \gamma_5) \) are chiral projectors and \( \sigma \equiv a/\alpha_s \) determines the lattice spacing in the fifth dimension \( \alpha_s \) relative to \( a \) [19].

The domain wall fermion matrix \( D_F \) is non-hermitean but it satisfies the relation

\[ D_F^\dagger = \gamma_5 R_5 D_F \gamma_5 R_5, \] (6)

with the reflection in the fifth dimension \( (R_5)_{x',s} = \delta_{x',N_s+1-s} \ (1 \leq s \leq N_s) \). This relation implies that the determinant of \( D_F \) is real and \( \tilde{D}_F = \gamma_5 R_5 D_F \) is hermitean. Using the hermitean Wilson fermion matrix \( \tilde{D} = \gamma_5 D \).
the hermiticity of $\tilde{D}_F$ in nicely displayed in an $s$-block form:

$$
\tilde{D}_F = \begin{pmatrix}
-a_{mf} P_L & 0 & 0 & \cdots & 0 & 0 & -\sigma P_{R_+} & \sigma \gamma + \tilde{D} \\
0 & 0 & 0 & \cdots & 0 & -\sigma P_{R_+} & \sigma \gamma + \tilde{D} & \sigma P_L \\
0 & 0 & 0 & \cdots & -\sigma P_{R_+} & \sigma \gamma + \tilde{D} & \sigma P_L & 0 \\
0 & 0 & 0 & \cdots & \sigma \gamma + \tilde{D} & \sigma P_L & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
-\sigma P_{R_+} & \sigma \gamma + \tilde{D} & \sigma P_L & \cdots & 0 & 0 & 0 & 0 \\
\sigma \gamma + \tilde{D} & \sigma P_L & 0 & \cdots & 0 & 0 & 0 & 0 & am_{f} P_R
\end{pmatrix}.
$$

(7)

The Pauli–Villars action is designed to cancel the contribution of the heavy fermions in the large $N_s$ limit. It
has a Gaussian form resulting after integration in the inverse of the fermion determinant. In order that the Gaussian
integrals be well defined let us first only consider the case of two degenerate fermion flavours when the Pauli–
Villars action is

$$
S_{PV} = \sum_{x,s,x',s'} \Phi^\dagger(x',s';x,s)am_{f} = \frac{1}{DF,am_{f} = 1}DF,am_{f} = 1 \Phi(x,s).
$$

(8)

As it is shown by the notation, the bare mass parameter $am_{f}$ is fixed here at

$$
am_{f} = 1.
$$

After performing the Grassmannian integrals over the fermion fields $\bar{\Psi}, \Psi$ and the Gaussian integrals over the Pauli–Villars fields $\Phi^\dagger, \Phi$ the result, for two degenerate fermion flavours, is the ratio of determinants

$$
\frac{\det(DF)}{\det(D_{F,am_{f} = 1}F,am_{f} = 1)} = \frac{\det(\tilde{D}_F^2)}{\det(\tilde{D}_{F,am_{f} = 1}^2)}.
$$

(9)

Here we used the fact that $\det(DF)$ is real and $\det(DF) = \det(\tilde{D}_F)$.

According to (9) the effective gauge action describing two degenerate flavours of fermions is a function of
the squared hermitean fermion matrix $\tilde{D}_F^2$. The same is also true in case of any number of flavours, as it will be
discussed in the next section. The non-zero matrix elements of $\tilde{D}_F^2$ are, in an $s$-block form:

$$
\begin{align*}
(\tilde{D}_F^2)_{s,s} &= 2\sigma^2 + 2\sigma D_r + \tilde{D}^2 + \delta_{s,1}(a^2m_f^2 - \sigma^2)P_L + \delta_{s,N_s}(a^2m_f^2 - \sigma^2)P_R, \\
(\tilde{D}_F^2)_{s,s+1} &= (\tilde{D}_F^2)_{s+1,s} = -\sigma^2 - \sigma D_r, \\
(\tilde{D}_F^2)_{1,N_s} &= (\tilde{D}_F^2)_{N_s,1} = am_{f}(\sigma + D_r),
\end{align*}
$$

(10)

where $D_r = \frac{1}{4}(\gamma_5 \tilde{D} + \tilde{D}\gamma_5)$ contains the Wilson term in the Wilson-fermion action:

$$
D_r(x',x) = \delta_{x',x}(4 - am_0) - \frac{1}{2} \sum_{\mu=1}^{4} \left[ \delta_{x',x+\hat{\mu}} U_{x\mu} + \delta_{x'+\hat{\mu},x} U_{x\mu}^\dagger \right].
$$

(11)

3. TSMB algorithm for domain wall fermions

The absolute value of the fermion determinant of an arbitrary (integer) number $N_f$ of domain wall fermion
flavours is, according to (9),

$$
|\det(\tilde{D}_F)|^{N_f} = |\det(\tilde{D}_F^2)|^{N_f/2}.
$$

(12)

Negative values of $N_f$ describe the Pauli–Villars fields. (The mass parameter $am_{f}$ is different for physical fermion
flavours and for Pauli–Villars fields, but this difference does not play a role in what follows.) For odd numbers of
flavours the sign of the determinant is neglected in (12). However, if the mass parameter is positive ($m_f > 0$) this
sign is expected to be irrelevant because of the relation of domain wall fermions to overlap fermions which have a positive determinant if the mass is positive [5,6].

Multi-boson algorithms [20] are based on polynomial approximations \( P_n \) satisfying

\[
\lim_{n \to \infty} P_n(x) = x^{-N_f/2}, \tag{13}
\]

which allow to represent the fermion determinant as

\[
\det(\tilde{D}_F^2) \simeq \frac{1}{\det P_n(\tilde{D}_F^2)}. \tag{14}
\]

Assuming that the polynomial roots occur in complex conjugate pairs, one can write \( P_n \) as

\[
P_n(\tilde{D}_F^2) \propto n \prod_{j=1}^{N_f/2} (\tilde{D}_F - \rho_j^*) (\tilde{D}_F - \rho_j). \tag{15}
\]

This leads to the multi-boson representation

\[
\det(\tilde{D}_F) \simeq \prod_{j=1}^{N_f/2} (\tilde{D}_F - \rho_j^*) (\tilde{D}_F - \rho_j)^{-1}
\]

\[
\propto \int [d\phi][d\phi^+] \exp \left\{ -\sum_{j=1}^{N_f/2} \phi_j^x \phi_j^x \left[ (\tilde{D}_F - \rho_j^*) (\tilde{D}_F - \rho_j) \right]_{XX} \right\}. \tag{16}
\]

Here \( \phi_j^x, (j = 1, 2, \ldots, n) \) are complex boson (pseudofermion) fields.

The two-step multi-boson algorithm [18] is based, instead of (13), on a polynomial approximation by a product of polynomials

\[
\lim_{n_2 \to \infty} P^{(1)}_{n_1}(x) P^{(2)}_{n_2}(x) = x^{-N_f/2}, \tag{17}
\]

where the first polynomial \( P^{(1)}_{n_1}(x) \) itself is an approximation to \( x^{-N_f/2} \), but it has a relatively low order. The multi-boson representation (16) is only used for the first polynomial \( P^{(1)}_{n_1} \). The correction factor \( P^{(2)}_{n_2} \) is

\[
\det(\tilde{D}_F) \simeq \frac{1}{\det P^{(1)}_{n_1}(\tilde{D}_F^2) \det P^{(2)}_{n_2}(\tilde{D}_F^2)} \tag{18}
\]

is realized in a stochastic noisy Metropolis correction step with a global accept-reject condition, in the spirit of [21]. In order to obtain the appropriate Gaussian vector for the noisy correction the inverse square root of \( P^{(2)}_{n_2} \) is also needed. This can be represented by another polynomial approximation

\[
P^{(3)}_{n_3}(x) \simeq \left( P^{(2)}_{n_2}(x) \right)^{-1/2}. \tag{19}
\]

A practical way to obtain \( P^{(3)} \) is to use a Newton iteration

\[
P^{(3)}_{k+1} = \frac{1}{2} \left( P^{(3)}_k + \frac{1}{P^{(3)}_k P^{(2)}_{n_2}} \right), \quad k = 0, 1, 2, \ldots \tag{20}
\]

The TSMB algorithm becomes exact only in the limit of infinitely high polynomial order: \( n_2 \to \infty \) in (17) and (18). Instead of investigating the dependence of expectation values on \( n_2 \) by performing several simulations, it is better to fix some relatively high order \( n_2 \) for the simulation and perform another correction in the “measurement” of expectation values by still finer polynomials. This is done by reweighting the configurations [22]. The
reweighting for general $N_f$ is based on a polynomial approximation $P_{n_4}^{(4)}$ which satisfies
\[
\lim_{n_4 \to \infty} P_{n_1}^{(1)}(x) P_{n_2}^{(2)}(x) P_{n_4}^{(4)}(x) = x^{-N_f/2}.
\] (21)

For more details see, for instance, [23,24].

Up to this point the particular form of the domain wall fermion action introduced in the previous section makes no difference compared to other applications of TSMB. The occurrence of negative number of flavours, as used for the Pauli–Villars fields, is the only new feature. However, for even number of flavours one can just use the form given by (9). In this case the first polynomial $P^{(1)}$ for the Pauli–Villars fields is exact and no corrections are needed. For odd number of flavours one has to deal with polynomial approximations of some integer power of $\sqrt{x}$ and the machinery of polynomial approximations works as usual.

Another peculiarity of domain wall fermions is that the fermion field has an extra index labeling the fifth coordinate. In practice this can easily lead to a situation where the storage of $n_1$ multi-boson fields in computer memory becomes a problem. (For typical simulation parameters including the values of $n_1$ see the recent studies in [25–27].) Fortunately, in cases if the storage of the multi-boson fields is problematic, one can organize the gauge field update in such a way that the dependence on the multi-boson fields is collected in a few auxiliary $3 \otimes 3$ matrix fields which can be easily stored in memory. The multi-boson fields can be kept on disk and have to be read before and written back after a complete boson field update. The duration of the input–output is negligible compared to the time of the update.

The auxiliary $3 \otimes 3$ matrix fields are spin-traces over the multi-boson fields which can be constructed as follows. The dependence of the effective gauge field action on the multi-boson fields can be summarized by the formula
\[
S_{\text{eff}}(U_{x\mu}, \phi) = \text{Re} \, \text{Tr} \left( U_{x\mu} S^{(1)}_{x\mu}(\phi) + U_{x\mu} S^{(2)}_{x\mu}(\phi) \right) + \cdots.
\] (22)

Here $U_{x\mu}$ is the gauge link variable to be updated. The omitted part denoted by the dots contain terms from the pure gauge action and other terms which do not depend on $U_{x\mu}$.

The traces over spinor indices appearing in (23), respectively, (24), are defined as
\[
\tilde{f}_{s',s;x;\mu} \equiv -\sum_{j=1}^{n_1} \text{Re} \, \rho_j \text{Tr}_{\phi} \left[ (\gamma_5 + \gamma_5 \gamma_\mu) \phi_{j;x,\mu} \phi_{j';x+\hat{\mu}}^\dagger \right] + (s \leftrightarrow s'),
\]
\[
f_{s',s;x;\mu}^{(r)} \equiv -\sum_{j=1}^{n_1} \text{Tr}_{\phi} \left[ \phi_{j;x,\mu}^\dagger \phi_{j';x+\hat{\mu}} \right] + (s \leftrightarrow s'),
\]
and
\[
S^{(1)}_{x\mu}(\phi) = \sum_{s=1}^{N_f} \left( \tilde{f}_{s',s;x;\mu} + x f_{s',s;x;\mu}^{(r)} \right) - \mu f_{1,N_f;x;\mu} - \sigma \sum_{s=1}^{N_f-1} f_{s,x+1;x;\mu}^{(r)}
\]
(23)
The indices on the multi-boson fields $\phi_{j,x}$ are as follows: $j$ is labeling the different multi-boson fields as they appear in (16), $x$ is the four-dimensional site and $s$ the fifth coordinate. The colour and spinor indices are not shown. After performing the trace over spinor indices the result is, of course, a $3 \otimes 3$ complex matrix.

Using the formulas (23)–(26) the effect of the multi-boson fields can be collected in the auxiliary $3 \otimes 3$ matrices.

The total number of $3 \otimes 3$ matrices to be stored in memory is $4 \cdot 3 \cdot 12 = 40$ per four-dimensional site, which is usually not a problem. The multi-boson fields can be updated one-by-one and kept otherwise on disk. This means that one has to store $\phi_{j,x}$ only for a single value of the multi-boson index $j$. Of course, due to the index $s$, the storage requirement is increasing with $N_s$.

4. Numerical simulation tests

Test runs have been performed for two degenerate quark flavours $N_f = 2$ on $8^3 \cdot 4$ lattices in the vicinity of the $N_t = 4$ thermodynamic crossover. The parameter sets have been chosen from the points in parameter space which were investigated in [15]. Typical parameters were: $\mu_0 = 1.9$, $\mu_f = 0.1$, $\sigma = 1.0$, 0.5 and $5.20 \leq \beta \leq 5.45$.

The first task is to determine the parameters of the necessary polynomials for different extensions of the fifth dimension $N_s$. These will be largely influenced by the condition number of the squared hermitean fermion matrix $\tilde{D}_F^2$, because the interval $[\epsilon, \lambda]$ where the polynomial approximations are optimized has to cover the eigenvalues of $\tilde{D}_F^2$ on a typical gauge configuration. A set of polynomial parameters for different $N_s$ is given in Table 1. As it is shown by the table, the largest eigenvalues of $\tilde{D}_F^2$ are only slightly increasing with $N_s$ but, in the covered range, the smallest ones decrease roughly proportional to $1/N_s$. Therefore, the condition number $\lambda/\epsilon$ is proportional to $N_s$. (For a discussion of bounds on the condition number see [28].)

Table 1 refers to the case $N_f = 2$ which is considered in the present Letter. As discussed in the previous section, an odd number of flavours would also require polynomial approximations for the Pauli–Villars fields. Since the mass parameter is kept there at $\mu_f = 1$, the corresponding condition numbers are much smaller and the polynomial orders substantially lower. In the runs shown by Table 1 the ratio of condition numbers is typically of the order

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$N_f$</th>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$n_3$</th>
<th>$n_4$</th>
<th>$\epsilon$</th>
<th>$\lambda$</th>
<th>$k_{QHB}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>8</td>
<td>44</td>
<td>240</td>
<td>300</td>
<td>300</td>
<td>0.011</td>
<td>50.0</td>
<td>5000</td>
</tr>
<tr>
<td>1.0</td>
<td>12</td>
<td>56</td>
<td>300</td>
<td>450</td>
<td>470</td>
<td>0.0071</td>
<td>57.0</td>
<td>7900</td>
</tr>
<tr>
<td>1.0</td>
<td>16</td>
<td>64</td>
<td>350</td>
<td>470</td>
<td>500</td>
<td>0.0052</td>
<td>58.0</td>
<td>9700</td>
</tr>
<tr>
<td>1.0</td>
<td>24</td>
<td>72</td>
<td>450</td>
<td>640</td>
<td>640</td>
<td>0.0032</td>
<td>59.0</td>
<td>12300</td>
</tr>
<tr>
<td>0.5</td>
<td>6</td>
<td>48</td>
<td>270</td>
<td>350</td>
<td>360</td>
<td>0.0071</td>
<td>43.0</td>
<td>6000</td>
</tr>
<tr>
<td>0.5</td>
<td>12</td>
<td>64</td>
<td>400</td>
<td>570</td>
<td>570</td>
<td>0.0046</td>
<td>44.0</td>
<td>10100</td>
</tr>
</tbody>
</table>
of 10. This would require, for instance, in case of the first line of the table polynomial orders \( n_1 = 32, \ n_2 = 60, \ n_3 = 90, \ n_4 = 100 \).

A first estimate of the computation work can be given in terms of the number of fermion-matrix vector multiplications needed for performing a sweep over the multi-boson and gauge fields. An approximate formula per update cycles is:

\[
N_{\text{MVM}} \simeq 2I_{\text{QHB}}c_{\text{QHB}}^{-1} + 6(n_1NB + NG) + 2NC(n_2 + n_3).
\]  

(27)

Here \( NB \) is the number of boson field update sweeps per cycle, \( NG \) the number of gauge field update sweeps and \( NC \) the number of noisy Metropolis accept-reject steps. It is assumed that a global quasi heatbath for the boson fields [29] is performed once per \( c_{\text{QHB}} \) cycles with a average number of matrix inverter iterations \( I_{\text{QHB}} \). (Note that \( I_{\text{QHB}} \) is a sum over \( n_1 \) inversions.) The relative contributions of the three terms in (27) are subject to optimization. The quasi heatbath is typically an important part of operations and \( I_{\text{QHB}} \) is characteristic for the total number of matrix vector multiplications \( N_{\text{MVM}} \). For the actual values occurring in the runs see Table 1 which shows that, for a given value of lattice spacing ratio \( \sigma \), \( I_{\text{QHB}} \) increases somewhat faster than \( N_1^{1/2} \).

In order to obtain an estimate for the total amount of necessary computation work, \( N_{\text{MVM}} \) has to be multiplied by the integrated autocorrelation length \( \tau_{\text{int}} \) given in number of update cycles. The measurement of autocorrelations requires high statistics and a lot of computer time and is beyond the scope of the present Letter. A rough estimate based on previous experience with TSMB tells that \( \tau_{\text{int}} \) is proportional to \( n_1 \). According to the table a very rough estimate for the increase of \( n_1 \) is \( n_1 \propto N_1^{1/2} \). Taking into account that a single matrix vector multiplication grows linearly with \( N_1 \) and \( \propto N_1^{1/2} \). The fast increase with \( N_1 \) favors domain wall fermion schemes with \( \sigma < 1 \) and relatively small \( N_1 \), as proposed in [30].

As these test runs show, the application of the TSMB algorithm for numerical simulations of domain wall quarks is possible. Simulations using alternative formulations based on overlap fermions, as proposed in [28,31], should also be feasible along the same lines. The interesting question about the computation speed compared to, say, Wilson quarks requires a more detailed analysis including the measurement of autocorrelations. Since the good chiral properties of domain wall fermions develop only sufficiently close to the continuum limit, the performance studies have to be finally performed on large lattices.

Acknowledgement

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References

Finite temperature spectral functions of the linear $O(N)$-model at large-$N$ applied to the $\pi-\sigma$ system

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Abstract

The thermal evolution of the spectral densities derivable from the two-point functions of the elementary and the quadratic composite fields of the $O(N)$ model is studied in the isosinglet channel and in the broken symmetry phase at infinite $N$. The results are applied with realistic parameter values to the $N=4$ case. They provide a reasonable description of the $\sigma$ meson at $T=0$. Threshold enhancement is observed around $T \sim 1.07m_\pi$. For higher temperatures the maximum of the spectral function in the single meson channel decreases and becomes increasingly rounded. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

There is increasing interest in finding signatures for the characterisation of strongly interacting matter around the transition temperature from the hadronic to the quark–gluon regime. Density and temperature induced features of the correlations in the soft pion spectra represent a frequently discussed issue. The majority of these pions comes from resonance decay. One expects that temperature induced shift in the position and width of the $\rho$ and $\sigma$ pole location is sensitively reflected in the spectra.

The effects related to the variation of the location and the width of the broad $\sigma$ resonance were investigated in the past decade systematically [1]. The main theoretical framework of these investigations was the $O(4)$ linear sigma-model. Various reorganisations of the perturbative series [2] were proposed for the self-energy calculations. Despite of the rather large value of the non-linear coupling $\lambda_R \approx O(30–70)$, one loop calculations gave important qualitative hints for the enhancement of the spectral function in the $\sigma$-channel near the two-pion threshold as the temperature approaches the transition region.

In the non-relativistic context it has been recognised long time ago that in scalar theories the hybridisation [3] of the two-point functions of the elementary and quadratic composite fields play essential role in the determination
of the elementary excitations in the whole temperature interval of the broken symmetry phase. Neutron scattering data on superfluid helium were successfully interpreted by taking into account the hybridisation [4].

Large $N$ techniques are known to account for this phenomenon correctly, while ordinary perturbative approaches usually miss it [3,5].

An additional point in favour of the use of the large-$N$ expansion is that it leads to a second order transition point in agreement with the well-known behaviour of the $O(N)$ model [6]. The lowest order computations done with any (even optimised) version of the conventional perturbation theory yield a first order phase transition when the temperature is increased.

Large-$N$ techniques are extensively studied also in connection with the out-of-equilibrium evolution and thermalisation of relativistic Bose-condensates [7–11], though the hybridisation in this respect did not receive till now much attention.

Our main goal with this investigation was to apply large-$N$ techniques for finding the finite temperature variation of the spectral function in the elementary and quadratic composite isosinglet channels, which influences the $\pi-\pi$ correlations measured in heavy ion collisions to a large extent. For this purpose the parametrisation of the linear $O(4)$-symmetric meson model is chosen to reproduce as closely as possible the accepted zero temperature fit to the complex mass, $M_\sigma - i \Gamma/2$ of this broad resonance [12].

In this Letter, after summarising the general formalism, mostly the results of the chirally invariant limiting case will be discussed, where the pions are massless, since the main ideas can be presented in this case very transparently. Results relevant for the case with explicit symmetry breaking will be presented shortly in the concluding part of the Letter.

We shall work in the leading order large-$N$ approximation to the Schwinger–Dyson equations of the $O(4)$ model in the broken symmetry phase. The effect of next-to-leading order corrections is the subject of our ongoing research.

2. The model and its renormalised dynamical equations

The Lagrangian density including a term reflecting explicit breaking of the $O(N)$ symmetry is the following:

$$L = \frac{1}{2} \left[ \partial_\mu \phi^a \partial^\mu \phi^a - m^2 \phi^a \phi^a \right] - \frac{\lambda}{24N} (\phi^a)^2 (\phi^b)^2 + \sqrt{N} h \phi^1. \tag{1}$$

The broken symmetry phase can be studied after an appropriate shift, by introducing the symmetry breaking background:

$$\phi^a \rightarrow (\sqrt{N} \Phi + \phi^1, \phi^i). \tag{2}$$

The quantum field representing the fluctuations of the order parameter is termed $\sigma$, while the modes transversal to it are the Goldstone pions.

The equation of state, which determines the absolute value of the condensate comes from the requirement of the vanishing quantum expectation for the coefficient of the term linear in $\phi^1$ in the shifted Lagrangian:

$$\left\langle \frac{\delta L}{\delta \phi^1} \right\rangle = 0 = \sqrt{N} \Phi \left[ m^2 + \frac{\lambda}{6} \phi^2 + \frac{\lambda}{6N} (\phi^a)^2 - \frac{h}{\Phi} \right]. \tag{3}$$

The quadratic fluctuations of the shifted fields are computed with an accuracy $O(N^1)$, therefore only the contribution of the pions is retained. Anticipating a non-zero Goldstone mass due to the explicit symmetry breaking (which will be verified below) one finds the following relation connecting renormalised quantities (only
contributions proportional to $N^0$ are kept):

$$m_R^2 + \frac{\lambda_R}{6} \Phi^2(T) + \frac{\lambda_R}{96\pi^2} m_G^2(T) \ln \frac{m_G^2(T)}{M_0^2} + \frac{\lambda_R T^2}{12\pi^2} \int dy \sqrt{y^2 - m_G^2(T)/T^2} = \frac{h}{\Phi(T)}. \quad (4)$$

Here the following renormalised couplings were introduced (momentum cut-off was applied for the regularisation of some divergent integrals):

$$\frac{m^2}{\lambda} + \frac{\Lambda^2}{96\pi^2} - \frac{m^2}{\lambda} \ln \frac{\Lambda^2}{M_0^2} = \frac{m_R^2}{\lambda_R} + \frac{1}{96\pi^2} \ln \frac{\Lambda^2}{M_0^2} = \frac{1}{\lambda_R}. \quad (5)$$

The issue of the normalisation point $M_0$ will be discussed below.

At leading order ($N = \infty$) the only contribution to the self-energy of the Goldstone modes comes from the tadpole diagram. Therefore one has:

$$G_G^{-1}(p) = p^2 - m^2 - \frac{\lambda}{6} \Phi^2(T) - \frac{\lambda}{6N} ((\phi^a)^2) = p^2 - \frac{h}{\Phi(T)}. \quad (6)$$

Here the sum representing the mass term was simplified in view of the equation of state (3). This equation implies that at $N = \infty$ the Goldstone particle is stable and its mass, $m_G^2(T) = h/\Phi(T)$ increases with the temperature.

With this identification the renormalised equation of state (4) can be cast into a more practical form by eliminating $m_R^2$ and $h$ in favour of the $T = 0$ value of the condensate $\langle \Phi_0 \rangle$ and of the pion mass ($m_G$):

$$\begin{align*}
\frac{\lambda_R}{6} \phi_0^2 & \left( \frac{m_G^4}{m_G^4} - 1 \right) + \frac{\lambda_R}{96\pi^2} \left[ \left( \frac{m_G^2}{m_G^2} - 1 \right) \ln \frac{m_G^2}{M_0^2} + \frac{m_G^2}{m_G^2} \ln \frac{m_G^2}{m_G^2} \right] \\
& + \frac{\lambda_R T^2}{12\pi m_G^2} \int dy \sqrt{y^2 - m_G^2/T^2} \left( e^y - 1 \right)^{-1} = \frac{m_G^2}{m_G^2} - 1.
\end{align*} \quad (7)$$

This form makes it clear that after $\lambda_R$ and $M_0/m_G$ are chosen in the process of renormalisation at $T = 0$, one finds from this equation $m_G(T)/m_G$ as a function of $T/m_G$ if the physical input $\phi_0^2/h = f_0^2/N m_G^2$ is made.

The $\sigma$ propagator receives $O(N^0)$ contribution from the infinite iteration of the bubble diagrams $b(p)$, where on both lines Goldstone fields are propagating:

$$G_H^{-1}(p) = p^2 - \frac{h}{\Phi(T)} - \frac{\lambda}{3} \Phi^2(T) \frac{1}{1 - \lambda b(p)/6}. \quad (8)$$

Without entering its derivation we give here also the expression for the leading large-$N$ propagator of the quadratic composite field $(\phi^a \phi^a - ((\phi^a)^2))(x, t)$:

$$F(p) = \frac{(p^2 - h/\Phi(T)) b(p)/6 + \Phi^2(T)/3}{(p^2 - h/\Phi(T))(1 - \lambda_R b(p)/6) - \lambda_R \Phi^2(T)/3}. \quad (9)$$

It has the same denominator as $G_H(p)$ making explicit the hybridisation of the two objects. The bubble contribution with external momentum $p_0$, $p$ is the sum of a zero temperature and a $T$-dependent part, $b(p) = b_0(p) + b_T(p_0, p)$, which read as follows:

$$b_0(p) = \frac{1}{16\pi^2} \left[ - \ln \frac{\Lambda^2}{M_0^2} + \ln \frac{m_G^2}{M_0^2} - \sqrt{1 - 4m_G^2/p^2} \ln \frac{1 - 4m_G^2/p^2 - 1}{1 - 4m_G^2/p^2 + 1} \right]. \quad (10)$$
\[
b_T(p) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{4q_1q_2} \left\{ \begin{array}{c}
(n_1 + n_2) \left[ \frac{1}{p_0 - \omega_1 - \omega_2 + i\epsilon} - \frac{1}{p_0 + \omega_1 + \omega_2 + i\epsilon} \right] \\
- (n_1 - n_2) \left[ \frac{1}{p_0 - \omega_1 + \omega_2 + i\epsilon} - \frac{1}{p_0 + \omega_1 - \omega_2 + i\epsilon} \right] \end{array} \right\},
\]

where \(n_i = 1/(\exp(\beta \omega_i) - 1)\) and \(\omega_1 = (q^2 + m_G^2)^{1/2}, \quad \omega_2 = ((q + p)^2 + m_G^2)^{1/2}\). The first term in the expression of \(b_0(p)\) is cancelled in the expression of the \(\sigma\) propagator by the divergence of the bare coupling \(\lambda\) and the inverse propagator, expressed in terms of the renormalised quantities is finite. Note, that \(b_0(p)\) has to be evaluated with \(m_G(T)\)!

The phenomenologically most interesting object is the spectral function of the order parameter field \(\sigma\), defined as

\[
\rho_H(p_0, p, T) = -\frac{1}{\pi} \text{Im} G_H(p_0, p, T).
\]

For non-zero \(T\) \(\rho_H\) has a singularity at \(p_0 = 0\) due to the Bose–Einstein factor \(n(\beta p)/2\). In the chiral limit \(h = 0\) this point coincides with the two-pion threshold. In order to make the effects related to the physical excitations visible around this point in Fig. 1 we show \(\rho_1(p_0, 0, T) = (1 - \exp(-p_0/2T))\rho_H(p_0, 0, T)\). The curves are drawn for various temperatures below \(T_c\) and a representative value of \(\lambda_R = 310\).

The temperature dependence of the order parameter needed for its evaluation is obtained from the equation of state which simplifies in this case [6] to

\[
\frac{\Phi^2(T)}{T_c^2} = \frac{1}{12} \left(1 - \frac{T^2}{T_c^2}\right), \quad T_c^2 = 12\Phi_0^2.
\]

The shape of the spectral density starts with a broad bump at \(T = 0\) which is shifted towards lower frequencies when the temperature is increased. Already for \(T = 0\) \(\rho_H\) has finite value at \(p_0 = 0\), where its value is gradually increasing. At a temperature \(T \sim 0.7T_c\) the bumpy structure at finite \(p_0\) completely disappears and a simple Lorentzian shape develops with its maximum located at the threshold.

This evolution can be transparently interpreted if the temperature dependent location of the physical pole of the \(\sigma\) propagator is found. The physical pole is located in the lower half of the complex \(p_0\)-plane, therefore one is faced with the analytical continuation of \(b_T(p)\) into the lower \(p_0\)-halfplane, since originally it is defined only
in the upper halfplane by the Landau-prescription: $\epsilon > 0$. As becomes clear, this threshold enhancement is the manifestation of the gradual chiral symmetry restoration as $T$ tends to $T_c$.

3. The physical quasiparticle excitation in the $\sigma$ channel

Below we give some details concerning the temperature variation of the excitation spectra and the interpretation of the spectral function with its help in the chiral limit ($h = 0$). One starts with the analysis at $T = 0$, where one fixes the renormalised parameters $m_R^2, \lambda_R$, remaining after $h$ was set to zero. The renormalised mass is related to the size of the condensate (or in view of (13) to $T_c$) by the well-known equation: $-m_R^2/\Phi_0^2 = \lambda_R/6$.

$\lambda_R$ will be chosen from a range where one finds for $\Gamma/M_\sigma$ values which are the closest to the experimental one. For this we determine the $T = 0$ pole of the $\sigma$ propagator by looking for the zeros of $G_H^{-1}$ in terms of renormalised quantities:

$$G_H^{-1}(p) = p^2 - \frac{\lambda_R}{3} \Phi_0^2 \frac{1}{1 - \lambda_R \ln(-p^2/M_0^2)/96\pi^2} = 0.$$  \hspace{1cm} (14)

The physical solution at rest of this equation is parametrised by putting $p = 0$, $p_0 = M_0 \exp(-i\varphi_0)$, $\pi/2 > \varphi_0 > 0$. (One can find its mirror solution in the third quarter.) The notation shows, that the absolute value of the pole is chosen for the normalisation point defined in (5) and used in (10). If a physically satisfactory solution is found for some value $\lambda_R$ with this normalisation point, the renormalisation group invariance of (14) ensures that for a different choice of $M_0$ the same ratio $\Gamma/M_\sigma$ is found at some appropriately shifted value of $\lambda_R$. In this way the actual value of $\lambda_R$ cannot be said to be large or small!

Since in view of (4) in the chiral limit one has a very simple equation for the critical temperature, our solution provides the mass $M_\sigma = M_0 \cos \varphi_0$ and the imaginary part $M_0 \sin \varphi_0$ in proportion of the critical temperature. It is interesting to note that using for $\sqrt{N} \Phi = 2\Phi$ the experimental value of $f_\pi$, Eq. (13) gives $T_c = 161$ MeV, while the lattice simulations yield $(173 \pm 8)$ MeV [13]. The agreement is much better than expected. The real and imaginary parts of the complex physical pole are shown in Fig. 2 as a function of $\lambda_R$.

![Fig. 2. The imaginary and real parts of the physical poles at $T = 0$ in the chiral limit and also for $h \neq 0$. Also shown is the logarithm of the tachyon pole position in proportion to the mass of $\sigma$ for various temperatures below $T_c$ and for $h = 0$.](image-url)
In addition to the physical zeros of the inverse propagator scalar theories are known to have a tachyonic zero on the positive imaginary axis \( p_0 = i M_L \) (see [8] and references therein), which makes the theory unstable. Therefore the theory can be used as an effective theory until the physical scale \( M_0 \) is acceptably smaller than \( M_L \). The position of the tachyonic zero is also found from (14). (The fact that the tachyonic pole shows up also in \( G_H \) not only in \( F \) is a manifestation of hybridisation.) It is clear from Fig. 2 that for \( \lambda_R \leq 400 \) one has \( M_L/M_0 > 4 \) and the effective approach is well justified. The representative value \( \lambda_R = 310 \) correspond to a ratio \( M_\sigma/\Gamma = 1 \) at \( T = 0 \). The value of \( M_\sigma \) is estimated to be \( 7 \Phi_0 \sim 350 \text{ MeV} \).

At finite temperature we are interested in the temperature dependence of the \( \sigma \)-pole at rest, therefore we set \( p = 0 \) in (11). The analytic continuation onto the lower plane can be carried out by adding to the expression of \( b_T \) valid in the upper halfplane a term which makes its imaginary part continuous when one approaches the real \( p_0 \)-axis either from the upper or the lower halfplane:

\[
b_T^c(p_0) = \frac{1}{8\pi^2} \int_0^\infty dx \frac{1}{e^x - 1} \left[ \frac{1}{z - x} - \frac{1}{z + x} \right], \quad z = \frac{p_0}{2T}, \quad \text{Im} \, p_0 > 0,
\]

\[
b_T^c(p_0) = b_T^c(p_0) - \frac{i}{4\pi} \exp(z) - 1, \quad \text{Im} \, p_0 < 0, \quad \text{Re} \, p_0 > 0. \quad (15)
\]

The roots are parametrised the same way as in the \( T = 0 \) case, \( p_0 = M(T) \exp(-i\varphi) \). After one determines \( \Phi(T) \) from Eq. (13), from the complex equation

\[
1 - \frac{\lambda_R}{96\pi^2} \left( \ln \frac{M^2(T)}{M_0^2} - i(2\varphi + \pi) \right) - \frac{\lambda_R}{3} \frac{\Phi^2(T)}{M^2(T)} e^{2i\varphi} - \frac{\lambda_R}{6} b_T^c(p_0 = M(T)e^{-i\varphi})
\]

\[
+ \frac{\lambda_R}{24\pi} \exp(M(T)\exp(-i\varphi)/2T) - 1 = 0 \quad (16)
\]

one can find the complex pole as a function of the temperature. All quantities are measured in units of \( T_c \sim \Phi_0 \).

Also at finite temperature one can study the tachyon solution in the upper halfplane using \( b_T^c(i M_L) \). One might wonder if the temperature dependence of the tachyonic pole does not restrict further the coupling range where the effective use of the scalar theory is consistent. The broadening in Fig. 2 of the line corresponding to the tachyonic pole reflects the slight decrease of its mass scale with increasing temperature. However, it is clear that in the region \( \lambda_R < 400 \), where the effective approach is consistent for \( T = 0 \), the position of the tachyon pole is practically unchanged.

Similarly, one can look directly for physical roots along the negative imaginary axis. Then using \( b_T^c \) from (15) one finds that above a well-defined \( T_{\text{imag}}(\lambda_R) < T_c \) the physical pole becomes purely imaginary: \(-iM(T)\). If \( \mu = M(T)/2T \ll 1 \), for the highest occupied low frequency region, one can use the expansion of the Bose–Einstein factors appearing in the last two terms of (16) with respect to the powers of \( \mu \) and keep the leading terms:

\[
1 - \frac{\lambda_R}{48\pi^2} \ln \frac{2T\mu}{M_0} + \frac{\lambda_R}{3} \frac{\Phi^2(T)}{4T^2} \frac{1}{\mu^2} - \frac{\lambda_R}{48\pi} \frac{1}{\mu} = 0. \quad (17)
\]

This equation could have been derived directly, if one would have applied from the beginning the same approximation in Eq. (11). Clearly, the region around the origin (the critical region) can be analysed also directly with help of this simpler equation. Note that one has to go one step beyond the classical approximation to achieve a consistent approach.

The general pattern of the trajectory of the physical pole with increasing temperature was the following. The real part of its position started to diminish when \( T \) was raised. In this way the broad bump of the spectral function moves towards the origin, and its width is increasing due to the slight increase of its imaginary part. Depending on \( \lambda_R \) the root reaches at some \( T_{\text{imag}} < T_c \) the negative imaginary axis and collides with its mirror root arriving from the third quarter. Each one is joining smoothly the trajectory of one of the pair of imaginary roots which
appear just at $T_{imag}$ and move opposite directions. We observe that the $\sigma$-bump gets lost in the background before the temperature reaches $T_{imag}$. The root approaching the origin along the imaginary axis produces in the spectral function a shrinking shape which becomes a Lorentzian only very close to $T_c$. Eventually for $T = T_c$ it builds up a term proportional to $\delta(p_0)/p_0$.

We see that the complex pole of the $\sigma$-propagator qualitatively accounts for the behaviour of the spectral function near and below the critical point as well. Note that the other pole moving away from the origin along the imaginary axis as well as the tachyon disappear from $G_H$ at $T_c$. They remain poles of $F$ only.

In the vicinity of the critical point, where $\xi = T/8\pi \Phi^2(T)$ is the dominant length scale and the condition $p_0, |p| \ll T$ is fulfilled, one can derive an equation also for the soft modes with non-zero momentum:

$$-3i|p|\xi^{-1} - \frac{p_0^2 - p^2}{|p|^2} \ln \frac{p_0 - |p|}{p_0 + |p|} = \frac{1}{4\pi} \frac{(p_0^2 - p^2)\xi}{T_c} \ln \frac{p_0^2 - p^2}{T_c^2} = 1.$$  \hfill (18)

Its solution in the approximation, when on the left-hand side only the first term is retained exhibits the form of dynamical scaling [14,15]: $p_0 = |p|\xi f(|p|\xi)$ with $\xi = 1$. The second term provides the leading correction to scaling. In $O(N)$ models, the dynamical exponent $z = d/2$ has been obtained for finite $N$ on the basis of scaling and renormalisation group arguments, where in our case $d = 3$ [16–18]. For the correct interpretation of the situation it is important that there are two distinct hydrodynamical regions in the $O(N)$ model for large-$N$ [16]. (The system studied in [16] can be regarded as a lattice regularised version of the linear $\sigma$-model.) In the true critical region $z = d/2$ is valid, in a precritical region $\bar{z} = 1 - 8S_d/Nd + O(1/N^2)$, where $S_d = 2/\pi^2$ for $d = 3$. The first region shrinks when $N$ becomes large and completely disappears at $N = \infty$. Furthermore, it has been found that $\bar{z}$ agrees with the dynamical exponent determined for the quasiparticles, that is outside the hydrodynamical regime, at least to $O(1/N)$ [19]. Details of the application of this analysis to the present case will be published in our forthcoming paper.

4. The case of the explicit breaking of chiral symmetry

Finally, we wish to discuss shortly the results of the application of the leading large-$N$ solution of the $O(N)$ model to the low lying effective $\pi-\sigma$ system where an explicit symmetry breaking is necessary.

Since in this case one cannot speak of any phase transition, it is more sensible to look for the roots of $G_H^{-1}(p_0, 0)$ in proportion to $\Phi_0 \equiv f'_{\sigma}/2$. The value $m^2_{\Phi_0}/m^2_{\Phi_0} \approx 9.06$ is fixed by phenomenology and choosing $\lambda_R$ with the restriction $M_0/M_\perp \leq 4$, one can find the zero temperature position of the $\sigma$-pole. In Fig. 2 its real and imaginary parts are already shown as a function of $\lambda_R$ together with the results obtained for $h = 0$. In the region of $\lambda_R$ allowed the ratio $M_\perp /T_\sigma$ moves away from the phenomenologically preferred range ($M_\sigma = 3.95 f_{\pi}, m_\sigma /T \sim 1.4$) emphasising the need for a next-to-leading order calculation.

Now $\Phi(T)$ is determined from (7). The spectral function of the single $\sigma$-channel is shown in the left of Fig. 3 for various values of the temperature in the transition range $T = (0-1.2)m_{G_0}$ with $\lambda_R = 400$. The location of its maximum at $T = 0$ is correlated with the location of the $T = 0$ pole. Maximal threshold enhancement [20] is observed in the close neighbourhood of 1.074 $m_{G_0}$ with the spectral function numerically exhibiting a cuspy form, with a very narrow width. However, when the temperature is further increased, the maximum is pulled away from the actual position of the threshold and a broad rounded structure is seen. We return to the detailed interpretation of this structure in terms of the pole trajectories in a forthcoming publication.

As we have mentioned above also the spectral function belonging to the propagator (9) of the composite field can be constructed to leading order in $N$. The corresponding spectral function reflects the excitations in the composite $(\phi^2)^2$ channel. Its temperature dependent deformation is displayed on the right hand of Fig. 3 for the same (realistic) parameters as used in case of $\rho_H$ above. Its behaviour is rather similar to the single meson spectral function. No signal of any meson-meson bound state can be observed. The only difference one notices is that for large
Fig. 3. The spectral functions for the linear sigma model with realistic couplings and explicit symmetry breaking. On the left figure the imaginary part in the $\sigma$-channel, on the right the same for the composite channel is shown.

frequencies this function decreases more slowly, which reflects mostly the large residuum of the tachyon pole in this propagator.

5. Conclusions

We have described the temperature dependence of the elementary excitations of the $O(4)$ model in the leading order of the $1/N$ expansion. For the chirally symmetric case a very suggestive picture of the complex pole evolution makes unique the interpretation of the change of shape of the single-particle spectral function when the temperature approaches the critical point.

The restrictions on the range of the scalar self-coupling $\lambda_R$ arising from the requirement of keeping distance from the tachyonic pole, for a fixed normalisation point $M_0$ prevent us in finding a fully realistic $\sigma$-particle mass at $T = 0$ when the physical pion mass is the input. We have found that the threshold enhancement in both the elementary and the composite spectral functions is maximal at some $T_{\text{enh}} \sim m_{G0}$. Beyond this temperature the cuspy maximum becomes rounded again. We believe that this qualitative feature remains valid when the next to leading order corrections will be included.

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References

Exact formula of probability and CP violation for neutrino oscillations in matter

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Abstract

Within the framework of the standard three neutrino scenario, we derive an exact and simple formula of the oscillation probability \( P(\nu_e \rightarrow \nu_{\mu}) \) in constant matter by using a new method. From this formula, it is found that the matter effects can be separated from the pure CP violation effects. Furthermore, the oscillation probability can be written in the form,

\[ P(\nu_e \rightarrow \nu_{\mu}) = A \cos \delta + B \sin \delta + C, \]

in the standard parametrization without any approximation. We also demonstrate that the approximate formula in high-energy can be easily reproduced from this as an example. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Just like the quark system, it has been shown from the atmospheric neutrino experiments \cite{1} and the solar neutrino experiments \cite{2} that neutrinos have finite mass and finite mixing. In this situation, it is extremely interesting to investigate the CP phase in the lepton sector. Fortunately, recent report from SNO experiment \cite{3} favors the LMA MSW \cite{28} solutions to the solar neutrino problem. This means that the measurements of CP phase may be possible because of the large 1–2 mixing angle and the large 1–2 mass difference.

In order to measure the CP phase, the long-baseline experiments such as the JHF experiment \cite{4} and the neutrino factory experiments \cite{5} are planned. In the past, the asymmetries \( \Delta P_{CP} = P(\nu_e \rightarrow \nu_{\mu}) - P(\bar{\nu}_e \rightarrow \bar{\nu}_{\mu}) \) and \( \Delta P_T = P(\nu_e \rightarrow \nu_{\mu}) - P(\nu_{\mu} \rightarrow \nu_e) \) have been considered as the main approach to measure the CP phase \( \delta \) \cite{6–9}. These are methods to measure the direct CP violation term which depends on \( \sin \delta \). However, the measurement of \( \Delta P_{CP} \) is not directly related to the discovery of CP violation, because of the fake CP violating effects from the earth matter. \( \Delta P_T \) is a pure T-violating observable, but it has its own experimental difficulties. So, alternative approach has been recently considered in \cite{10–14}. This is an attempt to obtain the information on the CP phase totally from the probabilities itself, not only the direct CP violation term but also the indirect CP violation.
term which depends on \( \cos \delta \). In these papers the oscillation probability is written approximately in the form,

\[
P(\nu_e \rightarrow \nu_\mu) \approx A \cos \delta + B \sin \delta + C.
\]

The extra information which is proportional to \( \cos \delta \) will lead us to the value of \( \delta \) in spite of the matter effect. In order to obtain more precise information, it is highly desirable to have an exact expression for \( P(\nu_e \rightarrow \nu_\mu) \). Some attempts to derive the exact formula have been made in the context of three neutrino scenarios [15–18]. These formulae are useful for numerical calculation. However, the precise CP dependence of \( P(\nu_e \rightarrow \nu_\mu) \) has not been investigated sufficiently [16].

To describe our approach, let us review the work of Naumov [19] and Harrison and Scott [6]. The Hamiltonian \( \tilde{H} \) in matter is related to \( H \) in vacuum as

\[
\tilde{H} = H + \frac{1}{2E} \text{diag}(a, 0, 0),
\]

where \( a \equiv 2\sqrt{2} G_F N_e E \), \( G_F \) is Fermi constant and \( N_e \) is the electron density in matter. In particular, taking the products of non-diagonal elements, one obtains the following identity, which we call Naumov–Harrison–Scott identity,

\[
\text{Im}(\tilde{H}_{e\mu} \tilde{H}_{\mu e} H_{e\tau}) = \text{Im}(H_{e\mu} H_{\mu e} H_{e\tau}),
\]

in CP-odd part, where

\[
\Delta_{ij} \equiv m_i^2 - m_j^2,
\]

is Jarlskog factor [20], \( J_{ij} \equiv U_{qi}^* U_{qj} U_{qi}^* U_{qj} \) and \( U \) is the Maki–Nakagawa–Sakata (MNS) matrix [21]. Here the quantities expressed by the tilde include the matter effects. We show that \( \text{Re} \tilde{J}^{ij}_{e\mu} \) has only a linear term in \( \cos \delta \).

\[1\] The calculation of \( \tilde{U} \) is performed by diagonalizing \( \tilde{H} \) in Ref. [17].

\[2\] The exact formula obtained in this method is very simple and the matter effects come in only through effective masses. We show that \( \text{Re} \tilde{J}^{ij}_{e\mu} \) in high energy can be easily reproduced from our exact formula as an example. Finally, we numerically calculate the coefficients \( A, B \) and \( C \).

### 2. Exact formula of the oscillation probability

The flavor and mass eigenstates are related by the MNS matrix \( \tilde{U}_{qi} \) in matter, where \( \alpha = e, \mu, \tau \) is the flavor index, \( i = 1, 2, 3 \) is the mass index. The amplitude for \( \nu_e \) to \( \nu_\mu \) transition is given by

\[
A(\nu_e \rightarrow \nu_\mu) = e^{-i p L} \sum_{i=1}^{3} \tilde{U}_{ei}^* e^{-i \frac{1}{2 E} \Delta_{ij} L} \tilde{U}_{\mu i}.
\]

The amplitude depends only on the products \( \tilde{U}_{ei} \tilde{U}_{\mu i}^* \). One of the important points in this Letter is that these products can be easily calculated from identities which we derive below.

From the unitarity relation and the other two relations,

\[
\tilde{H}_{e\mu} = H_{e\mu} = \frac{p}{2E},
\]

\[
\tilde{H}_{e\tau} \tilde{H}_{\mu \tau} - \tilde{H}_{e\mu} \tilde{H}_{e\tau} = \frac{q}{(2E)^2}.
\]
three identities on the products $\tilde{U}_{ei}\tilde{U}^*_{i\mu}$ can be obtained as follows:

$$\sum_{i=1}^{3} \tilde{U}_{ei}\tilde{U}^*_{i\mu} = \sum_{i=1}^{3} U_{ei}U^*_{i\mu} = 0,$$

(9)

$$\sum_{i=1}^{3} \lambda_i \tilde{U}_{ei}\tilde{U}^*_{i\mu} = \sum_{i=1}^{3} m^2_i U_{ei}U^*_{i\mu} = p,$$

(10)

cyclic

$$\sum_{(ijk)} \lambda_i \lambda_j \lambda_k \tilde{U}_{ei}\tilde{U}_{ej}\tilde{U}^*_{i\mu} \tilde{U}^*_{j\nu} = \sum_{(ijk)} m^2_i m^2_j U_{ei}U^*_{i\mu} U_{ej}U^*_{j\nu} = q,$$

(11)

cyclic

where $p$ and $q$ are constants determined by the parameters in vacuum and the sum is over $(ijk) = (123), (231), (312)$. We use the relation $\tilde{U}_{ei}\tilde{U}_{ej} - \tilde{U}_{ei}\tilde{U}_{ej} = \tilde{U}^*_{i\mu}(\det \tilde{U})$, etc., obtained from the formula $\tilde{U}^+ = \tilde{U}^{-1} = \tilde{U}(\det \tilde{U})^{-1}$, where $\tilde{U}$ represents the cofactor matrix.

Solving the simultaneous equations for the products $\tilde{U}_{ei}\tilde{U}^*_{i\mu}$, we obtain

$$\tilde{U}_{ei}\tilde{U}^*_{i\mu} = p\lambda_i + q, \quad (12)$$

where $(ijk)$ takes (123), (231), (312). From the definition $J^j_{ei\mu} = \tilde{U}_{ei}\tilde{U}^*_{i\mu}(\tilde{U}_{ej}\tilde{U}^*_{j\mu})^*$, the exact formula of the oscillation probability is given by

$$P(\nu_e \rightarrow \nu_\mu) = -4 \sum_{(ij)} \text{Re} J^j_{ei\mu} \sin^2 \left(\frac{\Delta_{ij} L}{4E}\right)$$

$$- 2 \sum_{(ij)} \bar{J} \sin \left(\frac{\Delta_{ij} L}{2E}\right),$$

(13)

where the sum is over $(ij) = (12), (23), (31)$ and

$$\text{Re} J^j_{ei\mu} = \frac{|p|^2 \lambda_i \lambda_j + |q|^2 + \text{Re}(pq^*) (\lambda_i + \lambda_j)}{\Delta_{ij} A_{12} A_{23} A_{31}},$$

(14)

$$\bar{J} = \frac{\text{Im}(pq^*)}{\Delta_{12} A_{23} A_{31}}.$$  

(15)

We find that the matter effects are confined in the effective masses only. We can obtain the probability for antineutrinos, $\nu_e \rightarrow \bar{\nu}_\mu$, by exchanging $a \rightarrow -a$ and $\delta \rightarrow -\delta$ in $\Delta_{ij}$ and $J^j_{ei\mu}$ of Eq. (13).

Let us comment on the relation between our result and that of other authors. The second identity (10) is also given in Ref. [22]. The third identity (11) is new and play an important role in deriving our exact formula. The similar expression to (12) is given in Ref. [18] as the result of the calculation of $e^{-iHt}$, although the CP phase has not been considered.\footnote{2} Next, $\text{Im}(pq^*)$ in (15) are rewritten as

$$\text{Im}(pq^*) = \frac{1}{(2E)} \text{Im}(He_{ei}H_{i\mu}He_{\mu\nu})$$

$$= \Delta_{12} A_{23} A_{31} J,$$

(16)

from (7) and (8). Naumov–Harrison–Scott identity is reproduced by substituting (16) into (15).

### 3. Separation of CP odd/even parts

In this section, we give a concrete expression for the oscillation probability and then, we study the dependence of the oscillation probability on the CP phase.

First let us consider the constants $p$ and $q$. We use the standard parametrization

$$U_{ei} = \begin{pmatrix}
-c_{23}s_{12} & s_{23}s_{12} & c_{13}s_{12} \\
-c_{23}s_{12} & s_{23}s_{12} & c_{13}s_{12} \\
-c_{23}s_{12} & s_{23}s_{12} & c_{13}s_{12}
\end{pmatrix}$$

(17)

where $\sin \theta_{ij} = s_{ij}$, $\cos \theta_{ij} = c_{ij}$. In addition, as the neutrino oscillation probabilities do not depend on the mass itself, but the mass square differences, we take $m^2_1 = 0$, $m^2_2 = \Delta_{21}$ and $m^2_3 = \Delta_{31}$ without loss of generality. So, $p$ and $q$ are given by

$$p = p_1 e^{-i\delta} + p_2, \quad q = q_1 e^{-i\delta} + q_2,$$

(18)

where $p_i$ and $q_i$ are real numbers;

$$p_1 = (\Delta_{31} - \Delta_{21}s_{12}^2)s_{23}s_{13}c_{13},$$

(19)

$$p_2 = \Delta_{21}s_{12}c_{12}c_{23}c_{13},$$

$$q_1 = -\Delta_{31}A_{21}c_{12}^2 s_{23}s_{13}c_{13},$$

(20)

$$q_2 = -\Delta_{31}A_{21}s_{12} c_{12}c_{23}c_{13}.$$  

Then, we have

$$|p|^2 = p_1^2 + p_2^2 + 2p_1 p_2 \cos \delta,$$

(21)

$$|q|^2 = q_1^2 + q_2^2 + 2q_1 q_2 \cos \delta,$$

(22)

We notice that the expression of $\tilde{U}^*U$ in (12) is also derived from Eq. (4) in Ref. [23] after some calculations [24].
Therefore, the oscillation probability can be written in the form

$$P(ν_e → ν_μ) = A \cos δ + B \sin δ + C,$$  \hspace{1cm} (25)

from (13)–(15). Note that $A$, $B$ and $C$ are independent of $δ$ and the oscillation probability is expressed only by the linear terms in $\cos δ$ and $\sin δ$ up to a constant as described below. This is one of our main results.

Here

$$A = \sum_{(ij)} A_{ij} \sin^2(\frac{\tilde{A}_{ij} L}{4E}),$$  \hspace{1cm} (26)

$$B = \sum_{(ij)} B_{ij} \sin(\frac{\tilde{A}_{ij} L}{2E}),$$  \hspace{1cm} (27)

$$C = \sum_{(ij)} C_{ij} \sin^2(\frac{\tilde{A}_{ij} L}{4E}),$$  \hspace{1cm} (28)

are expressed by the products of the oscillation part dependent on $L$ and $A_{ij}$, $B_{ij}$ and $C_{ij}$. And then, $A_{ij}$, $B_{ij}$ and $C_{ij}$ are given by

$$A_{ij} = -4 \left[ 2p_1 p_2 \lambda_i \lambda_j + 2q_1 q_2 
+ (p_1 q_2 + q_1 p_2) (\lambda_i + \lambda_j) \right]$$
$$\times \left[ \tilde{\lambda}_{ij} \tilde{\lambda}_{12} \tilde{\lambda}_{23} \tilde{\lambda}_{31} \right]^{-1},$$  \hspace{1cm} (29)

$$B_{ij} = -2 \left[ \frac{p_1 q_2 - p_2 q_1}{\tilde{\lambda}_{12} \tilde{\lambda}_{23} \tilde{\lambda}_{31}} \right],$$  \hspace{1cm} (30)

$$C_{ij} = -4 \left[ \left( \frac{p_1^2 + p_2^2}{2} \right) \lambda_i \lambda_j + (q_1^2 + q_2^2) \right]$$
$$\times \left[ \tilde{\lambda}_{ij} \tilde{\lambda}_{12} \tilde{\lambda}_{23} \tilde{\lambda}_{31} \right]^{-1},$$  \hspace{1cm} (31)

as the function of the masses and mixing angles. Since the effective masses $\lambda_i$ shown in [15–17] do not depend on $δ$, the coefficients $A$, $B$ and $C$ are independent of $δ$.

Our analytic result given in (25) should be compared with the result of [12] depicted in Fig. 1 obtained numerically. The trajectory becomes an ellipse in the bi-probability space when $δ$ changes from 0 to $2\pi$. The CP dependence of the exact form of $P(ν_e → ν_μ)$ given in (25) becomes much simpler than the result in [16].

By solving (25) for $\sin δ$ and $\cos δ$ one obtain

$$\sin δ = \frac{B(P - C) ± A \sqrt{A^2 + B^2 - (P - C)^2}}{A^2 + B^2},$$  \hspace{1cm} (32)

$$\cos δ = \frac{A(P - C) ± B \sqrt{A^2 + B^2 - (P - C)^2}}{A^2 + B^2}.$$  \hspace{1cm} (33)

Thus, we can determine the value of CP phase except for the ambiguity of the sign from the measurement of the probability. The sign ambiguity is understood as follows. If we measure the probability of the neutrino at a fixed energy and a baseline, we find the solutions on a “line a”. As shown in Fig. 1, there are two intersections $X$ and $Y$ of “line a” with the CP trajectory. This is the reason why the ambiguity due to the sign appears in the analytic solutions (32) and (33).

In order to resolve the sign ambiguity, we need to measure more than two kinds of probabilities, for example, neutrino and antineutrino. We denote $P$ and $\bar{P}$ of the oscillation probabilities for neutrino and antineutrino, respectively, as

$$P = A \cos δ + B \sin δ + C,$$  \hspace{1cm} (34)

$$\bar{P} = \bar{A} \cos δ + \bar{B} \sin δ + \bar{C}.$$  \hspace{1cm} (35)

Then, CP phase can be determined by

$$\sin δ = \frac{(\bar{A}P - A\bar{P}) - (\bar{A}C - A\bar{C})}{\bar{A}B - AB},$$  \hspace{1cm} (36)

$$\cos δ = \frac{(\bar{B}P - B\bar{P}) - (\bar{B}C - B\bar{C})}{\bar{B}A - BA},$$  \hspace{1cm} (37)

without the ambiguity of the sign. This means that the solution is at $X$, the intersect of “line a” and “line b”.

---

Fig. 1. An example of CP trajectory. We take $P$ for the horizontal axis and $\bar{P}$ for the vertical axis. The value of $δ$ changes from 0 to $2\pi$. 
Although the value of CP phase is determined, in principle, in (36) and (37), there remain other ambiguities included in $A$, $B$, $C$ and $\bar{A}$, $\bar{B}$, $\bar{C}$. The methods to resolve these ambiguities are discussed in the references for example [12,25–27]. We discuss the ambiguities due to the sign of mass-squared differences in Section 5.

4. Simple derivation of approximate formula

In the previous section, we have shown that the exact formula of the oscillation probability can be expressed as $P(\nu_e \rightarrow \nu_\mu) = A \cos \delta + B \sin \delta + C$. In this section, we demonstrate that the approximate formula seen in [10,11] is easily derived as an example in the case of $m_3 < m_2 \ll m_1$. One obtains the approximate formula for other patterns of mass hierarchy in the same way.

Let us first consider the coefficient $B$ of $\sin \delta$. $B$ is expressed in the form of the sum as (27). Note that, under the condition $x + y + z = 0$, the identity

$$\sin 2x + \sin 2y + \sin 2z = -4 \sin x \sin y \sin z,$$

holds, and $B$ from (27) is rewritten in the form of product as

$$B = \sum_{(ij)} B' \sin \left( \frac{\Delta_{ij} L}{2E} \right)$$

$$= -4B' \sin \left( \frac{\Delta_{12} L}{4E} \right) \sin \left( \frac{\Delta_{23} L}{4E} \right) \sin \left( \frac{\Delta_{31} L}{4E} \right).$$

Next, let us consider the coefficient $A$ of $\cos \delta$. Under the same condition as in deriving $B$, the identity

$$\sin^2 x = -(\sin x \sin y \cos z + \sin x \cos y \sin z)$$

holds and $A$ is rewritten as

$$A = \sum_{(ij)} A_{ij} \sin^2 \left( \frac{\Delta_{ij} L}{2E} \right)$$

$$= -\sum_{(ijk)} (A_{jk} + A_{kj}) \cos \left( \frac{\Delta_{jk} L}{4E} \right) \sin \left( \frac{\Delta_{jk} L}{4E} \right) \sin \left( \frac{\Delta_{ij} L}{4E} \right).$$

Substituting (19) and (20) for $p$ and $q$ in (29)–(31), $A$, $B$ and $C$ are rewritten with the masses and the mixing angles as

$$A = \sum_{(ijk)} \frac{-8J_r \Delta_{21} [\Delta_{31} \lambda_k (\lambda_k - \Delta_{31}) + A_k^{(1)}]}{\Delta_{jk}^2 \Delta_{ki}^2}$$

$$\times \cos \left( \frac{\Delta_{ij} L}{4E} \right) \sin \left( \frac{\Delta_{jkl} L}{4E} \right) \sin \left( \frac{\Delta_{kl} L}{4E} \right),$$

$$B = \frac{8 \Delta_{12} \Delta_{23} \Delta_{31}}{\Delta_{12} \Delta_{23} \Delta_{31}} J_r$$

$$\times \sin \left( \frac{\Delta_{12} L}{4E} \right) \sin \left( \frac{\Delta_{23} L}{4E} \right) \sin \left( \frac{\Delta_{31} L}{4E} \right),$$

$$C = \sum_{(ij)} (-4) \left[ s_{13}^2 (s_{23}^2 c_{13}^2 \Delta_{31}^2 \lambda_i \lambda_j + C_{ij}^{(1)} + C_{ij}^{(2a)}) \right]$$

$$+ C_{ij}^{(2b)} \left( \Delta_{ij} \Delta_{12} \Delta_{23} \Delta_{31} \right)^{-1}$$

$$\times \sin^2 \left( \frac{\Delta_{ij} L}{4E} \right),$$

where $J_r = s_{12} c_{13}^2 s_{23} c_{13} s_{13}^2 c_{13}^2$, and

$$A_k^{(1)} = \Delta_{21} \left[ \Delta_{31} \lambda_k (c_{12}^2 - s_{12}^2) + \lambda_k^2 s_{12}^2 - \Delta_{31}^2 c_{12}^2 \right].$$

$$C_{ij}^{(1)} = \Delta_{21} \Delta_{31} \left[ -\lambda_i (\lambda_j s_{12}^2 - \Delta_{31} c_{12}^2) - \lambda_j (\lambda_i s_{12}^2 + \Delta_{31} c_{12}^2) \right] s_{23}^2 c_{13}^2,$$

$$C_{ij}^{(2a)} = \Delta_{21}^2 (\lambda_i s_{12}^2 + \Delta_{31} c_{12}^2)$$

$$\times (\lambda_j s_{12}^2 + \Delta_{31} c_{12}^2) s_{23}^2 c_{13},$$

$$C_{ij}^{(2b)} = \Delta_{21}^2 (\lambda_i - \Delta_{31}) (\lambda_j - \Delta_{31}) s_{12}^2 c_{12}^2 c_{13}^2,$$

Note that these expressions are still exact. In the limit of small $\Delta_{21}$, terms given in Eqs. (47)–(50) are higher order in $\Delta_{21}$ and can be ignored. The superscripts of $A$ and $C$ stand for the power of $\Delta_{21}$, and (2a) represents the term proportional to $s_{13}^2$ and (2b) is the term independent of $s_{13}^2$.

Finally, we obtain the well-known approximate formula by neglecting the smallest effective mass. In the high energy neutrino the smallest effective mass is $\lambda_1 \simeq \Delta_{21}$. Other effective masses $\lambda_2$ and $\lambda_3$, correspond to $a$ or $\Delta_{31}$. Accordingly, $A$, $B$ and $C$ are
approximated by
\[
A = \frac{8 J_\nu \Delta_{21} \Delta_{31}}{a(\Delta_{31} - a)} \cos \left( \frac{\Delta_{31} L}{4E} \right) \sin \left( \frac{a L}{4E} \right) \times \sin \left( \frac{(\Delta_{31} - a)L}{4E} \right),
\]
\[
B = \frac{8 J_\nu \Delta_{21} \Delta_{31}}{a(\Delta_{31} - a)} \sin \left( \frac{\Delta_{31} L}{4E} \right) \sin \left( \frac{a L}{4E} \right) \times \sin \left( \frac{(\Delta_{31} - a)L}{4E} \right),
\]
\[
C = \frac{4 \Delta_{31}^2}{(\Delta_{31} - a)^2} \Delta_{23}^2 \Delta_{13}^2 \sin^2 \left( \frac{(\Delta_{31} - a)L}{4E} \right),
\]
under the condition \(\Delta_{21}/\Delta_{31} < s_{13}\). When \(s_{13}\) is smaller than \(\Delta_{21}/\Delta_{31}\), the term \(C_{ij}^{(2)}\) independent of \(s_{13}\) becomes the dominant term. Although the approximate formula derived here is in agreement with the ones seen in [10,11], the derivation is rather simple. Moreover, one can easily reproduce the approximate formula in low energy [7].

5. Numerical analysis of CP odd/even part

In this section, we investigate the values of the coefficients \(A\), \(B\) and \(C\) in cases of neutrino and antineutrino using the exact expressions. We also investigate them changing the signs of \(\Delta_{31}\) and \(\Delta_{21}\).

In this numerical analysis, we take \(\theta_{12} = \pi/4\), \(|\Delta_{21}| = 10^{-3} \text{ eV}^2\), \(\theta_{23} = \pi/4\) and \(|\Delta_{31}| = 3 \times 10^{-3} \text{ eV}^2\) to be consistent with the LMA MSW solution to the solar neutrino problem [2,3] and the zenith-angle dependences of atmospheric neutrinos [1]. We also take \(\theta_{13} = 0.05\) within the upper bound of CHOOZ experiment [29]. The baseline length is typically taken to be \(L = 2900 \text{ km}\) and the matter density is taken to be 3.2 \text{ g}/\text{cm}^3.

In Fig. 2 we show the coefficients \(A\), \(B\) and \(C\) changing with the energy \(E\). We observe that the sign of \(A\) is opposite for example in Fig. 2(a) and (d). We also observe that \(A\) and \(B\) have the opposite sign but \(C\) has the same sign comparing Fig. 2(a) with (e). In addition, some peaks have appeared in all graphs of Fig. 2 with the change of energy. In case of \(\Delta_{31} > 0\), the peaks around 6 GeV in Fig. 2(a) for neutrinos are enhanced compared with those in Fig. 2(b) for antineutrinos. Inversely, in case of \(\Delta_{31} < 0\), the peaks in Fig. 2(d) for antineutrinos are enhanced compared with those in Fig. 2(c) for neutrinos.

These features are understood qualitatively from the approximate formula (51)–(53). First let us consider the sign of \(A\), \(B\) and \(C\). As we found from (51)–(53), when the signs of both \(\Delta_{31}\) and \(a\) change, the sign of \(A\) becomes opposite and the signs of \(B\) and \(C\) do not change. On the other hand, when the sign of \(\Delta_{21}\) changes, the signs of both \(A\) and \(B\) change while the sign of \(C\) does not change. Next, let us consider the magnitude of the peak around 6 GeV. These are strongly affected by the denominator \((\Delta_{31} - a)\). Since the signs of \(\Delta_{31}\) and \(a\) are opposite in Fig. 2(a) and (d), the denominator \((\Delta_{31} - a)\) becomes small and the magnitude of the peaks are enhanced. On the other hand, since the signs of \(\Delta_{31}\) and \(a\) are the same in Fig. 2(b) and (c), the peaks are suppressed. Finally, let us explain the position of the peak in Fig. 2(a) and (d) around 6 GeV. Roughly, the peak energy is determined by the following:

\[
\sin \left[ 1.27 \left( \frac{\Delta_{31} - a}{1 \text{ eV}^2} \right) \left( \frac{L}{1 \text{ km}} \right) \left( \frac{E}{1 \text{ GeV}} \right)^{-1} \right] \sim 1
\]

\[
\rightarrow E \simeq 6 \text{ GeV} \quad (\text{at } L = 2900 \text{ km}).
\]

As pointed out by Parke and Weiler [8], and Lipari [13], the peak energy is lower than the energy of 1–3 MSW resonance since the baseline length is short compared with the earth diameter. The above discussions on Fig. 2(a)–(e) can be applied to other figures.

We have studied how the magnitude of \(A\), \(B\) and \(C\) change due to the sign of the mass squared differences. In the case of \(m_1 < m_2 < m_3\), the coefficients have been investigated in Ref. [10] by using the approximate formula. These correspond to Fig. 2(a) and (b). The sign of \(\Delta_{31}\) is determined from the leading term \(C\) as pointed out by many authors (for example, see [10]). On the other hand, the sign of \(\Delta_{21}\) is determined from next leading terms \(A\) or \(B\). This means that the sign of \(\Delta_{21}\) is simultaneously determined in addition to the CP phase. It may be interesting as the first observation of the sign of \(\Delta_{21}\) using artificial neutrino beam.
Fig. 2. A, B, C at \( L = 2900 \) km. The graphs of the left and right side correspond to the neutrino and the antineutrino, respectively. The solid lines, the dotted lines and the dashed lines are for \( A, B \) and \( C \) in all graphs. And from top to bottom, \((\Delta_{31} > 0, \Delta_{21} > 0), (\Delta_{31} < 0, \Delta_{21} > 0), (\Delta_{31} > 0, \Delta_{21} < 0)\) and \((\Delta_{31} < 0, \Delta_{21} < 0)\) cases.
6. Summary

We have studied neutrino oscillations in constant matter within the framework of the three neutrino scenario. We summarize the results obtained in this Letter.

(i) We have derived an exact expression of the oscillation probability by using a new method. We have calculated \( \tilde{U} \tilde{U}^* \) from the identities without directly calculating single \( \tilde{U} \). Not only the derivation but also the result becomes simple and the matter effects enter only through effective masses.

(ii) We have obtained the CP-dependence of the oscillation probability exactly by using the standard parametrization. It has been shown that the oscillation probability is in the form,

\[
P(\nu_e \to \nu_\mu) = A \cos \delta + B \sin \delta + C.
\]

We have also demonstrated that the approximate formula in high energy can be easily reproduced from our result.

Finally, let us comment on the oscillation probabilities for other channels. These probabilities are easily derived in the same way as \( P(\nu_e \to \nu_\mu) \). \( P(\nu_e \to \nu_\tau) \) has the same CP-dependence as \( P(\nu_e \to \nu_\mu) \). However, \( P(\nu_\mu \to \nu_\tau) \) has the term which depends on \( \cos 2\delta \) in addition to the linear terms in \( \sin \delta \) and \( \cos \delta \).

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Bilarge leptonic mixing from Abelian horizontal symmetries

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Abstract

We construct and present a model for leptonic mixing based on higher-dimensional operators, using the Froggatt–Nielsen mechanism, and Abelian horizontal symmetries (flavor symmetries) of continuous and discrete type. Our model naturally yields bilarge leptonic mixing, coming from both the charged leptons and the neutrinos, and an inverted neutrino mass hierarchy spectrum. The obtained values of the parameters, i.e., the leptonic mixing parameters and the neutrino mass squared differences, are all consistent with the atmospheric neutrino data and the Mikheyev–Smirnov–Wolfenstein large mixing angle solution for the solar neutrino problem.

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Keywords: Leptonic mixing; Neutrino masses; Discrete symmetries; Higher-dimensional operators

1. Introduction

Predicting the pattern of fermion masses and mixings from a fundamental gauge theory is one of the major challenges in particle physics. In such an approach, the observed hierarchy of the fermion masses is usually understood in terms of some symmetry breaking interaction. In fact, due to their plausible Majorana nature, the extreme smallness of the neutrino masses could be associated with a violation of the $B-L$ symmetry. Thus, the neutrinos can shed light on the origin of the fermion masses and mixings, since most grand unified theories (GUTs) based on SO(10) or $E_6$ and also string theories indeed expect the $B-L$ symmetry to be broken [1]. In building models, it is therefore particularly important to naturally reproduce the neutrino mass squared differences and the leptonic mixing parameters that have been determined by the atmospheric [2,3] and solar [4] neutrino data.

The neutrino mass squared differences are generally defined as
\[
\Delta m_{\alpha\beta}^2 = m_{\alpha}^2 - m_{\beta}^2,
\]

where $m_{\alpha\beta}$ is the mass of the $\alpha$th neutrino mass eigenstate. We will here assume that there are three neutrino flavors, and therefore, three neutrino flavor states $\nu_\alpha$ ($\alpha = e, \mu, \tau$) and also three neutrino mass eigenstates $\nu_a$ ($a = 1, 2, 3$). The unitary leptonic
mixing matrix\(^1\) is then given by

\[ U = (U_{\alpha a}) \equiv U^L U^\nu , \]

where the unitary mixing matrix \( U^L (U^\nu) \) rotates the left-handed charged lepton fields (the neutrino fields) so that the charged lepton mass matrix (the neutrino mass matrix) becomes diagonal. Thus, the leptonic mixing matrix acquires contributions from both the charged leptons and the neutrinos. These contributions usually add in a non-trivial way (see Appendix A). For three neutrino flavors, in the so-called standard parameterization, the leptonic mixing matrix reads

\[ U = \begin{pmatrix}
S_1C_3 & S_2C_2 & S_3e^{-i\delta} \\
-S_1C_1 - S_1S_2C_3e^{i\delta} & C_1C_3 - S_1S_2S_3e^{i\delta} & S_2C_1 \\
S_1S_3 - S_2S_3C_1e^{i\delta} & -S_1S_3 - S_2S_3C_1e^{i\delta} & S_2C_3
\end{pmatrix} , \]

where \( S_a \equiv \sin \theta_a \), \( C_a \equiv \cos \theta_a \) (for \( a = 1, 2, 3 \)), and \( \delta \) is the physical \( CP \) phase. Here \( \theta_1 \equiv \theta_{23} \), \( \theta_2 \equiv \theta_{12} \), and \( \theta_3 \equiv \theta_{13} \) are the leptonic mixing angles. Recent results suggest that among the possible solutions to the solar neutrino problem, the Mikheyev–Smirnov–Wolfenstein (MSW) [6] large mixing angle (LMA) solution is somewhat preferred to the MSW small mixing angle (SMA) solution, the MSW low mass (LOW) solution, and the vacuum oscillation (VAC) solution [7–9]. Actually, a global solar two-flavor neutrino oscillation analysis including the latest SNO data strongly favors the MSW LMA solution [10]. However, the MSW LMA solution excludes maximal solar mixing at the 95% confidence level [8] (and now even at the 99.73% confidence level [9]), and therefore, it also disfavors the scenario of so-called bimaximal mixing [11]. Thus, we will instead, most probably, have a bilarge mixing scenario in which the solar mixing angle \( \theta_{12} \) is large, but not maximal, and the atmospheric mixing angle \( \theta_{23} \) is approximately maximal. It is interesting to observe that there are very strong indications that the leptonic mixing is large, whereas, on the other hand, it has turned out experimentally that the quark mixing is small [12].

In this Letter, we will investigate a model, which yields in a technically natural way bilarge leptonic mixing, reproduces the observed mass hierarchy of charged leptons, and leads to an inverted neutrino mass hierarchy spectrum. This will be achieved by generating lepton mass matrix textures, where the mixing of the charged leptons is comparable with the mixing of the quarks and the mixing of the neutrinos is essentially bimaximal. The striking difference between the bilarge leptonic mixing and the small quark mixing will then be accounted for by the neutrinos (mainly) and the charged leptons (partly).

This Letter is organized as follows: in Section 2, we will introduce a model by adding to the standard model (SM) a set of extra fields and horizontal symmetries that will give rise to specific effective Yukawa interactions for the charged leptons. Next, minimizing the corresponding scalar potential, we will naturally obtain a mass matrix texture for the charged leptons, which is in agreement with experimental data. In Section 3, we will extend the representation content of the model in order to also obtain a realistic mass matrix texture for the neutrinos. (In Appendix A, we will explicitly calculate the leptonic mixing angles coming from the diagonalizations of the charged lepton and neutrino mass matrices, respectively.) Finally, in Section 4, we will present a summary as well as our conclusions.

2. Charged leptons

2.1. Horizontal symmetries

We will here consider an extension of the SM in which the lepton masses arise from higher-dimension-al operators [13] via the Froggatt–Nielsen mechanism [14]. (For recent studies, see, e.g., Ref. [15].) We will write, in a self-explanatory notation, the lepton doublets as \( L_\alpha = (\nu_\alpha L e_\alpha R) \), where \( \alpha = e, \mu, \tau \) and the right-handed charged leptons as \( E_\alpha = e_R \), where \( \alpha = e, \mu, \tau \). Suppose that the part of the scalar sector, which transforms non-trivially under the SM gauge group, consists of two Higgs doublets \( H_1 \) and \( H_2 \), where \( H_1 \) couples to the neutrinos and \( H_2 \) to the charged leptons.\(^2\) (For simplicity and without loss of generality, the quark sector will be left out in our entire discussion.) Let us first restrict our discussion

\(^1\) The leptonic mixing matrix is sometimes called the Maki–Nakagawa–Sakata (MNS) mixing matrix [5].

\(^2\) This can easily be achieved by imposing a discrete \( \mathbb{Z}_2 \) symmetry under which \( H_2 \) and \( E_\alpha (\alpha = e, \mu, \tau) \) are odd and \( H_1 \) and the rest of the SM fields are even.
to the generation of the charged lepton masses. In order to obtain the structure of the charged lepton mass matrix from an underlying symmetry principle, we will further extend the scalar sector by SM singlet scalar fields $\phi_i$ ($i = 1, 2, \ldots, 8$) and $\theta$ and assign the fields gauged horizontal U(1) charges $Q_1$, $Q_2$, and $Q_3$ as in Table 1.

In the rest of the Letter, it is always understood that the Higgs doublets $H_1$ and $H_2$ are total singlets under transformations of the additional symmetries. Note that our model is kept anomaly-free, since the fermions transform as vector-like pairs under the extra U(1) charges. The charges $(Q_1, Q_2)$ of the charged lepton–antilepton pairs are given by Table 2.

We observe that these charges forbid dimension-four Yukawa coupling terms for the coupling of the first generation to the second and third generations. A realistic charged lepton mass matrix will arise if, in addition to the U(1) charges, we introduce a set of discrete symmetries $\mathcal{D}_i$ ($i = 1, 2, \ldots, 5$), which are, at this level, not plagued with chiral anomalies. The $\mathbb{Z}_2$ symmetry

\[
\mathcal{D}_2: \begin{align*}
E_\mu &\rightarrow -E_\mu, \\
\phi_1 &\rightarrow -\phi_1, \\
\phi_2 &\rightarrow -\phi_2
\end{align*}
\]

sets to leading order the $e-e$-element of the charged lepton mass matrix equal to zero. It has been pointed out that bimaximal leptonic mixing corresponds to a permutation symmetry of the second and third generation [16]. Thus, we will introduce the three permutation symmetries

\[
\mathcal{D}_3: \begin{align*}
L_\mu &\rightarrow -L_\mu, \\
E_\mu &\rightarrow -E_\mu, \\
\phi_1 &\leftrightarrow \phi_2, \\
\phi_3 &\leftrightarrow \phi_4
\end{align*}
\]

\[
\mathcal{D}_4: \begin{align*}
L_\mu &\rightarrow -L_\mu, \\
\phi_1 &\leftrightarrow \phi_2, \\
\phi_5 &\leftrightarrow \phi_6
\end{align*}
\]

\[
\mathcal{D}_5: \begin{align*}
\phi_2 &\rightarrow -\phi_2, \\
\phi_4 &\rightarrow -\phi_4, \\
\phi_6 &\rightarrow -\phi_6, \\
\phi_7 &\rightarrow -\phi_7
\end{align*}
\]

Then, the most general charged lepton mass terms, which are invariant under all symmetry transformations of our model, are given by the higher-dimensional operators

\[
\mathcal{L} = \sum_i H_i [ (Y_{\text{eff}}^1)_{\alpha \beta} + (Y_{\text{eff}}^2)_{\alpha \beta} ] E_\beta + \text{h.c.},
\]

where the relevant effective Yukawa interaction matrices $Y_{\text{eff}}^1$ and $Y_{\text{eff}}^2$ are on the forms

\[
Y_{\text{eff}}^1 = \begin{pmatrix}
0 & B(\phi_3 - \phi_4) & B(\phi_3 + \phi_4) \\
A(\phi_1 - \phi_2) & C(\phi_3 - \phi_6) & 0 \\
A(\phi_1 + \phi_2) & 0 & C(\phi_5 + \phi_6)
\end{pmatrix},
\]

(5a)

\[
Y_{\text{eff}}^2 = \text{diag}(0, D\phi_7, D\phi_8).
\]

(5b)

Here, the dimensionful coefficients $A$, $B$, $C$, and $D$ are given by

\[
A = Y_a \frac{\theta^2}{M_1^2},
\]

(6a)

\[
B = Y_b \frac{\theta^2}{M_1^2},
\]

(6b)

\[
C = Y_c \frac{1}{M_1^2},
\]

(6c)

\[
D = Y_d \frac{\theta}{M_1^2},
\]

(6d)

where the quantities $Y_a$, $Y_b$, $Y_c$, and $Y_d$ are arbitrary order unity coefficients and $M_1$ is the high mass scale.

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<th>$E_\mu$</th>
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<tbody>
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<td>$(Q_1, Q_2)$</td>
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<td>$(0, 1)$</td>
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<tr>
<td>$L_e$</td>
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<td>$L_\mu$</td>
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<td>$(1, −1)$</td>
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<td>$L_\tau$</td>
<td>$(0, −1)$</td>
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</table>
of the intermediate Froggatt–Nielsen states. Actually, in Section 2.3, the mass scale $M_1$ will be related to the breakdown scale of the extra symmetries by a small expansion parameter.

2.2. The scalar potential

The most general renormalizable scalar potential, involving only the fields $\phi_i$ ($i = 1, 2, \ldots, 6$), which is invariant under transformations of the horizontal symmetries given in Section 2.1, reads

$$V = \mu_1^2 (|\phi_1|^2 + |\phi_2|^2) + 2\mu_2^2 (|\phi_3|^2 + |\phi_4|^2) + \lambda_1^2 (|\phi_5|^2 + |\phi_5|^2)$$

$$+ \lambda_2^2 (|\phi_6|^2 + |\phi_6|^2) + \lambda_3^2 (|\phi_1|^2 |\phi_2|^2 + |\phi_3|^2 |\phi_4|^2 + |\phi_5|^2 |\phi_6|^2)$$

$$+ M_1^2 (\phi_1^† \phi_2 + \phi_1^† \phi_2 + \phi_1^† \phi_3) + M_2 (\phi_1^† \phi_2 + \phi_1^† \phi_3) + M_3 (\phi_1^† \phi_2 + \phi_1^† \phi_3), \quad (7)$$

where all coefficients are real. Due to the symmetries of our model, the remaining scalar fields enter relevant terms in the potential $V$ only via quartic couplings in form of absolute squares of these fields, which means that they can be combined into the coefficients $\mu_i$ ($i = 1, 2, 3$). Therefore, we can choose the coefficients in the potential to fulfill $\mu_i^2 < 0$ ($i = 1, 2, 3$), $\lambda_i > 0$ ($i = 1, 2, 3$), and $a, b, \ldots, f > 0$, which yield after spontaneous symmetry breaking (SSB) non-vanishing vacuum expectation values (VEVs) that satisfy

$$|\langle \phi_1 \rangle| = |\langle \phi_2 \rangle|, \quad |\langle \phi_3 \rangle| = |\langle \phi_4 \rangle|,$$

$$|\langle \phi_5 \rangle| = |\langle \phi_6 \rangle|.$$ 

If $\lambda_i < 0$ ($i = 1, 2, 3$), then we obtain pairwise relatively real VEVs, i.e.,

$$\frac{\langle \phi_1 \rangle}{\langle \phi_2 \rangle} \frac{\langle \phi_3 \rangle}{\langle \phi_4 \rangle} \in \{-1, 1\}.$$

Next, choosing $m_1 < 0$ and $m_2, m_3 > 0$, we obtain

$$\frac{\langle \phi_1 \rangle}{\langle \phi_2 \rangle} = 1 \quad \text{and} \quad \frac{\langle \phi_3 \rangle}{\langle \phi_4 \rangle} = -1,$$

i.e., the relative sign between $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$ is equal to the relative sign between $\langle \phi_3 \rangle$ and $\langle \phi_4 \rangle$ and opposite to the relative sign between $\langle \phi_5 \rangle$ and $\langle \phi_6 \rangle$. In Section 2.3, we will show that this alignment mechanism reconciles the permutation symmetry $D_5$ with an approximate diagonal form of the charged lepton mass matrix.

2.3. The charged lepton mass matrix

Suppose that the SM singlet scalar fields acquire their VEVs at a high mass scale and thereby give rise to a small expansion parameter

$$\epsilon \simeq \frac{\langle \phi_i \rangle}{M_1} \simeq \frac{\langle \theta \rangle}{M_1} \simeq 10^{-1}, \quad (8)$$

where $i = 1, 2, \ldots, 8$. Such small hierarchies can arise from large hierarchies in supersymmetric theories when the scalar fields acquire their VEVs along a “D-flat” direction [17]. As a consequence of the permutation symmetries $D_3, D_4,$ and $D_5$, the lowest energy state is two-fold degenerate. Applying the results of Section 2.2 and inserting Eq. (8) into Eqs. (6) and the result thereof into Eqs. (5), we obtain the two possible charged lepton mass matrices

$$M_{\ell} \simeq m_{\ell} \begin{pmatrix} 0 & \epsilon^2 & 0 \\ \epsilon^2 & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (9)$$

and

$$M_{\ell} \simeq m_{\ell} \begin{pmatrix} 0 & 0 & \epsilon^2 \\ 0 & 1 & 0 \\ \epsilon^2 & 0 & \epsilon \end{pmatrix}. \quad (10)$$

where $m_{\ell}$ is the tau mass and only the order of magnitude of the matrix elements have been indicated. Note that a permutation of the second and third generations, $L_\mu \leftrightarrow L_\tau$, $E_\mu \leftrightarrow E_\tau$, leads from one solution to the other. Let us take the first one. Diagonalization of $M_{\ell}$ [in Eq. (9)] then gives for the charged lepton masses
the order-of-magnitude relations\(^3\)

\[
\frac{m_e}{m_\mu} \simeq \epsilon^2 \simeq 10^{-2} \quad \text{and} \quad \frac{m_\mu}{m_\tau} \simeq \epsilon \simeq 10^{-1},
\]

which approximately fit the experimentally observed values [i.e., \((m_e/m_\mu)_{\text{exp}} \simeq 4.8 \times 10^{-3}\) and \((m_\mu/m_\tau)_{\text{exp}} \simeq 5.9 \times 10^{-3}\) [12]]. Here \(m_e\) and \(m_\mu\) are the electron and muon masses, respectively. The charged lepton mass matrix \(M_\ell\) are the electron and muon masses, respectively. The charged lepton mass matrix \(M_\ell\) is diagonalized by a rotation of the left-handed charged lepton fields in the 1–2-plane by an angle \(\theta_{12}^{\ell} \simeq 6^\circ (\theta_{12}^{\mu} = \frac{\epsilon}{\sqrt{\lambda_2 \epsilon^2}} + O(\epsilon^3), \text{where } \epsilon \simeq 0.1, \theta_{13}^\ell = 0, \text{and } \theta_{23}^\ell = 0)\), which will finally give a contribution to all leptonic mixing angles (see Appendix A).

3. The neutrino mass matrix

Let us now turn our discussion to the neutrino mass matrix. As intermediate Froggatt–Nielsen states we will assume two heavy SM singlet Dirac fermions \(F\), which have masses of the order \(M_1\). In order to account for the smallness of the neutrino masses, we will furthermore introduce three SM singlet Dirac fermions \(N_\ell\), \(N_\mu\), and \(N_\tau\), which have masses of the order of some relevant high mass scale \(M_2\). From the assignment of the charges \((Q_1, Q_2)\) to the lepton doublets, the structure of the \((Q_1, Q_2)\) charges associated with the effective neutrino mass matrix follows immediately as in Table 3.

Note that the given representation content so far forbids any neutrino mass term. We therefore introduce the additional SM singlet scalar fields \(\phi_9, \phi_{10}, \phi_{11}\), and \(\phi_{12}\) and we assign the charges \(Q_1, Q_2, \text{and } Q_3\) to the fields as in Table 4.

It is easily verified that the results for the charged lepton mass matrix remain unchanged by this new representation content. The leading order tree-level realizations of the higher-dimensional operators, which generate the neutrino masses, are shown in Figs. 1 and 2. After SSB, the effective neutrino mass matrix \(M_\nu\) will be on the approximate bimaximal mixing form [11]

\[
M_\nu = \begin{pmatrix}
A' & B' & -B' \\
B' & 0 & 0 \\
-B' & 0 & 0
\end{pmatrix},
\]

where

\[
A' = Y'_{\nu} \frac{(H_1)^2}{M_2} \langle \phi_{11}\rangle \langle \phi_{12}\rangle |\theta|,
\]

\[
B' = Y'_{\nu} \frac{(H_1)^2}{M_2} \langle \phi_{10}\rangle M_1,
\]

and the quantities \(Y'_{\nu}\) and \(Y'_{\nu}\) are arbitrary order unity coefficients. Note that the permutation symmetry \(D_3\) establishes \(\langle |\phi_{10}\rangle | = |\langle |\phi_{10}\rangle |\), and hence, the magnitudes of the \(e-\mu\)- and \(e-\tau\)-entries in the effective neutrino mass matrix \(M_\nu\) are exactly degenerate. The possible relative phase \(\varphi\) between these entries can be eliminated by, e.g., the field redefinition \(L_\tau \rightarrow e^{i\varphi} L_\tau\),
φτ

m

matrix from Eqs. (13). Thus, diagonalizing the neutrino mass also applies to the indices charged lepton sector. If we assume that all SM singlet since it does not affect the mixing angles in the

Fig. 2. The dimension-eight operators, generating the e-e-element in the effective neutrino mass matrix.

since it does not affect the mixing angles in the charged lepton sector. If we assume that all SM singlet scalar fields acquire their VEVs at the breakdown scale of the additional horizontal symmetries, then Eq. (8) also applies to the indices \( i = 9, 10, 11, 12 \). Therefore, we can parameterize the effective neutrino mass matrix \( M_\nu \) with the same expansion parameter \( \epsilon \) that was used for the charged lepton mass matrix \( M_L \) and we find the relative suppression ratio \( \Delta m^2 / \Delta m^2_{\text{best-fit}} \simeq 10^{-2} \) from Eqs. (13). Thus, diagonalizing the neutrino mass matrix \( M_\nu \) in Eq. (12), we obtain the following spectrum of the neutrino masses\(^4\)

\[
m_1 \simeq m_2 \quad \text{and} \quad m_3 = 0,
\]

which is on the inverted hierarchical form (i.e., \( m_3 \ll m_1 \simeq m_2 \Rightarrow 0 \simeq |\Delta m^2_{21}| \ll |\Delta m^2_{32}| \simeq |\Delta m^2_{31}| \)). Using the MSW LMA and atmospheric neutrino mass squared differences\(^5\) [3,9]

\[
\Delta m^2_{\odot} \equiv |\Delta m^2_{21}| \simeq 3.7 \times 10^{-5} \text{ eV}^2 \sim 10^{-5} \text{ eV}^2, \\
\Delta m^2_{\text{atm}} \equiv |\Delta m^2_{32}| \simeq 2.5 \times 10^{-3} \text{ eV}^2 \sim 10^{-3} \text{ eV}^2.
\]

\(^4\) The neutrino mass spectrum is: \( m_1 = |\lambda'_1|, m_2 = |\lambda'_2|, m_3 = |\lambda'_3| \), where \( \lambda'_1 \simeq -B' \sqrt{2}, \lambda'_2 \simeq B' \sqrt{2} \), and \( \lambda'_3 = 0 \) are the eigenvalues of the matrix \( M_\nu \).

\(^5\) Here \( \Delta m^2_{21} = \lambda'_2 - \lambda'_1 \), i.e., \( |\Delta m^2_{21}| \simeq 2 \sqrt{2} |A'B'| \) and \( |\Delta m^2_{32}| \simeq |\Delta m^2_{31}| \simeq 2 |B'|^2 \).

and a VEV of the order of the electroweak scale \( \langle H_1 \rangle \simeq 10^2 \text{ GeV} \), we obtain from Eqs. (13) the high mass scale \( M_2 \simeq 10^{13} \text{ GeV} \) (as well as \( m_1 \simeq m_2 \simeq 0.05 \text{ eV} \) and \( \epsilon \simeq 0.1 \)). The entry \( A' \) of the matrix in Eq. (12) induces a deviation from maximal solar mixing, which is of the order \( 0.1^\circ (\theta^0_{12} = \frac{\pi}{4} + \frac{1}{4\sqrt{2}} \epsilon^2 - \frac{1}{9\sqrt{2}} \lambda^0 \epsilon^4 + O(\epsilon^{10})) \), where \( \epsilon \simeq \sqrt{A'/B'} \simeq 0.11, \theta^0_{13} = 0, \text{ and } \theta^0_{23} = 45^\circ \). This deviation can be neglected in comparison with the contribution coming from the charged lepton sector, which is about \( 6^\circ \) (see Section 2.3), resulting in a change of the mixing angles \( \theta_{12} \) and \( \theta_{13} \) by approximately \( 4^\circ \), while the atmospheric mixing angle \( \theta_{23} \) practically stays maximal (see Appendix A). Thus, the leptonic mixing angles are

\[
\theta_{12} \simeq 41^\circ, \quad \theta_{13} \simeq 4^\circ, \quad \text{and} \quad \theta_{23} \simeq 45^\circ.
\]

Hence, our model predicts the charged lepton mass hierarchy spectrum, an inverted neutrino mass hierarchy spectrum, bilarge leptonic mixing, as well as it reproduces the mass squared differences to lie within the ranges preferred by the MSW LMA solution\(^6\) and atmospheric neutrino data.\(^7\) In particular, it yields a significant deviation from maximal solar mixing. However, the solar mixing angle is bounded from below by approximately \( 41^\circ \) and it is therefore still too close to maximal to be in the 95% (or 99.73%) confidence level region of the MSW LMA solution\(^8\) [8,9].

4. Summary and conclusions

In summary, we have presented a model built upon extra fields, horizontal (flavor) symmetries, and higher-dimensional operators including the Froggatt–Nielsen mechanism. This model naturally yields the well-known mass matrix textures

\[
\begin{pmatrix}
0 & \epsilon^2 & 0 \\
\epsilon^2 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\epsilon^2 & 1 & -1 \\
1 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}
\]

\(^6\) 99.73% C.L.: \( 1.9 \times 10^{-5} \text{ eV}^2 \leq \Delta m^2_{\odot} \leq 2.7 \times 10^{-4} \text{ eV}^2 \) [18]; best-fit: \( \Delta m^2_{\odot} \simeq 3.7 \times 10^{-5} \text{ eV}^2 \) [9].

\(^7\) 90% C.L.: \( 1.6 \times 10^{-3} \text{ eV}^2 \leq \Delta m^2_{\text{atm}} \leq 4.0 \times 10^{-3} \text{ eV}^2 \); best-fit: \( \Delta m^2_{\text{atm}} \simeq 2.5 \times 10^{-3} \text{ eV}^2 \) [3].

\(^8\) 99.73% C.L.: \( 0.22 \leq \tan^2 \theta_{23} \leq 0.71 \Rightarrow 25^\circ \leq |\theta_{23}| \leq 45^\circ \) [18]; best-fit: \( \tan^2 \theta_{23} \simeq 0.37 \Rightarrow |\theta_{23}| \simeq 31^\circ \) [9].
for the charged leptons and the neutrinos, respectively, which involve the same small expansion parameter $\epsilon \simeq 0.1$. These textures reproduce the mass hierarchy among the charged leptons very accurately as well as they give rise to the most probable values of the mass squared differences for the neutrinos coming from solar and atmospheric neutrino data. In addition, assuming no $CP$ violation, i.e., $\delta = 0$, the model gives bilarge leptonic mixing, i.e., $\theta_{12} \simeq 41^\circ$, $\theta_{13} \simeq 4^\circ$, and $\theta_{23} \simeq 45^\circ$, which is in very good agreement with the present experimental data. The mixing angle $\theta_{12}$ is on the borderline of being compatible with the MSW LMA solution, which, of course, has, however, been further strengthened by recent SNO results [10] and the CHOOZ experiment [19], whereas $\theta_{13}$ is on the borderline of being compatible with the present experimental data. $\theta_{23}$ was further strengthened by recent SNO results [10] and the MSW LMA solution, which has, however, been below the CHOOZ upper bound, and $\theta_{23}$ fits perfectly the best-fit value from the Super-Kamiokande collaboration of their atmospheric neutrino data.

As a final conclusion, we have shown in Appendix A that the leptonic mixing angles are all dependent on the "solar" mixing angle $\theta_{12}$ in the charged lepton sector in a non-trivial way.

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Appendix A. The leptonic mixing matrix: a special case

The leptonic mixing matrix can be written as

$$U = O_{23}(\theta_{23}) O_{13}(\theta_{13}, \delta) O_{12}(\theta_{12})$$

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & C_{23} & S_{23} \\
0 & -S_{23} & C_{23}
\end{pmatrix}
\]

where

\[
\begin{pmatrix}
C_{13} & 0 & S_{13} e^{-i\delta} \\
0 & 1 & 0 \\
-S_{13} e^{i\delta} & 0 & C_{13}
\end{pmatrix}
\] \times

\[
\begin{pmatrix}
C_{12} & S_{12} & 0 \\
-S_{12} & C_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\left( C_{13} \right) \left( \theta_{12} \right) \\
\left( 0 \right) \left( \theta_{12} \right) \\
\left( -S_{13} e^{i\delta} \right) \left( \theta_{12} \right)
\end{pmatrix}
\]

\[
\begin{pmatrix}
C_{12} \left( \theta_{12} \right) \\
S_{12} \left( \theta_{12} \right) \\
-S_{12} \left( \theta_{12} \right)
\end{pmatrix}
\]

Thus, inserting Eqs. (A.2) into the definition of the leptonic mixing matrix, $U = U_{aa} \equiv U^{\ell \ell} U^{\nu}$, we find that

$$U = O_{12}^{\ell \ell} O_{13}^{\ell \ell} O_{23}^{\ell \ell}$$

Furthermore, assuming that we have only a small mixing coming from the mixing angle $\theta_{12}$ in the charged lepton sector ($\theta_{12}^{\ell} = 0, \theta_{23}^{\ell} = 0$) and bimaximal mixing in the neutrino sector ($\theta_{12}^{\nu} = 45^\circ, \theta_{13}^{\nu} = 0, \theta_{23}^{\nu} = 45^\circ$), we then obtain

$$U = O_{12}^{\ell \ell} (\theta_{12}^{\ell}) O_{23}^{\nu} (\theta_{23}^{\nu}) O_{13}^{\nu} (\theta_{13}^{\nu})$$

The mixing angles (in the standard parameterization) of a $3 \times 3$ orthogonal mixing matrix can be read off as follows [20]:

$$\theta_{12} = \arctan \frac{U_{e2}}{U_{e1}}$$

$$\theta_{13} = \arcsin U_{e3}$$

$$\theta_{23} = \arctan \frac{U_{\mu3}}{U_{\tau3}}$$

Thus, inserting the appropriate matrix elements of the matrix $U$ in Eq. (A.4) into Eqs. (A.5)–(A.7), we finally
obtain
\[ \theta_{12} = \arctan \frac{\cos \theta_{12}^\ell - \frac{1}{\sqrt{2}} \sin \theta_{12}^\ell}{\cos \theta_{12}^\ell + \frac{1}{\sqrt{2}} \sin \theta_{12}^\ell}, \]  
(A.8)
\[ \theta_{13} = -\arcsin \left( \frac{1}{\sqrt{2}} \sin \theta_{12}^\ell \right), \]  
(A.9)
\[ \theta_{23} = \arctan \cos \theta_{12}^\ell. \]  
(A.10)

When \( \theta_{12}^\ell \) is small (\( \theta_{12}^\ell \ll 1 \)), we have\(^9\)
\[ \theta_{12} = \frac{\pi}{4} - \frac{1}{\sqrt{2}} \theta_{12}^\ell - \frac{1}{6\sqrt{2}} \theta_{12}^\ell^3 + O(\theta_{12}^\ell^5), \]  
(A.11)
\[ \theta_{13} = -\frac{1}{\sqrt{2}} \theta_{12}^\ell + \frac{1}{12\sqrt{2}} \theta_{12}^\ell^3 + O(\theta_{12}^\ell^5), \]  
(A.12)
\[ \theta_{23} = \frac{\pi}{4} - \frac{1}{4} \theta_{12}^\ell + \frac{1}{24} \theta_{12}^\ell^3 + O(\theta_{12}^\ell^6). \]  
(A.13)

Note that all leptonic mixing angles receive contribution from the small mixing angle \( \theta_{12}^\ell \) in the charged lepton sector. Furthermore, we observe that \( \theta_{12} \) has first order corrections in \( \theta_{12}^\ell \), whereas \( \theta_{23} \) has only second order corrections in \( \theta_{12}^\ell \). The mixing angle \( \theta_{13} \) is directly proportional to the mixing angle \( \theta_{12}^\ell \) (when \( \theta_{12}^\ell \) is small), which means that if \( \theta_{12}^\ell \) is small, then \( \theta_{13} \) will also be small. In fact, \( |\theta_{13}| \ll 13.1^{\circ} \) has to be fulfilled in order for the mixing angle \( \theta_{13} \) to be below the CHOOZ upper bound \( \sin^2 2\theta_{13} \lesssim 0.10 \) (i.e., \( |\theta_{13}| \lesssim 9.2^{\circ} \)) [21].

References

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\(^9\) Introducing a small deviation \( \eta \) from maximal solar mixing in the neutrino sector (i.e., \( \theta_{12}^\ell = 45^{\circ} \rightarrow \theta_{12} = 45^{\circ} + \eta \), where \( \eta \ll 1 \)), we find that \( \theta_{12} = \frac{\pi}{4} + \eta - \frac{1}{6\sqrt{2}} \theta_{12}^\ell - \frac{1}{6\sqrt{2}} \theta_{12}^\ell^3 + O(\theta_{12}^\ell^5) \), whereas \( \theta_{12}^\ell \) and \( \theta_{23} \) remain unchanged.
On the failure of spin–statistics connection in quantum gravity

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Abstract

Many years ago Friedman and Sorkin [Phys. Rev. Lett. 44 (1980) 1100; Gen. Rel. Grav. 14 (1982) 615] established the existence of certain topological solitonic excitations in quantum gravity called (topological) geons. Geons can have quantum numbers like charge and can be tensorial or spinorial having integer or half-odd integer spin. A striking result is that geons can violate the canonical spin–statistics connection. Such violation induces novel physical effects at low energies. The latter will be small since the geon mass is expected to be of the order of Planck mass. Nevertheless, these effects are very striking and include CPT and causality violations and distortion of the cosmic microwave spectrum. Interesting relations of geon dynamics to supersymmetry are also discussed. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

The spin–statistics connection asserts that tensorial particles (those with integral spin) obey Bose statistics and spinorial ones obey Fermi statistics. It has a central role in determining properties of matter including its stability and is generally regarded as a fundamental result of quantum physics. It has a counterpart in 2 + 1 dimensions where particles of fractional spin $\theta$ are asserted to obey fractional statistics with the same $\theta$.

Our understanding of this connection however is not perfect. It has been proved using the axioms of relativistic local quantum field theories (RQFTs) [4]. It has also been established for Skyrme-like solitons and ’t Hooft–Polyakov monopoles [5], and particles of fractional spin in 2 + 1 dimensions [6], apparently using physical principles which are mutually different and different too from those of RQFT’s. In the literature, we also encounter proofs of this connection using yet other considerations [7]. In addition, no such theorem can be established in conventional nonrelativistic physics. The standard spin–statistics connection cannot also be established in generality for the gravitational topological excitations known as geons [2,3,8–10], although certain novel spin–statistics connections can be proved for them [8].

The lack of spin–statistics connection in quantum gravity attracts attention. Nonrelativistic physics is a limiting form of RQFT’s and therefore its loss there is attributed to an imperfect limiting procedure. But this escape route
is not available for quantum gravity with its enormous energy scales where RQFT loses its validity. It is rather the latter which is a limiting form of a unified model for gravity and elementary particles.

Studies of the common principles underlying the different approaches to this connection suggest that it needs the possibility of creation–annihilation processes. Nonrelativistic models incorporating such events have been devised (see Balachandran et al. papers in [3]), they also naturally correlate spin and statistics. For geons, these processes can occur only with topology change, but even in their presence, the desired relation can be recovered but imperfectly, for a limited class of geons [10]. An alternative algebraic approach to quantum gravity and geon statistics has also been devised with physical inputs like cluster decomposition [8]. It predicts definite spin–statistics connection in $2 + 1$ dimensions (with its probable extension to higher dimensions), which, however, does not necessarily assert that spinorial geons are fermions or tensorial geons are bosons.

In summary then, there are strong indications that the canonical spin–statistics connection fails in quantum gravity. We can then inquire how this failure percolates to interactions of elementary particles. We initiate the study of this issue in this Letter. Quantum gravity effects cannot be important for low energy phenomenology unless they are enhanced by coherent processes involving large numbers, experiments are very accurate or proposals of “large extra dimensions” [11] are intimations of reality. But their study is important even if they are tiny as they challenge concepts of traditional quantum physics. If quantum gravity and string physics are judged by their verifiable predictions, there is no reason to pursue those enterprises [12]. An added reason for our work here that it lets us model spin–statistics violation in a particular way and derive bound on the violation parameters.

Geons are discoveries of Friedman and Sorkin [1]. Their existence has far-reaching implications for quantum gravity. We begin with a brief introduction to geons and their spin–statistics properties in Section 2 and follow it up in Section 3 with the effective interaction they generate among (say) standard model particles. They can be written down using guess work on operator product expansions, but we go a bit beyond that by identifying processes that fix their coefficients. The leading interaction is simple. Geons can be charged or neutral, spinorial or tensorial. Let the particle symbols also denote their fields. A spin- $rac{1}{2}$ charged geon can then interact with the electron via the coupling

$$L'(x) = \eta(e^\dagger G + G^\dagger e)(x).$$

$G$ here is a Bose field: otherwise this interaction is not very striking. Similar interactions can happen between a tensorial fermion $G$ and standard particles.

An interesting consequence of these interactions is that it can lead to effective supersymmetry and quantum Hall effect-like phenomena at low energies.

Interactions of this sort, or more generally even the existence of these exotic $G$ fields, are not compatible with RQFT. We provisionally take the following stand about this point. Geons are massive, with mass of the order of Planck mass, and we look only at low energy processes where they can be handled nonrelativistically. There is then no inconsistency. At energies where relativity is important, we presume that new effects enter the picture, perhaps dictated by the extended structure of geons. Our models are no good for these energies.

Interactions such as $H'(x)$ have physical consequences. These are briefly outlined in Sections 4 and 5. In particular, we discuss level distortions and black body spectrum. There are in addition violations of causality and CPT which are also pointed out. The three appendices are devoted to technical calculations.

Summarizing, the main results of the Letter are conceptual and concern the above strikingly novel interaction between geons and standard elementary particles. These interactions, studied in the non-relativistic approximation and low energies here, arise from the mediation of black holes. They violate non-relativistic causality which requires energy densities at spatially separated points to commute at a fixed time. They are not CPT-invariant either. Nevertheless, the emergent nonrelativistic physics has energy levels bounded below and no obvious inconsistency. The influences of such an interaction on energy levels and black body radiation are investigated, but unfortunately no convincing signal characteristic of the new interaction and large enough to be detected has been found. Causality violations can have a sensitive impact on dispersion relations [19] and latter can possibly detect the novel interactions if the Planck scale is in the TeV range.
2. What are geons

Elementary approaches to gravity work with spacetimes $X \times \mathbb{R}$ with $\mathbb{R}$ accounting for time $t$, and the spatial slice $X \times \{t\} \approx X$ being $\mathbb{R}^D$, except during treatment of black holes.

In the 70s, Friedman and Sorkin [1] initiated studies of asymptotically flat spatial slices (diffeomorphic to) $X$ different from $\mathbb{R}^D$. They pointed out that there are classes of manifolds $X$ called prime manifolds which are perfect infrastructures for describing elementary solitonic excitations in quantum gravity. There is only one such orientable manifold for $D = 2$ and that is the plane with a handle. It leads to the 2D geon. A plane with $n$ handles then gives the excitation of $n$ 2D geons. For $D = 3$, there are an infinity of basic manifolds (connected sums of $\mathbb{R}^3$ and closed prime manifolds) and an infinity of geons. A deep result of Friedman and Sorkin was that quantisation of geons, just like the quantisation of two-flavour Skyrmions [13], is not unique, and a certain class of geons can be quantized to give spinorial particles. The underlying primes are known as spinorial primes. Don Witt [14] later extended this work by an exhaustive study of spinorial primes. Later studies [3,9] revealed that in 2D, the geon for a plane with a handle can be quantized to have any spin.

Meanwhile Sorkin [2] studied the spin–statistics connection for geons and argued that no such relation can exist in the absence of topology change. This result was elaborated by Sorkin and coworkers [3] and others [8,9] and the generic failure of the spin–statistic connection in quantum gravity was firmly established. As indicated earlier, a correlation between spin and statistics can be shown with enough physical inputs, but still as a rule it fails to be conventional.

3. Effective interactions mediated by geons

Geons of pure gravity can be tensorial or spinorial, but will be a singlet under the standard model group. Geons with nontrivial standard model quantum numbers can occur when the standard model interactions are also included. In what follows, we have in mind geons of this enhanced theory which violate the spin–statistics connection.

We are after novel interactions among elementary particles induced by this violation in quantum gravity. Processes leading to such couplings are not abundant. Those involving black holes seem to be the sole mediations for this purpose. Black hole processes conserve quantum numbers like charge and angular momentum expressible as flux integrals over a sphere at infinity. But they need not conserve statistics. For this reason the following transition can occur. If $G$ is a spinorial boson with the same charge as the electron $e$, a black hole can absorb $e$ and emit $G$ as Fig. 1 illustrates, and vice versa. This process leads to a direct $e$–$G$ coupling because of vacuum fluctuations involving the creation and annihilation of black holes. A virtual black hole can thus mediate $e$–$G$ mixing (this process could be especially important should the scenarios presented in the last two references of [11] prove relevant).

Several calculations along these lines exist for gravity-induced proton decay [15,16] where a proton, for example, is converted by a black hole into $e$ plus tensorial particles like photons. Accurate calculations are not possible because of lack of control of quantum gravity. The importance of such research for the present work is to show that geons will certainly mix with standard model particles and suggest estimates for the coefficients in operator product expansions.

![Fig. 1](image-url)

We conclude that black hole fluctuations in the vacuum induce $G-e$ couplings with the leading term $\lambda e^\dagger G + \lambda^* G^\dagger e$ at low energies. That is for a charged spin-1/2 geon. A neutral spin-1 geon with real field $G_{\mu}$ can likewise couple to the photon field $A_{\mu}$ by the term constant $\times (\partial_{\mu} G_{\nu} - \partial_{\nu} G_{\mu})(\partial^\mu A^\nu - \partial^\nu A^\mu)$. We can write other similar quadratic couplings of geons and low energy excitations.

We can assume $\lambda$ to be $> 0$ in $G-e$ coupling by writing $\lambda = |\lambda| e^{i\chi}$ and absorbing the phase $\chi$ into the definition of $G$. In Fig. 1 we can exchange the particles, so that $\lambda$ has to be a symmetrical function of $m_e$ and $m_G$ where $m_A$ is the mass of particle $A$. We assume that $m_e \ll m_G$ and keep its leading term in $m_e/m_G$. The symmetry is lost in this approximation. Dimensional considerations then show that $\lambda = m_e f(\frac{m_e}{m_G})$, where we retain dependence of $f$ only on the single mass ratio $m_e/m_G$, ignoring other elementary particles. According to [15] (see also [16]),

$$f(\frac{m_e}{m_G}) = (\frac{m_e}{m_G})^K$$

(times a factor of order 1),

where the integer $K = 2$ (spin of $G$) = 1. We let $K$ be free for caution.

The conventional choice for $m_G$ is Planck mass $m_{Pl} \sim 10^{19}$ GeV. That gives $m_e/m_G \sim 10^{-22}$. In models with “large extra dimensions” [11], $m_G$ can be low. For the TeV scale gravity the same ratio becomes $m_e/m_G \sim 10^{-12}$.

More favorable values of $\lambda$ can be got by changing $e$ to a heavier particle. Already with a neutron we gain a factor of $10^3$:

$$m_n/m_G \approx 10^{-19} \text{ if } m_G \sim m_{Pl},$$

$$\approx 10^{-9} \text{ if } m_G \sim 1 \text{ TeV.}$$

(1)

In this case, if $K = 1$ or 2, the effects studied below are within experimental reach [17,18].

4. Level distortions

In this section, we explore the effects of the interaction in Section 3 on energy levels. They get shifted as is to be expected. This effect is illustrated using the harmonic oscillator system. It is a simple, but basic system where the new physics can be understood with relative transparency and then applied to other situations. It is also an approximation to quantum field theory where we retain only one mode each of a geon and a standard model field and only terms in the Lagrangian density quadratic in these fields (see below).

4.1. A boson and a fermion

There are new important features encountered when more than one fermion or boson is considered in the Hamiltonian. We will, therefore, study them later.

Let us consider a system that has only two degrees of freedom, represented by creation operators $(b^\dagger, f^\dagger)$ and annihilation operators $(b, f)$. Commutation (anticommutation) relations are taken to be

$$[b, b^\dagger] = 1, \quad \{f, f^\dagger\} = 1,$$

where curly brackets mean anti-commutators as usual. Assuming the existence of the common vacuum, $|0\rangle$, which is annihilated by both $f$ and $b$, the Hilbert space is spanned by the linear combinations of the following states:

$$|n\rangle = \frac{1}{\sqrt{n!}} (b^\dagger)^n |0\rangle, \quad f^\dagger |n\rangle = \frac{1}{\sqrt{n!}} (b^\dagger)^n f^\dagger |0\rangle, \quad b |0\rangle = f |0\rangle = 0.$$

(3)

As has already been mentioned in the introduction, a relativistic particle with half-integer (integer) spin can only be successfully described in a local field theoretic formalism if it has fermionic (bosonic) commutation rules. Since the excitations that we want to study are very heavy ($m \approx M_{Pl}$), they should admit a non-relativistic description. In
this case there is no connection between spin and statistics and we will, therefore, assume that one of the operators
(it does not matter for the moment which one) represents the excitation with the “wrong” statistics (e.g., either \( b \)’s
are spin half or \( f \)’s are spin 0, 1, etc.). (The spin degrees of freedom are being ignored.)

At this point, a few comments are in order. The Hamiltonian of the model is

\[
H = \omega_b b \dagger b + \omega_f f \dagger f + gb \dagger f + gf \dagger b,
\]

where we can assume that \( g > 0 \) as pointed out earlier.

We can obtain (4), for example, from a Hamiltonian density \( \mathcal{H} \) with standard free field terms for \( G \) and \( e \) and
an additional interaction \( \mathcal{H}' \):

\[
\mathcal{H} = -\frac{1}{2m} G \dagger \nabla^2 G - e \dagger (\vec{\alpha} \cdot \vec{p} + \beta m_e) e + \mathcal{H}', \quad \mathcal{H}' = \eta (G \dagger e + e \dagger G).
\]

To get (4) we then mode expand \( G \) and \( e \) so as to diagonalize the free field terms. On retaining just one mode in
these expansions (pretending that they are discrete) and including \( \mathcal{H}' \), we get (4).

Below we will diagonalize (4). The generic eigenstates of \( H \) are not created from the vacuum by simple linear
expressions in \( b \dagger \) and \( f \dagger \) and their powers. Rather they are created from the vacuum by complicated expressions
involving \( b \dagger \) and \( f \dagger \). For this reason, the mode expansion diagonalizing the Hamiltonian \( \int d^3x \mathcal{H}(x) \) is unknown
to us.

The problem can be seen in yet another manner. The Hamiltonian for the \( G, e \) fields has the form

\[
\int d^3x \left( G \dagger + e \dagger \right)(x) \mathcal{\tilde{H}} \left( \begin{array}{c} G \\ e \end{array} \right)(x), \quad \mathcal{\tilde{H}} = \left( \begin{array}{cc} -\frac{1}{2m} \nabla^2 & \eta \\ \vec{\alpha} \cdot \vec{p} + \beta m_e & 0 \end{array} \right),
\]

where \( G \) and \( e \) are, say, Bose and Fermi fields, and \( \mathcal{\tilde{H}} \) is the “single particle Hamiltonian”. Normally we would
expand \( G \) and \( e \) in terms of eigenstates of \( \mathcal{\tilde{H}} \) with creation and annihilation operators of appropriate statistics as
coefficients. But that does not work now. The eigenstates are given by \( \mathcal{\tilde{H}} \Psi_n = E_n \Psi_n \) and have the form

\[
\Psi_n = \left( \begin{array}{c} \beta_n \\ \phi_n \end{array} \right).
\]

They mix Bose and Fermi modes, \((\beta_n, 0)\) and \((0, \phi_n)\) not being eigenstates. But then, we do not know what statistics
to assign to \( a_n \) in the expansion

\[
\left( \begin{array}{c} G \\ e \end{array} \right) = \sum a_n \Psi_n.
\]

Let us return to study of the spectrum of this Hamiltonian. It is easy to construct the exact eigenstates of this
model (see Appendix A). The Schrödinger equation is easily solved by using the ansatz

\[
|\psi_n \rangle = (\alpha_n (b \dagger)^{f \dagger} + \phi_n (b \dagger^{f \dagger}))|0\rangle, \quad H |\psi_n \rangle = E_n |\psi_n \rangle.
\]

The spectrum is given by two series of states labeled by the non-negative integer \( n \) with \( \alpha_n(x) \sim x^n \) and
\( \phi_n(x) \sim x^{n+1} \) (plus the vacuum state, \(|0\rangle \) which remains an eigenstate even for \( g \neq 0 \)). Energies of the pair of
nth states are (see (A.9))

\[
E_n^\pm = \frac{1}{2} \left( \omega_b (2n+1) + \omega_f \pm \sqrt{\left(\omega_b - \omega_f\right)^2 + 4g^2(n+1)} \right).
\]

This is the complete spectrum of the system. As \( g \) tends to 0 each state smoothly goes into one eigenstate of the
unperturbed Hamiltonian. It is interesting that perturbative ground state may or may not remain as such depending
on the value of \( g \). The condition for the existence of negative eigenvalues in the spectrum is

\[
no_b^2 + \omega_b \omega_f < |g|^2.
\]
The minimal value of $|g|$ when this happens is, therefore, $|g| = \sqrt{\omega_b \omega_f}$, while the approximate number of negative eigenvalues is $n_- = (|g|^2 - \omega_b \omega_f)/\omega_b^2$. It is worth mentioning that a similar Hamiltonian with both $f$ and $b$ considered to be bosons will not have a ground state once $g$ is sufficiently large ($|g| > \sqrt{\omega_b \omega_f}$).

A striking feature of these levels is that they are $b$–$f$ mixtures. For a small $g$ and $\omega_f > \omega_b$ we find the following behaviour:

$$
\alpha_n^+ = \left(1 - \frac{g^2(n+1)}{2(\omega_b - \omega_f)^2} + \cdots\right) \frac{1}{\sqrt{n!}},
\phi_n^+ = \frac{g}{(\omega_f - \omega_b)} \frac{1}{\sqrt{n!}},
$$

$$
\alpha_n^- = -\frac{g}{(\omega_f - \omega_b)} \frac{\sqrt{n+1}}{\sqrt{n!}},
\phi_n^- = \left(1 - \frac{g^2(n+1)}{2(\omega_b - \omega_f)^2} + \cdots\right) \frac{1}{\sqrt{(n+1)!}}.
$$

(in case of $\omega_f < \omega_b$, one makes the interchanges $\alpha_n^+ \leftrightarrow \alpha_n^-$, $\phi_n^+ \leftrightarrow \phi_n^-$).

A point worthy of attention is that level degeneracy for the generic level is not affected as $g$ is turned on (with the exception of the occasional coincidence of energies for some special values of $g$). As it is increased from zero, each level adiabatically and smoothly evolves. Degeneracy of the levels is therefore not affected when $g$ becomes non-zero.

However, there is one special case $g = g_s = \sqrt{\omega_b \omega_f}$ which makes (8) into an equality. In this case $E_{n=0} = 0$ as it is an eigenvalue for the perturbative vacuum $|0\rangle$ (remember that it is still an eigenstate). So in this case the “ground” state becomes degenerate, with two states of the same energy 0 being

$$
|0\rangle, \quad \left(-\frac{\omega_b}{\omega_f + \omega_b} f^+ + \frac{\omega_f}{\omega_f + \omega_b} b^+\right)|0\rangle.
$$

4.2. Connection to supersymmetry and QHE

It has been pointed out to us by Joseph Samuel that the Hamiltonian (4) is an element of the graded algebra of a supergroup, with $SO(2)$ as its underlying classical group. One sees this from the anticommutator

$$
[b^\dagger f, f^\dagger b]^+ = b^\dagger b + f^\dagger f.
$$

The graded algebra has $b^\dagger b$ and $f^\dagger f$ as even generators and $b^\dagger f$ and its adjoint as odd generators.

There is also an interesting connection of (4) to the Dirac Hamiltonian in a plane with a perpendicular uniform magnetic field as has also been pointed out to us by Samuel. In this case the three-dimensional zero-mass Dirac Hamiltonian in a magnetic field, $\vec{a} \cdot \vec{P}$, becomes $\alpha^+ \pi^- + \alpha^- \pi^+$, $\{\alpha^+, \alpha^-\} = 1$, $[\pi^+, \pi^-] = eB$ where $B$ is the magnetic field along the perpendicular direction. This Hamiltonian can be identified with the last two terms in (4).

4.3. One boson and two fermions

In the hydrogen atom, there are two levels with principal quantum number $n = 1$ corresponding to spin up and spin down. Their creation operators $f_i^\dagger$ ($i = 1, 2$) in the second-quantized formalism anticommute. Similarly we can associate fermionic oscillators to bound state levels of any spinorial particle.

This association is in conventional physics in the absence of the disturbing presence of geons. With geons in the spectrum there are additional interactions which spoil such simple associations. The simplest model that we can consider is a generalization of the Hamiltonian (4) to the two fermionic modes $f_{1,2}$ interacting with a single
bosonic geon \( b \):

\[
H = \omega_1 f_1^1 f_1 + \omega_2 f_2^1 f_2 + \Omega b^1 b + g_1 (f_1^1 b + b^1 f_1) + g_2 (f_2^1 b + b^1 f_2).
\]  

(15)

This Hamiltonian is analyzed in Appendix B. One can show that for the most interesting case when both fermions are degenerate, i.e., \( \omega_1 = \omega_2 \), it is possible to find the eigenfunctions and exact spectrum of the model. We shall show that the operator for the level corresponding to two fermionic excitations has the form (B.3):

\[
|\psi\rangle = \left\{ \alpha(b^1) f_1^1 f_2^1 + \phi_1(b^1) f_1^1 + \phi_2(b^1) f_2^1 + \Psi(b^1) \right\} |0\rangle.
\]  

(16)

It goes over to just \( f_1^1 f_2^1 \) as the interaction with the geon is switched off.

Let us next consider the case where \( \omega_1 = \omega_2 \) and \( g_1 = g_2 = g \). In this case, (15) is invariant under the exchange of \( f_1 \) with \( f_2 \). In order to study the possible Pauli principle violation in this case, one should consider what happens when the two fermionic operators are exchanged. Then \( |\psi\rangle \) becomes \( |\psi'\rangle \), where

\[
|\psi'\rangle = \left\{ \alpha(b^1) f_1^1 f_2^1 + \phi_1(b^1) f_1^1 + \phi_2(b^1) f_2^1 + \Psi(b^1) \right\} |0\rangle.
\]  

(17)

Thus \( |\psi\rangle \) will be an eigenstate of the permutation operator with \(-1\) eigenvalue only if \( \Psi = 0 \) and \( \phi_1 = -\phi_2 \). In general, if \( \omega_1 \neq \omega_2 \) and/or \( g_1 \neq g_2 \), this is not the case. However, we have checked that for \( \omega_1 = \omega_2 \) and \( g_1 = g_2 \), the eigenvalue series that goes to \( E_3 = 2\omega + n\Omega \) in the limit \( g_{1,2} \to 0 \), this is precisely the case: the eigenvectors of the (B.4) matrix have the structure \( (X_n(b), Y_n(b), -Y_n(b), 0) \). As we expect this result to be generally true, we conclude that there is no apparent Pauli principle violation in this model.

Actually, we can argue that models like this cannot violate Pauli principle unless level degeneracy is affected as \( g \) (= \( g_1 = g_2 \)) becomes nonzero. That is because (15) for \( \omega_1 = \omega_2 \) and \( g_1 = g_2 = g \) is symmetric under exchange of \( f_1 \) and \( f_2 \) and hence its eigenstates can be organized in irreducible representations (IRRs) of the permutation group \( S_2 \). At \( g = 0 \), the energy eigenstates with two fermions change sign under their exchange: they transform by the nontrivial IRR of \( S_2 \). By continuity, this IRR will persist if \( g \) is made nonzero. New effects can arise if level degeneracy is changed when \( g \) becomes nonzero, so that there is a state symmetric under \( S_2 \) degenerate with this IRR. But that does not happen in our model.

5. Black body spectrum

As it is written now, either of the frequencies \( \omega_b, f \) may be taken to correspond to a geon, the actual choice is dictated by the low energy effect that one wants to study. In the case of effects in atomic systems, one assumes that \( \omega_b \sim M_{\text{Pl}} \) and \( \omega_f = \sqrt{m^2 + k^2} \), \( \omega_f \ll \omega_b \). In this case the geon is a spin half bosonic excitation in gravity. However, it is very interesting to look at the case of the microwave background radiation, where the geon must be the spin-1 excitation and hierarchy of scales is opposite: \( \omega_b \sim \kappa_{\text{photon}} \) and \( \omega_f \ll \omega_f \sim \omega_f \sim M_{\text{Pl}} \).

In order to determine the effect of geons on the Planck spectrum one has to determine the correction to the thermal distribution function for the photons. Since the spectrum of the Hamiltonian (7) is known, the task reduces to the computation of the proper partition function (see Appendix C for details), which can be done perturbatively in \( |g|^2 \). The first non-trivial correction to the distribution function \( n \) turns out to be given by (C.7),

\[
n(\omega) = n_0(\omega) - n_0(\omega) \frac{|g|^2}{M_{\text{Pl}}^2} + \frac{e^{\beta\omega}}{(e^{\beta\omega} - 1)^2} \frac{\beta |g|^2}{M_{\text{Pl}}} + \cdots, \quad n_0(\omega) = \frac{1}{e^{\beta\omega} - 1},
\]  

(18)

where \( n_0 \) is the free bosonic Planck distribution.

One can immediately see that the expansion of (18) in powers of \( |g|^2 \) is singular as \( \omega_b \) goes to \( 0 \): the first correction diverges as \( 1/\omega_b^2 \). This is the same problem that plagues any theory that has massless modes at non-zero temperature. However, in this case a more careful treatment is needed. The reason for that is that the proliferation of the soft photons in the usual, free case does not lead to a divergent energy density: the phase space volume
scales like $\omega^3 d\omega$ and the denominator of the bosonic distribution produces a $1/\omega$ factor thus keeping the product finite. For us, the next order correction produces terms like $1/\omega^k$, where $k$ is roughly proportional to the order of the perturbative expansion. One possible solution to this problem is the following: in the case of the photon, due to dimensional considerations one should consider $g \sim k^2/M_{Pl}$, which will keep the answer finite in the limit $k = \omega_b \to 0$. This behaviour of $g$ is supported by the fact that the leading gauge invariant geon–photon coupling involves derivatives being constant $\times (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu G_\nu - \partial_\nu G_\mu)$ for a spin-1 geon as indicated earlier. Numerically, one can stop at the first order if $(\beta\omega_b)^2 \gg \beta |g|^2/(\omega_f)$. Using the suggestion above this condition reduces to just $T \ll M_{Pl}$ which is obviously satisfied.

The first correction to $n_0$ in the formula (18) is just a "grey body" factor, while the second one is more important at low frequencies of the photon. However, at this point one has to use the expression $\sim k^2/M_{Pl}$ for $g$, which makes the corrections look like

$$\Delta n(\omega) = -n_0(\omega) \frac{k^4}{M_{Pl}^4} + \left(\frac{e^{\beta\omega}}{e^{\beta\omega} - 1}\right)^2 \frac{\beta k^4}{M_{Pl}^3}.$$  

(19)

In the limit $k = \omega \to 0$ the second term becomes $k^2/(\beta M_{Pl}^4)$ and it is clearly insignificant for small $k$. It seems then, that even though there are some corrections to the "background radiation" following from the Hamiltonian described above they are too tiny to be detected in the experimentally accessible region.

It is important to make sure that whatever correction that the observed distribution function gets is a signature of the effect in hand and not of some other origin. While it is almost certainly impossible to establish this rigorously, one elementary test is possible here—what if the "mixed" mode is a boson as well? Now, for comparison let us consider the similar situation in the case when we have two bosonic operators coupled the same way as in the Hamiltonian (4):

$$H = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + g a_1^+ a_2 + g^* a_2^+ a_1.$$  

(20)

Here both $a_1$ and $a_2$ are bosons.

The correction to order $|g|^2$ to the distribution function can be calculated from (C.10), the result is identical to that of (C.7) to order $|g|^2$, with the same "grey body" factor and correction to the low energy behaviour. Thus, unfortunately, the difference between (C.7) and (C.10), therefore, comes only in the next order of perturbation theory. While the full expressions are given by (C.11), this difference in powers of $|g|^2$ is

$$\Delta n(\omega)_b + g - \Delta n(\omega)_b = \left\{ \frac{\beta}{M_{Pl}^3} \left( \frac{2 - 6e^{\beta\omega}}{(e^{\beta\omega} - 1)^2} \right) + \frac{6}{M_{Pl}^4} \left( \frac{e^{\beta\omega}}{(e^{\beta\omega} - 1)^2} \right) \right\} g^4 + \cdots.$$  

(21)

where $\Delta n(\omega)_b + g$ is $n(\omega) - n_0(\omega)$ in (18), while $\Delta n(\omega)_b$ is given in (C.11). These corrections become identical in the ultraviolet limit $\beta\omega \to \infty$ and the only difference comes in the subleading order in $1/M_{Pl}$. For that reason, it is difficult to propose an experimental test which would be able to see these corrections.

6. Final remarks: CPT, causality

6.1. CPT

In the course of proving the CPT theorem, anti-commutativity of fermionic fields and commutativity of tensorial ones are explicitly used [4]. But this feature need not hold for geons. CPT thus can fail in the presence of geons. The failure will be by small numbers like $10^{-19}$, $10^{-9}$ or its powers [cf. Eq. (1)]. Detailed calculations may be possible by allowing for mixing of quarks and leptons for instance with geons and integrating out the latter, but we have not done this work.
6.2. Causality

If $\Psi$ is a spinorial fermion field and $G$ a spinorial boson field, for example, the term $H_I(x) = \lambda (G^\dagger \Psi + \Psi^\dagger G)(x)$ in the Hamiltonian density $H(x)$ does not commute for the space-like separations: $[H(x), H(y)] \neq 0, (x - y)^2 > 0$. As $H(x) = H_0(x) + H_I(x)$, where $H_0(x)$ commutes for spacelike separations, $H(x)$ and $H(y)$ neither commute nor anti-commute for space-like separations. Thus Hamiltonian density, an observable, violates local causality.

Our model is in reality valid only nonrelativistically, so that we have to interpret this statement as asserting that the Hamiltonian densities at distinct spatial points at the same time do not commute. (Hence they cannot be “simultaneously measured”.)

The implications of this microscopic violation of causality are not adequately clear to us. It does have a phenomenological implication: forward dispersion relations will not be correct. Such violations of causality also occur in noncommutative geometry, in particular of D-branes in string physics [20]. Basically, this causality violation in our model is controlled by the intrinsic non-locality of the geons, and this non-locality is similar to having a fundamental length $l_f \sim 1/M_{Pl}$ in the theory. There are some indications [19] that forward dispersion relations can be a sensitive probe of $l_f$ provided it is not too small, say if $1/l_f$ is in the TeV range. Nonlocality will also spoil the analyticity of scattering amplitudes and its implications by small corrections. Investigations of these effects would be of great interest, being characteristic manifestations of intrusions of quantum gravity or string physics into elementary particle theory.

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Appendix A

In this appendix we construct eigenstates and find eigenvalues for the Hamiltonian (4). We start by writing an arbitrary eigenstate $|\psi\rangle$ as

$$|\psi\rangle = (\alpha(b^\dagger) f^\dagger + \phi(b^\dagger))|0\rangle,$$

$$H|\psi\rangle = E|\psi\rangle. \quad (A.1)$$

Using commutation relations (2) one has

$$[b, \phi(b^\dagger)] = \frac{\partial \phi(b^\dagger)}{\partial b^\dagger}, \quad [b, \alpha(b^\dagger)] = \frac{\partial \alpha(b^\dagger)}{\partial b^\dagger}, \quad (A.2)$$

and Eq. (A.2) becomes

$$\left\{ \omega_f \alpha(b^\dagger) f^\dagger + \omega_b \frac{\partial \alpha(b^\dagger)}{\partial b^\dagger} b^\dagger + g \alpha(b^\dagger) b^\dagger + \omega_b \frac{\partial \phi(b^\dagger)}{\partial b^\dagger} b^\dagger + g \frac{\partial \phi(b^\dagger)}{\partial b^\dagger} f^\dagger \right\}|0\rangle = E \left\{ \alpha(b^\dagger) f^\dagger + \phi(b^\dagger) \right\}|0\rangle. \quad (A.3)$$

By rewriting this in the matrix form one gets:

$$\omega_f \alpha(b^\dagger) + \omega_b \frac{\partial \alpha(b^\dagger)}{\partial b^\dagger} b^\dagger + g \frac{\partial \phi(b^\dagger)}{\partial b^\dagger} = E \alpha(b^\dagger), \quad (A.4)$$

$$g \alpha(b^\dagger) b^\dagger + \omega_b \frac{\partial \phi(b^\dagger)}{\partial b^\dagger} b^\dagger = E \phi(b^\dagger). \quad (A.5)$$
At this point let us assume the following behaviour of the functions \( \alpha \) and \( \phi \):

\[
\alpha(b^+) = \alpha_n(b^+)^n, \quad \phi(b^+) = \phi_n(b^+)^{(n+1)}.
\]

(A.7)

Here \( \alpha_n \) and \( \phi_n \) are (complex) numbers. It is easy to see that with this choice Eq. (A.6) reduce to a simple eigenvalue problem for a 2 \times 2 matrix:

\[
\begin{pmatrix}
\omega_f + n\omega_b & g(n + 1) \\
g & (n + 1)\omega_b
\end{pmatrix}
\begin{pmatrix}
\alpha_n \\
\phi_n
\end{pmatrix}
=E_n
\begin{pmatrix}
\alpha_n \\
\phi_n
\end{pmatrix}.
\]

(A.8)

The eigenvalue equation is quadratic, and yields the following two sets of solutions:

\[
E_n^\pm = \frac{1}{2}(\omega_b(2n + 1) + \omega_f \pm \sqrt{(\omega_b - \omega_f)^2 + 4g^2(n + 1)}).
\]

(A.9)

With these values of energy the coefficients \( \alpha_n \) and \( \phi_n \) can be determined from the normalization condition \( n!|\alpha_n|^2 + (n + 1)!|\phi_n|^2 = 1 \) and Eq. (A.8). They are found (upto an over-all phase) to be

\[
\alpha_n^\pm = \frac{\Delta \omega \pm \sqrt{\Delta \omega^2 + 4g^2(n + 1)}}{(\Delta \omega \pm \sqrt{\Delta \omega^2 + 4g^2(n + 1)})^2 + 4g^2(n + 1))^{1/2}}\sqrt{n!},
\]

(A.10)

\[
\phi_n^\pm = \frac{1}{2g}
\]

(A.11)

where \( \Delta \omega = \omega_f - \omega_b \). These expressions characterize the complete spectrum of the model. In the limit \( g \to 0 \), this smoothly goes to the unperturbed spectrum for which \( g = 0 \). This ensures that we have found all of the eigenstates of the system.

**Appendix B**

A similar treatment can be applied to the case of two fermionic modes coupled to a boson as in the Hamiltonian (15). The most generic ansatz for an energy eigenstate is

\[
|\psi\rangle = \left[\alpha(b^+) f_1^+ f_2^+ + \phi_1(b^+) f_1^+ + \phi_2(b^+) f_2^+ + \Psi(b^+)\right]|0\rangle.
\]

(B.1)

Applying Hamiltonian (15) and using relations similar to those of (A.2) and (A.3), we get

\[
(\omega_1 + \omega_2)\alpha(b^+) + \Omega_2 \frac{\partial \alpha(b^+)}{\partial b^+} b^+ - g_2 \frac{\partial \phi_1(b^+)}{\partial b^+} b^+ + g_1 \frac{\partial \phi_2(b^+)}{\partial b^+} = E\alpha(b^+),
\]

\[
-g_2 \omega_1 b^+ + \omega_1 \phi_1(b^+) + \Omega_2 \frac{\partial \phi_1(b^+)}{\partial b^+} b^+ + g_1 \frac{\partial \psi(b^+)}{\partial b^+} = E\phi_1(b^+),
\]

\[
+g_1 \omega_1 b^+ + \omega_2 \phi_2(b^+) + \Omega_2 \frac{\partial \phi_2(b^+)}{\partial b^+} b^+ + g_2 \frac{\partial \psi(b^+)}{\partial b^+} = E\phi_2(b^+),
\]

\[
g_1 \phi_1(b^+) b^+ + g_2 \phi_2(b^+) b^+ + \Omega_2 \frac{\partial \psi(b^+)}{\partial b^+} b^+ = E\Psi(b^+).
\]

(B.2)

As in the previous case we first assume the power-law behaviour of coefficients \( \alpha, \phi_1, \phi_2, \Psi \) and write

\[
\alpha(b^+)_n = \alpha_n(b^+)^n, \quad \phi_1(b^+)_n = \phi_1_n(b^+)^{n+1}, \quad \phi_2(b^+)_n = \phi_2_n(b^+)^{n+1}, \quad \Psi(b^+)_n = \Psi_n(b^+)^{n+2}.
\]

(B.3)
The corresponding 4 × 4 matrix equation can be read off Eq. (B.2):

\[
\begin{pmatrix}
\omega_1 + \omega_2 + \Omega n & -g_2(n + 1) & g_1(n + 1) & 0 \\
-g_2 & \omega_1 + \Omega(n + 1) & 0 & g_1(n + 2) \\
+g_1 & 0 & \omega_2 + \Omega(n + 1) & g_2(n + 2) \\
0 & g_1 & g_2 & \Omega(n + 2)
\end{pmatrix}
\begin{pmatrix}
\alpha_n \\
\phi_{1n} \\
\phi_{2n} \\
\phi_{nn}
\end{pmatrix}
= E
\begin{pmatrix}
\alpha_n \\
\phi_{1n} \\
\phi_{2n} \\
\phi_{nn}
\end{pmatrix}.
\]  
(B.4)

The eigenvalues can be obtained from the fourth order secular equation and are quite complicated for generic values of the frequencies and coupling constants. Nevertheless, for the physically interesting case of degenerate \((\omega_1 = \omega_2 = \omega)\) fermions coupled to a bosonic “geon”, the eigenvalue equation for the matrix (B.4) splits into the product of two quadratic ones and yields

\[
(\omega + \Omega(n + 1) - E)(\Omega(n + 2) - E) - (g_1^2 + g_2^2)(n + 2) = 0,
\]
(B.5)

\[
(2\omega + \Omega n - E)(\omega + \Omega(n + 1) - E) - (g_1^2 + g_2^2)(n + 1) = 0.
\]  
(B.6)

The corresponding energy eigenvalues are

\[
E_{1,2} = \frac{1}{2}\left[\omega + \Omega(2n + 3) \pm \sqrt{\omega^2 - 4\Omega^2(n + 2) + 4g^2(n + 2)}\right],
\]
\[
E_{3,4} = \frac{1}{2}\left[3\omega + \Omega(2n + 1) \pm \sqrt{(\omega - \Omega)^2 + 4g^2(n + 1)}\right],
\]  
(B.7)

But these do not exhaust all the energy eigenstates. Thus, if we look at the limiting case of \(g_{1,2} \to 0\), we find

\[
E_1 = \omega + \Omega(n + 1), \quad E_2 = \Omega(n + 2), \quad E_3 = 2\omega + \Omega(n), \quad E_4 = \omega + \Omega(n + 1).
\]  
(B.8)

Taking into account that vacuum state \(|0\rangle\) remains an eigenstate with energy 0, we see that there are three eigenvalues that are missing in the above sets, namely \(\Omega, \omega_1, \omega_2\). This has happened because while solving (B.2) we assumed (B.3), which is not the only possibility. Assuming that \(\alpha(b^\dagger) = 0\), one can show that the following equations for \(\phi_1, \phi_2\) and \(\Psi\) result:

\[
g_2\phi_1(b^\dagger) = g_1\phi_2(b^\dagger).
\]

These equations bring in the “missing” energies.

**Appendix C**

In this appendix we give detailed calculations of the influence of the boson–fermion mixing on the black body radiation spectrum (microwave background).

The standard way to do so is to introduce chemical potentials \(\mu_f, \mu_b\) for fermionic and bosonic excitations of the model. Then

\[
n_b(\omega_b) = -\left.\frac{\partial \Omega}{\partial \mu_b}\right|_{\mu_f=0},
\]  
(C.1)

where \(n_A(\omega_A)\) is the mean number of particles of type A with energy \(\omega_A\), and the potential \(\Omega\) is given by

\[
\Omega = -\frac{1}{\beta}\ln Z, \quad Z = \text{tr} e^{-\beta(H - \mu_f a_f - \mu_b b^\dagger b)}.
\]  
(C.2)

Because our initial \(H\) is quadratic, the addition of number operators leads just to the effective changes \(\omega_b \to \omega_b - \mu_b\) and \(\omega_f \to \omega_f - \mu_f\). Since the trace can be computed over any set of complete states, one can use these
new values in the expression for the spectrum (7) and just sum over $n$:

$$
Z = 1 + \sum_{n=0}^{\infty} \left( e^{-\beta E_0^+} + e^{-\beta E_0^-} \right)
= 1 + 2 \sum_{n=0}^{\infty} e^{-\frac{\beta}{2} ((2\pi n + 1)\omega_b + \omega_f)} \chi \left( \frac{\beta}{2} \sqrt{(\omega_b - \omega_f)^2 + 4|g|^2(n + 1)} \right).
$$

(C.3)

As the coupling parameter $g$ goes to 0, the above expression tends to

$$
1 + \sum_{n=0}^{\infty} \left( e^{-\beta(n+1)\omega_b} + e^{-\beta(n\omega_b + \omega_f)} \right)
= \frac{1 + e^{-\beta\omega_f}}{1 - e^{-\beta\omega_b}} = Z_f \times Z_b,
$$

(C.4)

where $Z_f$ and $Z_b$ are free fermionic and free bosonic partition functions. Expanding in powers of $|g|^2$, one gets

$$
Z = Z_f Z_b + 2 \frac{e^{-\frac{\beta}{2} (\omega_f + \omega_b)} Z_b \beta \text{sh} \left( -\frac{\beta}{2} |\omega_b - \omega_f| \right)}{|\omega_b - \omega_f|} |g|^2 + \ldots
$$

(C.5)

Rewriting the expression for the partition function as

$$
Z = Z_f Z_b \left( 1 + 2 e^{-\frac{\beta}{2} (\omega_f + \omega_b)} \frac{Z_b \beta \text{sh} \left( -\frac{\beta}{2} |\omega_b - \omega_f| \right)}{|\omega_b - \omega_f|} |g|^2 + \ldots \right)
$$

(C.6)

and assuming that $\omega_f \sim M_{Pl}$, the effective correction to the distribution function is

$$
\Omega = \Omega_0 - \frac{e^{-\beta\omega_b} Z_b}{(M_{Pl} - \omega_b)} |g|^2 + \ldots = \Omega_0 - \frac{n_b(\omega_b)}{(M_{Pl} - \omega_b)} |g|^2 + \ldots,

n(\omega) - n_0(\omega) = -n_0(\omega) \frac{|g|^2}{M_{Pl}^2} + \frac{e^{\beta\omega} \beta |g|^2}{(e^{\beta\omega} - 1)^2 M_{Pl}^2}.
$$

(C.7)

Here $n_0$ is the free bosonic distribution function.

Consider next the case of two bosons coupled in the same way as in the Hamiltonian (4):

$$
H = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + g a_1^+ a_2 + g^* a_2^+ a_1.
$$

(C.8)

Here both $a_1$ and $a_2$ are bosons. Contrary to the previous case this model is exactly solvable and the spectrum is

$$
E_{n_1, n_2} = \omega_1 n_1 + \omega_2 n_2, \quad n_1, n_2 \in \mathbb{N},
$$

$$
\omega_{\pm} = \frac{1}{2} \left( \omega_1 + \omega_2 \right) \pm \sqrt{\left( \omega_1 - \omega_2 \right)^2 + 4g^2}.
$$

(C.9)

Here we take $\omega_1 = \omega$ to be the “photonic” frequency and $\omega_2$ to be that of the mixing bosonic mode. The distribution function for the photon is

$$
n(\omega) = \frac{1}{e^{\beta\omega} - 1} \frac{\partial \omega_{-}}{\partial \omega} + \frac{1}{e^{\beta\omega_{+}} - 1} \frac{\partial \omega_{+}}{\partial \omega}.
$$

(C.10)

If one expands this expression in coupling constant $g$ in the limit $\omega = \omega_1 \ll \omega_2 \sim M_{Pl},$ the result is identical to that of (C.7) to order $|g|^2$, with the same “grey body” factor and correction to the low energy behaviour. Unfortunately, the difference between (C.7) and (C.10), therefore, comes only in the next order of perturbation theory. After some
algebra one finds

\[
\Delta n(\omega)_{b+g} = \left\{ -\frac{1}{M^2_{\text{Pl}}} (e^{\beta \omega} - 1) + \frac{\beta e^{\beta \omega}}{M^2_{\text{Pl}}} (e^{\beta \omega} - 1)^2 \right\} g^2 \\
+ \frac{\beta^2}{M^4_{\text{Pl}}} (e^{2\beta \omega} + e^{\beta \omega}) + \frac{\beta}{M^2_{\text{Pl}}} \left( \frac{3 - 4e^{2\beta \omega} - 6e^{\beta \omega}}{(e^{\beta \omega} - 1)^3} \right) + \frac{6}{M^4_{\text{Pl}}} \left( \frac{e^{\beta \omega} + 1}{(e^{\beta \omega} - 1)^2} \right) \bigg] 8^4 + \cdots,
\]

\[
\Delta n(\omega)_{b+b} = \left\{ -\frac{1}{M^2_{\text{Pl}}} (e^{\beta \omega} - 1) + \frac{\beta e^{\beta \omega}}{M^2_{\text{Pl}}} (e^{\beta \omega} - 1)^2 \right\} g^2 \\
+ \frac{\beta^2}{M^4_{\text{Pl}}} (e^{2\beta \omega} + e^{\beta \omega}) + \frac{\beta}{M^2_{\text{Pl}}} \left( \frac{-4e^{2\beta \omega} - 3e^{\beta \omega}}{(e^{\beta \omega} - 1)^3} \right) + \frac{6}{M^4_{\text{Pl}}} \left( \frac{1}{(e^{\beta \omega} - 1)} \right) \bigg] 8^4 + \cdots,
\]

(C.11)

where \(\Delta n(\omega)_{b+g}\) is the left-hand side of (C.7) while \(\Delta n(\omega)_{b+b}\) is the difference of (C.10) and \(n_0(\omega)\). These corrections become identical in the ultraviolet limit \(\beta \omega \to \infty\) and the only difference comes in the subleading order in \(1/M_{\text{Pl}}\).

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Dynamical behavior of dilaton in de Sitter space

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Abstract

We study the dynamical behavior of the dilaton in the background of three-dimensional Kerr–de-Sitter space which is inspired from the low-energy string effective action. The perturbation analysis around the cosmological horizon of Kerr–de-Sitter space reveals a mixing between the dilaton and other fields. Introducing a gauge (dilaton gauge), we can disentangle this mixing completely and obtain one decoupled dilaton equation. However, it turns out that this belongs to the tachyon. The stability of de Sitter solution with $J = 0$ is discussed. Finally we compute the dilaton absorption cross section to extract information on the cosmological horizon of de Sitter space. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

Recently an accelerating universe has proposed to be a way to interpret the astronomical data of supernova [1–3]. The inflation is employed to solve the cosmological flatness and horizon puzzles arisen in the standard cosmology. Combining this observation with the need of inflation leads to that our universe approaches de Sitter geometries in both the infinite past and the infinite future [4–6]. Hence it is very important to study the nature of de Sitter (dS) space and the dS/CFT correspondence [7]. However, there exist difficulties in studying de Sitter space. First there is no spatial infinity and global timelike Killing vector. Thus it is not easy to define the conserved quantities including mass, charge and angular momentum appeared in asymptotically de Sitter space. Second the dS solution is absent from string theories and thus we do not have a definite example to test the dS/CFT correspondence. Finally it is hard to define the $S$-matrix because of the presence of the cosmological horizon.

Accordingly most of works on de Sitter space were concentrated on the massive scalar propagation and its quantization [8–11]. Also the bulk-boundary relation for the scalar was introduced to study the dS/CFT correspondence [12]. Hence it is important to find a model which can accommodate the de Sitter space solution. In this work we introduce an interesting model which is motivated from the low-energy string action in $(2 + 1)$ dimensions [13]. This model includes a nontrivial scalar so-called the dilaton.1 Actually, we will use the dilaton to investigate the nature of the cosmological horizon in de Sitter space.

1 Previously we are interested in anti-de-Sitter black hole like the BTZ black hole [14]. In the BTZ black hole and the three-dimensional black string, the role of the dilaton was discussed in Ref. [15].
It is known that the cosmological horizon is very similar to the event horizon in the sense that one can define its thermodynamic quantities using the same way as is done for the black hole. Two important quantities to understand the black hole are the Bekenstein–Hawking entropy and the absorption cross section (greybody factor). The former specifies the entropy for the cosmological horizon explicitly. The entropy for the cosmological horizon was first discussed in literature [18]. However, there exist a few of the wave equation approaches to find the greybody factor for the cosmological horizon [11]. A similar work for the four-dimensional Schwarzschild–de-Sitter black hole appeared in [19]. But it focused mainly on obtaining the temperature of the horizon for all spherically symmetric black holes [17]. This can be obtained by solving the wave equation explicitly. The entropy for the cosmological horizon was first discussed in literature [18]. However, there exist a few of the wave equation approaches to find the greybody factor for the cosmological horizon [11]. A similar work for the four-dimensional Schwarzschild–de-Sitter black hole appeared in [19] but it focused mainly on obtaining the temperature of the eternal black hole. Also the absorption rate for the four-dimensional Kerr–de-Sitter black hole was discussed in [20].

In this Letter we compute the absorption cross section of the dilaton in the background of three-dimensional de Sitter (dS3) space with the cosmological horizon. For this purpose we first confine the wave equation only to the southern diamond where the time evolution of waves is properly defined even if this area does not include spatial infinity. And then we solve the wave equation to find the greybody factor in the low-energy and low-temperature limits.

The organization of this Letter is as follows. In Section 2 we briefly review our model inspired from the low-energy string action and its Kerr–de-Sitter solution. We introduce the perturbation to study the cosmological horizon in the background of Kerr–de-Sitter space in Section 3. In Section 4 we perform the potential analysis to check whether the de Sitter solution is or not stable. In Section 5 we calculate the absorption cross section for \( j = 1 \), 2-angular modes of the dilaton explicitly. Finally we discuss our results in Section 6.

## 2. Kerr–de-Sitter solution

We start with the low-energy string action in string frame [13]

\[
S_{\text{string}} = \int d^3 x \sqrt{-g} e^\Phi \left( R + (\nabla \Phi)^2 + \frac{8}{k} - \frac{1}{12} H^2 \right),
\]

where \( \Phi \) is the dilaton, \( H_{\mu \nu \rho} = 3 \delta_{\mu \nu} B_{\rho \parallel} \) is the Kalb–Ramond field, and \( k \) the cosmological constant. This action was widely used for studying the BTZ black hole and the black string [15]. Although \( k \) was originally proposed to be positive, here we assume to extend it to be negative for our purpose. The equations of motion lead to

\[
R_{\mu \nu} - \nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu \rho \nu \kappa} H^{\rho \nu \kappa} = 0,
\]

\[
\nabla^2 \Phi + (\nabla \Phi)^2 - \frac{8}{k} - \frac{1}{6} H^2 = 0,
\]

\[
\nabla_\mu H^{\mu \nu \rho} + (\nabla_\mu \Phi) H^{\mu \nu \rho} = 0.
\]

The Kerr–de-Sitter solution to Eqs. (2)–(4) is found to be [9]

\[
\bar{H}_{\mu \nu} = -\frac{i}{\ell} \bar{\Phi} = 0,
\]

\[
\bar{g}_{\mu \nu} = \begin{pmatrix}
-\left( M - r^2 / \ell^2 \right) & -J/2 & 0 \\
-J/2 & r^2 & 0 \\
0 & 0 & f^{-2}
\end{pmatrix}
\]

with the metric function \( f^2 = M - r^2 / \ell^2 + J^2 / 4 \ell^2 \). The above Kerr–de-Sitter solution is obtained from the BTZ black hole [14] by replacing both \( M \) and \( \ell^2 \) by \(-M\) and \(-\ell^2\). The metric \( \bar{g}_{\mu \nu} \) is singular at \( r = r_{\pm} \).

\[
r^2_{\pm} = \frac{M \ell^2}{2} \left( 1 \pm \left[ 1 + \left( \frac{J}{M \ell} \right)^2 \right]^{1/2} \right)
\]

with \( M = (r^2_+ + r^2_-) / \ell^2 = (r^2_+ - r^2_-) \) and \( J = 2 r_+ r_-(\ell^- / \ell_+) \). For convenience we introduce \( r_{\pm} = -r^2_{\pm} > 0 \) due to \( r^2_+ < 0 \) in Kerr–de-Sitter space. We note that in \((2 + 1)\) dimensions, there is no black hole horizon for Kerr–de-Sitter space because the black hole degenerates to a conical singularity at the origin \( r = 0 \). This singularity gives rise to some difficulties to analyze the wave equation in the southern diamond of Kerr–de-Sitter space. In this work we consider the cosmological horizon \( r_c = r_+ \) with interest. For convenience, we list the Hawking temperature \( T_c \), the area of...
3. Perturbation around Kerr–de-Sitter solution

To study the propagation of all fields in Kerr–de-Sitter space specifically, we introduce the small perturbation fields [15]

\[ H_{\phi r} = \tilde{H}_{\phi r} + \mathcal{H}_{\phi r}, \quad \Phi = 0 + \varphi, \]

\[ g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu} \]

around the background solution of Eq. (5). For convenience, we introduce the notation \( h_{\mu\nu} = h_{\mu\nu} - \tilde{g}_{\mu\nu} \hbar / 2 \) with \( \hbar = h^\rho_{\rho} \). And then one needs to linearize Eqs. (2)–(4) to obtain

\[
\delta R_{\mu\nu}(h) - \nabla_\mu \nabla_\nu \varphi - \frac{1}{2} \tilde{H}_{\mu\rho\sigma} \mathcal{H}^{\rho\sigma} + \frac{1}{2} \tilde{H}_{\mu\rho\sigma} \mathcal{H}^{\rho\sigma} h^{\rho\sigma} = 0, \quad (10)
\]

\[
\nabla^2 \varphi - \frac{1}{6} (2 \tilde{H}_{\mu\rho\sigma} \mathcal{H}^{\mu\rho\sigma} - 3 \tilde{H}_{\mu\rho\sigma} \mathcal{H}^{\mu\rho\sigma} \tilde{h}^{\rho\sigma}) = 0, \quad (11)
\]

\[
\nabla_\mu \mathcal{H}^{\mu\nu\rho} - (\nabla_\nu \hbar_\rho) \mathcal{H}^{\mu\nu\rho} + (\nabla_\mu \hbar_\rho) \mathcal{H}^{\mu\nu\rho} = 0, \quad (12)
\]

where the Lichnerowicz operator \( \delta R_{\mu\nu}(h) \) is given by

\[
\delta R_{\mu\nu} = -\frac{1}{2} \nabla^2 h_{\mu\nu} + \nabla_\sigma (h^\sigma_{\mu\nu}) - \nabla_\sigma h_\mu h^\sigma_{\nu} + \nabla_\sigma (\nabla_\mu h_\nu) h^{\rho\sigma}. \quad (13)
\]

These are the bare perturbation equations. It is desirable to examine whether we make a choice of perturbation and gauge which can simplify Eqs. (10)–(12) significantly. For this purpose we wish to count the physical degrees of freedom. A symmetric traceless tensor has \( D(D + 1)/2 - 1 \) in \( D \) dimensions. \( D \) of them are eliminated by the gauge condition. Also \( D - 1 \) are eliminated from our freedom to take further residual gauge transformations. Thus gravitational degrees of freedom are \( D(D + 1)/2 - 1 - D - (D - 1) = D(D - 3)/2 \). In three dimensions we have no propagating degrees of freedom for \( h_{\mu\nu} \). Also two-form \( B_{\mu\nu} \) has no physical degrees of freedom for \( D = 3 \). Hence the physical degree of freedom in the Kerr–de-Sitter solution is just the dilaton \( \varphi \).

Considering the \( t \) and \( \phi \)-symmetries of the background spacetime Eq. (5), we can decompose \( h_{\mu\nu} \) into frequency (\( \omega \)) and angular (\( j = 0, 1, 2, \ldots \)) modes in these variables

\[ h_{\mu\nu}(t, \phi, r) = e^{-i\omega t} e^{ij\phi} H_{\mu\nu}(r). \quad (14) \]

For simplicity, one chooses the same perturbation as in Eq. (14) for Kalb–Ramond field and dilaton as

\[ \mathcal{H}_{\phi r}(t, \phi, r) = \tilde{H}_{\phi r} H(t, \phi, r) \]

\[ = \tilde{H}_{\phi r} e^{-i\omega t} e^{ij\phi} \tilde{H}(r), \quad (15) \]

\[ \varphi(t, \phi, r) = e^{-i\omega t} e^{ij\phi} \tilde{\varphi}(r). \quad (16) \]

Since the dilaton is only a propagating mode, we are interested in the dilaton Eq. (11). Note that Eq. (10) is irrelevant to our analysis, because it belongs to the redundant relation. Eq. (11) can be rewritten as

\[ \nabla^2 \varphi + \frac{4}{\ell^2} (h - 2\mathcal{H}) = 0. \quad (17) \]

If we start with full six degrees of freedom of Eq. (14), we should choose a gauge. Conventionally, we choose the harmonic gauge \( (\nabla_\mu h^{\mu\rho} = 0) \) to describe the propagation of gravitons in \( D \geq 3 \) dimensions [21]. It turns out that a mixing between the dilaton and other fields of \( h, \mathcal{H} \) is not disentangled completely with the harmonic gauge condition. Here we focus on the propagation of the dilaton \( \varphi \). Fortunately, if we introduce the dilaton gauge \( (\nabla_\mu h^{\mu\rho} = h^{\mu\rho} \Gamma_\mu^{\rho\nu}) \), this difficulty may be resolved. Actually, this gauge was designed for the dilaton propagation [22]. We attempt to disentangle the last term in Eq. (17) by using both the dilaton gauge and Kalb–Ramond equation (12). Each component \( (\rho = t, \phi, r) \) of the dilaton gauge condition gives rise to

\[ t: \quad \left( \partial_t + \frac{1}{r} \right) h^{tt} - i\omega h^{t\phi} + \frac{1}{2} i \omega h g^{tt} = 0 \]

\[ - \frac{1}{2} i j h g^{t\phi} = 0. \quad (18) \]
\[ \phi: \left( \partial_r + \frac{1}{r} \right) h^{\phi r} - i \omega h^{\phi t} + i j h^{\phi \phi} + \frac{1}{2} \omega h \bar{g}^{\phi t} = 0, \]
\[ r: \left( \partial_r + \frac{1}{r} \right) h^{r r} - i \omega h^{r t} + i j h^{r \phi} = 0. \]

And each component \((v, \rho)\) of the Kalb–Ramond equation (12) leads to
\[ t\phi: -\partial_r (\phi + \mathcal{H} - h^{\phi t} - h^{\phi r}) + \frac{1}{r \ell^2} \left( -M + \frac{3 \rho^2}{\ell^2} + \frac{f^2}{4 r^2} \right) h'^r + i \omega h'^r = 0, \]
\[ t\phi: \partial_r h'^r + i \omega h'^t = 0, \]
\[ r\phi: -i \omega (\phi + \mathcal{H} - h^{\phi t} - h^{\phi r}) + \frac{1}{r} h'^t - \frac{2 r f^2}{\ell^2} h'^r = 0. \]

Solving six equations (18)–(23), one finds an important constraint
\[ \partial_{\mu} (2 \phi + 2 \mathcal{H} - h) = 0, \quad \mu = t, \phi, r \] (24)
which leads to \( h - 2 \mathcal{H} = 2 \omega \). This means that \( h \) and \( \mathcal{H} \) belong to the redundant field if one chooses the perturbations along Eqs. (14)–(16). It confirms that our counting for degrees of freedom is correct. We note that the harmonic gauge with the Kalb–Ramond equation (12) leads to the same constraint as in Eq. (24). As a result, Eq. (17) becomes a decoupled dilaton equation
\[ \nabla^2 \phi + \frac{8}{\ell^2} \phi = 0 \] (25)
which can be rewritten explicitly as
\[ \left[ f^2 \partial_r^2 + \left( \frac{1}{r^2} \partial_r r f \right) \right] \bar{\phi} - \frac{J j \omega}{r^2 f} + \frac{\omega^2}{f} = 0. \]

It is noted that if the last term is absent, Eq. (26) corresponds to the wave equation of a free scalar in Kerr–de-Sitter space. Comparing this dilaton equation with the massive scalar equation
\[ \nabla^2 \phi_m - m^2 \phi_m = 0, \] (27)

it seems that the dilaton propagates on Kerr–de-Sitter space with the tachyonic mass \( m^2 = -8/\ell^2 \). However, this observation is not the whole of story in de Sitter space.

4. Stability analysis for de Sitter space

It is not easy to solve the wave equation (26) of the dilaton on the southern diamond including the cosmological horizon \( r = r_c \) and the origin \( r = 0 \). The main difficulty comes from the fact that the black hole horizon degenerates to give a conical singularity at \( r = 0 \). In other words, the Kerr–de-Sitter solution represents a spinning point mass \( M \) in de Sitter space. This makes it hard to express the solution to the wave equation on the southern diamond in terms of the hypergeometric function. Then we cannot calculate the dilaton absorption cross section to test the cosmological horizon. Hence, hereafter, we confine ourselves to the pure de Sitter solution with \( J = 0 \). Then the origin is just that of the coordinate and thus there is nothing to worry about singularity on the southern diamond. Even though we give up the Kerr–de-Sitter background, we can study the nature of de Sitter space using the dilaton. Eq. (26) reduces to the differential equation for \( r \)
\[ (1 - r^2) \bar{\phi''} + \left( \frac{1}{r} - 3 r \right) \bar{\phi'} + \left( \frac{\omega^2}{1 - r^2} - \frac{j^2}{r^2} - m^2 \right) \bar{\phi} = 0, \] (28)

where the prime (‘) denotes the differentiation with respect to its argument and for simplicity we take \( \ell \) to be 1. The original equation from (26) with \( J = 0, \ell \neq 1 \) takes the same form as in Eq. (28) if \( r/\ell \to \tilde{r}, \omega \to \omega, m \to \tilde{m} \phi \). This information will be used for obtaining the absorption cross section in Section 5. From Eq. (28) it is obscure to know how the dilaton wave propagates in the southern diamond. In order to show it clearly, we must transform the wave equation into the Schrödinger-like equation by introducing
a tortoise coordinate \( r^* \). Then we can get information through the potential analysis. We introduce \( r^* = g(r) \) with \( g'(r) = 1/r (1 - r^2) \) to transform Eq. (28) into the Schrödinger-like equation with the energy \( E = \omega^2 \) [11]

\[
-\frac{d^2}{dr^2} \psi + V_\phi(r) \psi = E \psi
\]  

with the potential

\[
V_\phi(r) = \omega^2 + r^2 (1 - r^2) \left[ m^2_\psi \frac{j^2}{r^2} - \frac{\omega^2}{1 - r^2} \right].
\]  

Considering \( r^* = g(r) = \int g'(r) \, dr \), one finds

\[
r^* = \ln r - \frac{1}{2} \ln(1 - r^2),
\]

\[
e^{2r^*} = \frac{r^2}{1 - r^2}, \quad r^2 = \frac{e^{2r^*}}{1 + e^{2r^*}}.
\]  

Here we confirm that \( r^* \) is a tortoise coordinate such that \( r^* \to -\infty \) \((r \to 0)\), whereas \( r^* \to \infty \) \((r \to 1)\). Let us express the potential as a function of \( r^* \)

\[
V_\phi(r^*) = \omega^2 + \frac{e^{2r^*}}{(1 + e^{2r^*})^2} \left[ m^2_\psi + \frac{1 + e^{2r^*}}{e^{2r^*}} j^2 - (1 + e^{2r^*}) \omega^2 \right].
\]  

First of all we mention that this potential is the energy-dependent potential. Let us consider the low-energy limit of \( \omega^2 \ll 1 \). For \( m^2_\psi = -8 \), \( j = 0 \), \( \omega = 0.1 \), the shape of this takes a potential well near \( r^* = 0 \). Due to the potential well, this potential induces an exponentially large dilaton which is obviously contradicted to the genuine small value of the perturbation. Hence the \( j = 0 \)(s)-mode of the dilaton seems to be unstable. Naively speaking, this means that the cosmological horizon in our model does not truly exist. However, we have some ambiguity to define this s-mode in de Sitter space. Hence it is not clear from s-mode analysis that the cosmological horizon is unstable.

For \( j \neq 0 \)-modes with the low-energy of \( \omega^2 \ll 1 \), one finds that \( V_\phi(r^* = 0) = -8 + j^2/2 + \omega^2/2 \). Hence for \( j = 1, 2, 3 \)-modes one finds the potential wells, which confirm that the cosmological horizon is unstable. For \( j \geq 4 \)-modes the potential well disappears. In addition one finds the potential step with its height \( \omega^2 + j^2 \) on the left-hand side. All potentials decrease exponentially to zero as \( r^* \) increases on the right-hand side. This means that we always develop a well-defined wave near the cosmological horizon of \( r^* = \infty \). But near \( r^* = -\infty \) \((r = 0)\) it is not easy to develop the genuine waves. It is expected that the scattering to give a finite absorption cross section will occur if \( E = \omega^2 \sim V_\phi(r^*) \). This case is possible if \( \omega^2 \gg j^2 \). This means that the relevant scattering in de Sitter space may be arisen if the frequency of the external field is larger than its angular momentum quantum number. This corresponds to the low-temperature limit of \( \omega > T \). In this case the scattering can be defined even for \( j = 1, 2, 3 \) cases.

5. Absorption cross section

The absorption coefficient by the cosmological horizon is defined by the ratio of the outgoing flux at \( r = 0 \) to the outgoing flux at \( r = r_c \) as

\[
A = \frac{\mathcal{F}_{\text{out}}(r = r_c)}{\mathcal{F}_{\text{out}}(r = 0)}
\]  

Up to now we do not insert the curvature radius \( \ell \) of \( \text{dS}_3 \) space. The correct absorption coefficient can be recovered by replacing \( \omega(m_\psi) \) with \( \omega(l(m_\psi \ell)) \). Here we do not repeat the procedure of the flux computation but refer Ref. [11]. Then the dilaton absorption cross section in three dimensions is defined by

\[
\sigma_{\text{abs}} = \frac{A}{\omega} = \left[ \frac{A_c^2 + B_c^2}{A_c B_c} \right] \frac{\ell}{\vert \omega_{\text{ref}, j} \vert^2},
\]  

where

\[
\vert \omega_{\text{ref}, j} \vert^2 = \frac{|\Gamma(1 + j)|^2 |\Gamma(i\omega\ell)|^2}{|\Gamma(2 + j/2 + i\omega\ell/2)|^2 |\Gamma(1/2 + i\omega\ell/2)|^2}.
\]  

We observe that \( s(j = 0) \)-mode cross section is ill-defined because \( \mathcal{F}_{\text{out}}(r = 0) = 0 \) for \( j = 0 \). Furthermore the normalization factor of \( \left[ \frac{A_c^2 + B_c^2}{A_c B_c} \right] \) is not fixed by the theory. In order to obtain the explicit form, let us calculate \( \vert \omega_{\text{ref}, j} \vert^2 \) according to values of the angularmomentum quantum number \( j \). One finds

\[
\sigma_{\text{abs}, j}^\text{dil} = \left[ \frac{A_c^2 + B_c^2}{A_c B_c} \right] \frac{\ell}{\omega \omega \sinh(\pi \omega \ell)}
\]
Noting the temperature of the cosmological horizon cross section, let us consider the low-energy scattering. In order to get a definite expression for the absorption cross section, we calculate the absorption cross section of the dilaton which propagates on the southern diamond of three-dimensional de Sitter space. One of the striking results is that the low-energy cross section of the dilaton is not defined properly. This mainly rests on being unable to calculate its finite flux at $r = 0$. This contrasts sharply to the cases found in the symmetric black holes whose $s$-wave cross sections are well-defined and proportional to the area of the event horizon [17].

On the other hand, the $(j \neq 0)$-angular modes of the dilaton can be used for exploring the dynamical aspects of the cosmological horizon. We expect from the black hole analysis that the low-energy limit $(\omega R \to 0)$ of the $(l \neq 0)$-angular mode absorption cross section is proportional roughly to $(\omega R)^{4l}$ for the $D = 7$ black hole which is induced from D3-branes [23]. For $D = 5$ black hole, it is proportional to $(\omega R)^{2l}$ [24]. However, one finds from Eqs. (38) and (41) that those for $j \neq 0$ in the low-energy limit of $\omega < 1$ are given by $(\omega \ell)^{-2}$ which implies that the absorption cross section is greater than the area of the cosmological horizon. This is not the case what we want to get. From the potential analysis in Section 4, it conjectures that for $j = 1, 2, 3$ cases, the low-energy absorption cross section with $E = (\omega \ell)^{2}$ is meaningless because these become the unstable case. This implies that to obtain the finite absorption cross section, $\omega \ell$ should be large such as $\omega \ell > j$. This corresponds to the low-temperature limit of $\omega > T_r$. The low-temperature limit is meaningful in de Sitter space since its cross section appears less than $A_{\text{ch}}$. For example, we find from Eqs. (39) and (42) that the absorption cross sections takes $\sigma_{\text{abs,}j}^{\text{dil,}l = 1} \sim (\omega \ell)^{-(2j + 3)}$ roughly. This is consistent with the potential analysis. According to the this, the potential height is proportional to $\omega^2 + j^2$ which implies that the absorption cross section decreases as $j$ increases. On the other hand, the free scalar absorption cross section takes the same form $(\omega \ell)^{-2}$ as in the dilaton in the low-energy limit, while it is given by $\sigma_{\text{abs,}j}^{\text{free,}l = 1} \sim (\omega \ell)^{-(2j + 1)}$ in the low-temperature limit [11]. As a result, we confirm that the low-temperature limit (not the low-energy limit) of $(j \neq 0)$-angular mode absorption cross section will
be used to test the cosmological horizon in de Sitter space. This contrasts sharply to the fact that the low-energy $s$-mode plays an important role to test the black hole event horizon.

In conclusion, to get information about the cosmological horizon, we have to inject the test field with high frequency into the de Sitter background.

Finally we mention that the AdS bulk absorption cross section can be also calculated from the two-point function of CFT defined on the boundary if one assumes the AdS/CFT correspondence using the boundary-bulk Green function [25]. Hence we propose that our results for the dS bulk space can be recovered from the Euclidean CFT by making use of the dS/CFT correspondence [8,10] and the corresponding boundary-bulk Green function [12].

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Entropy and area of black holes in loop quantum gravity

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Abstract

Simple arguments related to the entropy of black holes strongly constrain the spectrum of the area operator for a Schwarzschild black hole in loop quantum gravity. In particular, this spectrum is fixed completely by the assumption that the black hole entropy is maximum. Within the approach discussed, one arrives in loop quantum gravity at a quantization rule with integer quantum numbers \( n \) for the entropy and area of a black hole.

The quantization of black holes was proposed long ago in the pioneering work [1], and from other points of view in [2,3]. The idea of [1] was based on the intriguing observation [4] that the horizon area \( A \) of a nonextremal black hole behaves in a sense as an adiabatic invariant. This last fact makes natural the assumption that the horizon area should be quantized. Once this hypothesis is accepted, the general structure of the quantization condition for large (generalized) quantum numbers \( N \) gets obvious, up to an overall numerical constant \( \alpha \). The quantization rule should be

\[ A_N = \alpha l_p^2 N. \]  

(1)

Indeed, the presence of the Planck length squared

\[ l_p^2 = G\hbar/c^3 \]  

(2)

in formula (1) is only natural. Then, for \( A \) to be finite in a classical limit, the power of \( N \) in expression (1) should be equal to that of \( \hbar \) in \( l_p^2 \). This argument, formulated in [5], can be checked, for instance, by inspecting any expectation value nonvanishing in the classical limit in ordinary quantum mechanics.

The subject of the present note is the entropy and spectrum of black holes in loop quantum gravity [6–10]. We confine below to a rather simplified version of this approach where the area spectrum of a spherical surface is

\[ A = \alpha l_p^2 \sum_{i=1}^{\nu} \sqrt{j_i(j_i + 1)}. \]  

(3)

Here a half-integer or integer “angular momentum” \( j_i \),

\[ j_i = \frac{1}{2}, 1, \frac{3}{2}, \ldots, \]  

(4)

is ascribed to each of \( \nu \) edges intersecting the surface. This set of edges, labeled by index \( i \), determines the surface geometry. The quantum numbers \( j_i \) assigned to these edges are constrained by the condition

\[ \sum_{i=1}^{\nu} j_i = n, \]  

(5)

where \( n \) is an integer. To each “angular momentum” \( j_i \), one ascribes \( 2j_i + 1 \) possible projections, from \(-j_i\) to \(j_i\).
to $+j$. We assume below that for a given $j$, all these projections have the same weight. With the vectors $\mathbf{j}$ being the only building blocks of the model, it is natural to consider that this $2j_i + 1$ degeneracy for each angular momentum $j_i$ corresponds to the spherical symmetry of the surface.

Some resemblance between expressions (1) and (3) is obvious, though in the last case the large number

$$N = \sum_{i=1}^{\nu} \sqrt{j_i(j_i + 1)}$$

is certainly no integer. As to the overall numerical factor $\alpha$ in (3), it cannot be determined without an additional physical input. This ambiguity originates from a free (so-called Immirzi) parameter [11,12] which corresponds to a family of inequivalent quantum theories, all of them being viable without such an input. One may hope that the value of this factor in (3) can be determined by studying the entropy of a black hole. This idea (mentioned previously in [13]) is investigated below.

We define the entropy $S$ of a spherical surface as the logarithm of the number of states of this surface with fixed $n$, $\nu$, and $v_j$, where $v_j$ is the number of edges with given $j$. Due to the mentioned $2j + 1$ degeneracy for each “angular momentum” $j$, the entropy is

$$S = \ln \left[ \prod_j (2j + 1)^{v_j} \prod_j v_j! \right] = \sum_j v_j \ln(2j + 1) + \ln(\sum_j v_j!) - \sum_j \ln v_j!. \quad (7)$$

The obvious constraints are

$$\sum_j v_j = \nu, \quad \sum_j jv_j = n. \quad (8)$$

Let us mention that the entropy arguments exclude for a black hole “empty” edges with $j_i = 0$. Obviously, if “empty” edges were allowed, the entropy would be indefinite even for fixed $N$ and $n$. In particular, with “empty” edges the Bekenstein–Hawking relation

$$S = \frac{A}{4l_p^2} \quad (9)$$

would not hold. Moreover, by adding an arbitrary number $v_0$ of “empty” edges in arbitrary order, the entropy could be made arbitrarily large without changing $N$ and $n$. Indeed, with “empty” edges allowed, the ratio $v! / v_0!$ grows indefinitely with $v_0$ at fixed values of $v_j$ with $j \neq 0$.

On the other hand, the same fundamental relation (9) dictates that the number of edges $\nu$ should be roughly on the same order of magnitude as the sum $n$ of “angular momenta”. Let us mention in this connection the model proposed in [13]. In this model the horizon is characterized by a single edge with $j = n/2$. Then the entropy grows with $n$ logarithmically,

$$S = \ln(2j + 1) = \ln(n + 1) \rightarrow \ln n,$$

while the area grows with $n$ linearly,

$$A \sim \sqrt{j(j + 1)} \sim \sqrt{n(n + 2)} \rightarrow n.$$

Since the requirement (9) for the classical limit $n \rightarrow \infty$ is grossly violated in it, the model of [13] has no physical meaning, or at least is incomplete. To save the model, some extra source of degeneracy should be included into it, but one cannot find in [13] any mention of such a degeneracy.

Thus, at least in the approach discussed (as distinct for example from that of [14]), relation (9) is an absolutely nontrivial constraint on a microscopic structure of theory.

It is natural to consider that the entropy of an eternal black hole in equilibrium is maximum. This argument is emphasized in [15], and used therein in a model of the quantum black hole as originating from dust collapse. Just the discussion of the assumption of maximum entropy is the main subject of the present Letter. More definite formulation of the problem considered below is as follows. With the quantum numbers $j_i$ being the only building blocks of the model, we are looking for such their distribution over the edges which results in the maximum entropy for a fixed total amount of the building material $n = \sum j_i$.

It is rather obvious intuitively that the entropy is maximum when all values of $j_i$ are allowed. To demonstrate that this is correct, we will consider a few more and more complex examples step by step, starting with the simplest choice for the quantum numbers $j_i$, where all of them are put equal to 1/2. Then $v_j = \nu \delta(j, 1/2)$, $v = 2n$, and

$$S = 2 \ln 2n. \quad (10)$$
With all \( j_i = 1/2 \) and \( v = 2n \), the area given by formula (3) equals

\[
A = a l_p^2 \sqrt{\frac{3}{2}} v = a l_p^2 \sqrt{3} n.
\]

(11)

Now, under the made assumption we obtain, due to formulae (9)–(11), the following value of the parameter \( \alpha \) of the theory:

\[
\alpha = \frac{8 \ln 2}{\sqrt{3}}.
\]

(12)

It should be pointed out that this is the value of the parameter \( \alpha \) derived previously in [16] within a Chern–Simons field theory, and that the typical value of \( j_i \) obtained therein is also 1/2 (see also [17,18]).

In fact, in this way one arrives at the quantization rule for the black hole entropy (and area) with integer quantum numbers \( \nu \) or \( n \) (see formula (10)), as proposed in [1]. Moreover, in this picture the statistical weight of the quantum state of a black hole is \( 2^\nu \) with integer \( \nu \), as argued in [2] (in the present case this integer \( \nu \) should be even).

Let us include now \( j = 1 \) in line with \( j = 1/2 \). Then the entropy reaches its maximum value

\[
S = 2 \ln 3 n = 2.197 n
\]

for \( v_{1/2} = n \), \( v_1 = n/2 \), with the mean value \( \langle j \rangle \) of angular momenta

\[
\langle j \rangle = n \frac{2}{\nu} = 0.667.
\]

(Here and below we retain in the expressions for entropy only leading terms, linear in the large parameter \( n \).) It is curious to compare these numbers with the analogous ones \( S = 2 \ln 2 n = 1.386 n \), \( \langle j \rangle = j = 1/2 = 0.5 \) for the pure \( j = 1/2 \) case.

But what happens if quantum numbers larger than 1 are also allowed? When \( j = 3/2 \) are included, in line with \( j = 1/2 \) and \( j = 1 \), the maximum entropy value

\[
S = 2.378 n
\]

is attained at \( v_{1/2} = 0.810 n \), \( v_1 = 0.370 n \), \( v_{3/2} = 0.150 n \). Both entropy and average angular momentum

\[
\langle j \rangle = 0.752
\]

increase again, but not too much, as compared to the previous ones \( S = 2.197 n \), \( \langle j \rangle = 0.667 \).

It is natural now to expect that the absolute maximum of entropy is reached when all values of quantum numbers are allowed. Let us consider this situation starting with the general formula (7). It is convenient to go over in it to new variables \( y_j \):

\[
v_j = ny_j,
\]

(13)

constrained in virtue of (5) by the obvious relation

\[
\sum_j j y_j = 1.
\]

(14)

Then, by means of the Stirling formula for factorials, we transform (7) to the following expression:

\[
S = n \left[ \sum_j y_j \ln(2j + 1) + \sum_j y_j \ln(\sum_{j'} y_{j'}) \right] - \sum_j y_j \ln y_j.
\]

(15)

Only the contribution proportional to the large number \( n \) is retained here. We have assumed also that the number of essential terms in the sums entering (7) (i.e., the number of the essential classes of the edges with the same \( j \)) is much smaller than \( n \). In fact, this number is on the order of \( \ln n \), and the leading correction to the approximate formula (15) is on the order of \( \ln^2 n \). Again, the situation with the leading correction here is different from that for the case when all \( j_i = 1/2 \), where the correction is just absent, and from that for the model considered in [17,18], where it is on the order of \( \ln n \).

We are looking for the extremum of expression (15) under the condition (14). The problem reduces to the solution of the system of equations

\[
\ln(2j + 1) + \ln \left( \sum_{j'} y_{j'} \right) - \ln y_j = \mu j,
\]

(16)

or

\[
y_j = (2j + 1)e^{-\mu j} \sum_{j'} y_{j'}.
\]

(17)

Here \( \mu \) is the Lagrange multiplier for the constraining relation (14). Summing expressions (17) over \( j \), we arrive at equation

\[
\sum_{j=1/2}^{\infty} (2j + 1)e^{-\mu j} = 1.
\]
or \[ \sum_{p=1}^{\infty} (p+1)z^p = 1, \quad p = 2j, \quad z = e^{-\mu/2}. \] (18)

Its solution is readily obtained:

\[ z = 1 - \frac{1}{\sqrt{2}} \quad \text{or} \quad \mu = -2 \ln z = 2.456. \] (19)

Let us multiply now Eq. (16) by \( y_j \) and sum over \( j \). Then, with the constraint (14) we arrive at the following result for the absolute maximum of the entropy for a given value of \( n \):

\[ S = \mu n = 2.456 n. \] (20)

This is the final term of the succession of previous values of entropy \( S \):

1.386\( n \), 2.197\( n \), 2.378\( n \).

Assuming that the entropy of an eternal black hole in equilibrium is maximum, we come to the conclusion that it is just (20), which is the true value of the entropy of a black hole.

To find the mean angular momentum \( \langle j \rangle \) in the state of maximum entropy, let us rewrite the constraint (14) as

\[ y_{1/2} = \frac{1}{\sqrt{2}/2} \sum_{p=1}^{\infty} (p+1)pz^{p-1} = 1. \] (21)

The sum in the last expression is also easily calculated, and with the value (19) for \( z \) we obtain \( y_{1/2} = 1/\sqrt{2} \). In its turn, this value of \( y_{1/2} \) together with Eq. (17) gives

\[ y_j = \frac{1}{2\sqrt{2}} (2j+1)2^{j-1}. \] (22)

and

\[ \sum_{j=1/2}^{\infty} y_j = \frac{\sqrt{2}+1}{2} = 1.207. \] (23)

Now, the mean angular momentum is

\[ \langle j \rangle = \frac{n}{v} = \frac{1}{\sum_{j=1/2}^{\infty} y_j}^{-1} = 2(\sqrt{2} - 1) = 0.828, \] (24)

which fits perfectly the succession of previous mean values \( \langle j \rangle \):

0.5, 0.667, 0.752.

Let us come back to the expression (3) for the black hole entropy. The sum (6) is conveniently rewritten as

\[ N = \sum_{j=1/2}^{\infty} \sqrt{\nu(j+1)}v_j. \]

With our formulae (19), (22), one can easily express this sum via \( n \):

\[ N = 1.471 n. \]

Thus, using the Bekenstein–Hawking relation (9), we obtain the following results in the loop quantum gravity for the area \( A \) of an eternal spherically symmetric black hole in equilibrium and for the constant \( \alpha \) of the area spectrum (3) of a spherical surface:

\[ A = 9.824 l^2_{pl} n = 6.678 l_p^2 N, \quad \alpha = 6.678. \] (25)

As to the mass \( M \) of a black hole, it is quantized in the units of the Planck mass \( m_p \) as follows:

\[ M^2 = \frac{0.614}{\pi} m_p^2 n. \] (26)

Let us present also for the sake of comparison the corresponding results of [16]:

\[ A_a = 8 \ln 2 l^2_{pl} n = 8 \ln \frac{2}{\sqrt{3}} l^2_{pl} N = 3.202 l_p^2 N, \]

\[ \alpha_a = 3.202, \quad M^2_a = \frac{\ln 2}{2\pi} m_p^2 n = \frac{0.347}{\pi} m_p^2 n. \]

Of course, the solution proposed in [16] looks at least more simple and elegant. On the other hand, the advantage of our solution is that it is based on a simple and natural physical conjecture.

It should be emphasized that in both cases one arrives at the quantization rule for the black hole entropy (and area) with integer quantum numbers \( n \), as proposed in [1].

In conclusion, let us comment briefly upon some previous investigations of the considered problem. In [19] the entropy is defined as logarithm of the number of microstates for which the sum (6) is between \( N \) and \( N + \Delta N, N \gg \Delta N \gg 1 \) (here and below the notations of the present article are used). The conclusion made in [19] is that the value of this logarithm is in the interval (0.96–1.38)\( N \). However, under the only condition \( N \gg 1 \), without any assumption made about the distribution of the angular momenta \( j \) over the edges, how can one arrive at the above numbers.
(0.96–1.38)N? The same question (in fact, objection) refers to the results obtained in [20].

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**Abstract**

The Weyl group symmetry $W(E_k)$ is studied from the points of view of the $E$-strings, Painlevé equations and $U$-duality. We give a simple reformulation of the elliptic Painlevé equation in such a way that the hidden symmetry $W(E_{10})$ is manifestly realized. This reformulation is based on the birational geometry of the del Pezzo surface and closely related to Seiberg–Witten curves describing the $E$-strings. The relation of the $W(E_k)$ symmetry to the duality of M-theory on a torus is discussed on the level of string equations of motion. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

In a recent paper [1], the second order (difference) Painlevé equations have been classified by using the geometry of algebraic surfaces. The classification falls into three types: rational, trigonometric and elliptic. Each case is associated with a special divisor corresponding to one of the Kodaira singular fibers of elliptic fibration (Table 1) [2]

\[
\begin{align*}
\text{ell.} & \quad I_0 \\
\text{tri.} & \quad I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow I_4 \rightarrow I_5 \rightarrow I_6 \rightarrow I_7 \rightarrow I_8 \rightarrow I_9 \\
\text{rat.} & \quad II \leftrightarrow III \leftrightarrow IV \leftrightarrow I^*_{0} \leftrightarrow I^*_{1} \leftrightarrow I^*_{2} \leftrightarrow I^*_{3} \leftrightarrow I^*_{4} \\
& \quad IV^* \leftrightarrow III^* \leftrightarrow II^*
\end{align*}
\]

(1)

In physics, the same\(^{1}\) diagram appeared as the RG flow of the $E$-strings in dimensions $d = 6$ [3,4], $d = 5$ [5,6] and $d = 4$ [7].

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\(^{1}\) The corresponding root systems are complement with each other.

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Table 1

<table>
<thead>
<tr>
<th>Fiber type</th>
<th>Singularity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_n$ ($n \geq 1$)</td>
<td>$\lambda_{n-1}$</td>
</tr>
<tr>
<td>II, III, IV ($n = 0, 1, 2$)</td>
<td>$A_n$</td>
</tr>
<tr>
<td>$I^*_n$ ($n \geq 0$)</td>
<td>$D_{n+4}$</td>
</tr>
<tr>
<td>II*, III*, IV* ($n = 8, 7, 6$)</td>
<td>$E_6$</td>
</tr>
</tbody>
</table>

Of course, this is not an accident, since both of them are described by the same geometry, namely the del Pezzo surface $B_9$ which is a blown up of $\mathbb{P}^2$ at 9 points. The difference is that in the $E$-string case the 9 points are chosen in special position so that the surface admits an elliptic fibration.

Furthermore, the $E_n$-series in the above diagram is also well-known in connection with the duality symmetry of M-theory compactified on a torus [8]. In this correspondence, the Weyl group part of the $U$-duality group is identified with the Cremona isometry $W(E_k)$ for del Pezzo $B_k$ [9]. (This duality was discussed in [10] and [11] from the point of view of Little String Theory and Borcherds symmetry, respectively.) The aim of this Letter is to examine the correspondence by closely looking the way how the Weyl group is realized in each case.

This Letter is organized as follows. In Section 2, we clarify the special role of the marked fiber at $u = \infty$ in the Seiberg–Witten geometry. (The importance of this marking was stressed in [12], see also [13,14].) We also give an example of duality map between two Seiberg–Witten curves corresponding to different spacetime dimensions. In Section 3, we give a reformulation of the elliptic Painlevé equation where the hidden $W(E_{10})$ symmetry is manifestly realized and the relation to the Seiberg–Witten geometry of $E$-strings is discussed. In Section 4, we study the Painlevé equations arising from a consistent truncation/reduction of the M-theory and compare the Painlevé Bäcklund transformations with the $U$-duality. Finally, Section 5 is devoted to the conclusions and discussions.

2. The marked fiber in Seiberg–Witten geometry

The equations for the $SU(2)$ $E_8$ flavor Seiberg–Witten curves with two mass parameters have been given by Minahan et al. [15]

\[ y^2 = x^3 - 2u(u^2 + m_1^2 x)(u^2 + m_2^2 x), \]
\[ y^2 = x^3 + u^2 x^2 - 2u(u^2 + \sin^2 m_1 x)(u^2 + \sin^2 m_2 x), \]
\[ y^2 = x^3 + (1 + k^2)u^2 x^2 - 2u(u^2 + \sin^2 m_1 x)(u^2 + \sin^2 m_2 x) + k^2 u^4 x. \]  

The discriminants and singular fibers are

\[ \Delta = u^8(\mu^2 + \cdots), \quad I^*_5 + 2I_1 + (\Pi)_{u=\infty}. \]

\[ (2) \]

\[ (3) \]  

\[ ^2 \text{Recently, a remarkable compact formula for the full 8-parameter elliptic Seiberg–Witten curve has been obtained in [16].} \]
where the leading term is of the form (5) with $v^{132}$.


which can be parametrized as

The $I_1^*$ ($= D_6$) singularity at $u = 0$ corresponds to the two mass deformation of $\Pi^*$ ($= E_8$). Note that the difference among the three cases (rat./tri./ell.) appears on the fiber at $u = \infty$. That is, the fiber is a cusp/nodal/smooth curve, respectively. The marked fiber curve at $u = \infty$ plays the role of the special divisor in (1).

To see the meaning of the marked fiber at $u = \infty$, let us consider the sections. For all three cases, the Mordell–Weil lattice are $A_1^* \oplus A_1^*$ [17]. In fact, we have the following generators of the sections [15],

rat. $x = -\frac{1}{v^2} u^2$, $y = i \frac{1}{v^3} u^3$,

tri. $x = -\frac{1}{\sin^2 \nu} u^2$, $y = i \frac{\cos \nu}{\sin^3 \nu} u^3$,

ell. $x = -\frac{1}{\sin^2 \nu} u^2$, $y = i \frac{\cn \nu \dn \nu}{\sn^3 \nu} u^3$,

where $v = m_1$ or $m_2$. Other sections can be obtained by addition and have the form at $u = \infty$ as

$x = a_2 u^2 + a_1 u + \cdots$, $y = b_3 u^3 + b_2 u^2 + \cdots$,

where the leading term is of the form (5) with $v = k_1 m_1 + k_2 m_2$ ($k_1, k_2 \in \mathbb{Z}$). Let us consider the elliptic case. The fiber at $u = \infty$ is a smooth curve

$y^2 = x^3 + (1 + k^2)x^2 + k^2 x$,

which can be parametrized as

$x = x(v) = -\frac{1}{\sn^2 \nu}$, $y = y(v) = i \frac{\cn \nu \dn \nu}{\sn^3 \nu}$.

Note that this is nothing but the leading term of the section. Hence the parameter $v$ indicates where the section and the marked fiber intersect [7,12,13]. This is true also in the rational and trigonometric cases. In the next section, we will see that the marked curve (which is not a fiber in general) also plays an essential role in the Painlevé equations.

Finally, we consider the relation between the curves in (3) and $SU(2)$ $N_f = 2$ Seiberg–Witten curve [18]. The $N_f = 2$ Seiberg–Witten curve

$y^2 = \left( x^2 - \frac{A^4}{64} \right) (x-u) + \frac{A^2}{4} M_1 M_2 x - \frac{A^4}{64} (M_1^2 + M_2^2)$,

and the elliptic case in (3) are both the generic curves with the $D_6$ singularity. Hence, they should be related with each other. In fact, up to simple change of variables $x, u$, these curves are equivalent. The relations of parameters are

$A^2 = 4 \left( \frac{1}{sn \ m_1} - \frac{1}{sn \ m_2} \right)$

$A^2 (M_1 + M_2) = 8 \frac{cn \ m_1 \ dn \ m_1}{sn^3 \ m_1}$, $A^2 (M_1 - M_2) = 8 \frac{cn \ m_2 \ dn \ m_2}{sn^3 \ m_2}$.

This mapping $(A, M_1, M_2) \leftrightarrow (k, m_1, m_2)$ can be interpreted as a kind of duality which connects different theories (in different dimensions).
3. The elliptic Painlevé equation

On a del Pezzo surface $B_k$ the Weyl group $W(E_k)$ acts as the Cremona isometry [19]. For the case of $k = 9$, the Weyl group $W(E_9)$ is the affine Weyl group of type $W(E^{(1)}_6)$ which contains the translation subgroup $\mathbb{Z}^8$ and this is the origin of the elliptic Painlevé equation [1]. This construction can be considered as an example of general strategy to construct discrete Painlevé equations by using affine Weyl groups [20].

We will reformulate the elliptic Painlevé equation in the form where the hidden $W(E_{10})$ symmetry is manifestly realized. Let $M$ be the space of $3 \times 10$ matrix

$$M = \left\{ X = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_{10} \\ y_1 & y_2 & y_3 & \cdots & y_{10} \\ z_1 & z_2 & z_3 & \cdots & z_{10} \end{bmatrix} \right\}. \quad (11)$$

Each column vector $P_i = (x_i : y_i : z_i)$ may be thought of as a projective coordinate of a point $P_i \in \mathbb{P}^2$. In view of this, we make an identification

$$\mathcal{M} = \text{PGL}(3) \backslash M / (\mathbb{C}^*)^{10}. \quad (12)$$

A representative of this coset can be taken as

$$X = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & u_5 & \cdots & u_{10} \\ 1 & v_5 & \cdots & v_{10} \end{bmatrix}, \quad (13)$$

where

$$u_i = \frac{\mu_{234}\mu_{13i}}{\mu_{134}\mu_{23i}}, \quad v_i = \frac{\mu_{234}\mu_{12i}}{\mu_{124}\mu_{23i}} \quad (i = 5, \ldots, 10), \quad (14)$$

and $\mu_{ijk}$ is the minor determinant of $X$ taking $i$, $j$ and $k$th columns. We have an action of the symmetric group $S_{10}$ which act as a permutation of the columns of $X$. In terms of the coordinates $(u_i, v_i)$, $i = 5, \ldots, 10$ the $S_{10}$-action can be written as follows [19]:

The actions of $s_1, s_2, s_3$ are given by

$$s_1(u_i) = \frac{1}{u_i}, \quad s_1(v_i) = v_i, \quad s_2(u_i) = v_i, \quad s_2(v_i) = u_i, \quad s_3(u_i) = \frac{u_i - v_i}{1 - v_i}, \quad s_3(v_i) = \frac{v_i}{v_i - 1}. \quad (15)$$

The action of $s_4$ is

$$s_4(u_5) = \frac{1}{u_5}, \quad s_4(v_5) = \frac{1}{v_5}, \quad s_4(u_i) = \frac{u_i}{u_5}, \quad s_4(v_i) = \frac{v_i}{v_5} \quad (i = 6, \ldots, 10) \quad (16)$$

and $s_i$ for $i = 5, \ldots, 9$ act as

$$s_i(u_{i+1}) = u_{i+1}, \quad s_i(u_{i+1}) = u_i, \quad s_i(u_j) = u_j \quad (j \neq i, i + 1), \quad s_i(v_{i+1}) = v_{i+1}, \quad s_i(v_{i+1}) = v_i, \quad s_i(v_j) = v_j \quad (j \neq i, i + 1). \quad (17)$$

Besides the permutations $s_i \in S_{10}$ ($i = 1, \ldots, 9$), there exist another important involution $s_0$ on the variables $(u_i, v_i)$, namely

$$s_0(u_i) = \frac{1}{u_i}, \quad s_0(v_i) = \frac{1}{v_i}. \quad (18)$$
Geometrically, this is a standard Cremona transformation with center \((P_1, P_2, P_3)\). By direct computation, we have
\[
(s_{03i})^2 = 1 \quad (i \neq 3) \quad \text{and} \quad (s_{033})^3 = 1.
\]
In summary, the transformations \(s_i, i = 0, 1, \ldots, 9\) defined by (15)–(17) and (18) give a birational representation of the Weyl group \(W(E_{10})\) on the field of rational functions \(\mathbb{C}(u_5, \ldots, u_{10}, v_5, \ldots, v_{10})\).

\[
E_{10}
\]

The construction of the elliptic Painlevé equation is very simple. The Weyl group \(W(E_{10})\) contains \(W(E_8^{(1)})\) generated by \(s_i\) \((i = 0, \ldots, 8)\). This group \(W(E_8^{(1)})\) has a translation subgroup \(\mathbb{Z}^8\). The birational action of these translations on \(\mathcal{M}\) is nothing but the Sakai’s elliptic Painlevé equation. The explicit action of these translations on the variables \((u_i, v_i)\) are too complicated and seems to be beyond our computational ability. We give an intermediate formula for one of the translations\(^3\)

\[
T = (pq)^2, \quad p = s_3s_4s_5s_2s_3s_4s_1s_2s_3s_0, \quad q = s_6s_7s_8s_5s_6s_7s_4s_5s_6.
\]

The result is given as follows:

\[
p(u_5, u_6, u_i) = \frac{\mu_{146}}{\mu_{156}} \left( \frac{\mu_{256}}{\mu_{246}} \frac{\mu_{356}}{\mu_{346}} \frac{\mu_{560}}{\mu_{460}} \right) \quad (i = 7, \ldots, 10),
\]

\[
p(v_5, v_6, v_i) = \frac{\mu_{145}}{\mu_{156}} \left( \frac{\mu_{256}}{\mu_{245}} \frac{\mu_{356}}{\mu_{345}} \frac{\mu_{560}}{\mu_{450}} \right) \quad (i = 7, \ldots, 10),
\]

\[
q(u_5, u_6, u_7, u_8, u_9, u_{10}) = \frac{1}{u_7} \quad \text{and} \quad q(v_5, v_6, v_7, v_8, v_9, v_{10}) = \frac{1}{v_7}.
\]

If the 9 points \(P_1, \ldots, P_9\) are in general position, there exist unique elliptic curve \(C \subset \mathbb{P}^2\) which pass through the 9 points. This curve \(C\) play the role of the fiber at \(u = \infty\) in the previous section and it is invariant under the action of \(W(E_8^{(1)})\). Using this curve \(C\) as a “ruler”, Sakai introduced another coordinates of the coset \(\mathcal{M}: \theta_1, \ldots, \theta_9, \tau\) and \((x : y : z) \in \mathbb{P}^2\), such that the matrix \(X\) is represented as

\[
X = \begin{bmatrix}
\wp(\theta_1) & \wp(\theta_2) & \cdots & \wp(\theta_9) & x \\
\wp'(\theta_1) & \wp'(\theta_2) & \cdots & \wp'(\theta_9) & y \\
1 & 1 & \cdots & 1 & z
\end{bmatrix}.
\]

Here \(\wp(\theta) = \wp(\theta, \tau)\) is the Weierstrass \(\wp\) function which parameterize the elliptic curve \(C\). In terms of Sakai’s coordinates, the action of \(W(E_8^{(1)})\) is given as follows \cite{1}. The \(S_9\) part is just the permutation of the parameters \(\theta_i\).

\(^3\)There exist 240 commuting translations corresponding to \(E_8\) roots. Any of them can be represented as a composition of 58 simple reflections. Among the 240 translations, only 8 of them are independent.
The only non-trivial one is $s_0$ which has been determined explicitly\(^4\) as

$$
s_0(\theta_i) = \theta_i' = \begin{cases} 
\theta_i + \frac{1}{3}(\theta_1 + \theta_2 + \theta_3), & i = 4, \ldots, 9, \\
\theta_i - \frac{1}{3}(\theta_1 + \theta_2 + \theta_3), & i = 1, 2, 3,
\end{cases}
$$

(24)

$$
s_0 \begin{bmatrix} x' \\
y' \\
z'
\end{bmatrix} = \begin{bmatrix} x_1' & x_2' & x_3' \\
y_1' & y_2' & y_3' \\
z_1'
\end{bmatrix} \begin{bmatrix} d_{23} & s_{12} & l_{12} \\
l_{23} & d_{12} & s_{13} \\
0 & d_{13} & l_{13}
\end{bmatrix},
$$

(25)

$$
l_{jk} = \det \begin{bmatrix} x & x_j & x_k \\
y & y_j & y_k \\
z & z_j & z_k
\end{bmatrix}, \quad d_{jk} = \det \begin{bmatrix} x_\ast & x_j & x_k \\
y_\ast & y_j & y_k \\
z_\ast & z_j & z_k
\end{bmatrix} \det \begin{bmatrix} x'_\ast & x'_j & x'_k \\
y'_\ast & y'_j & y'_k \\
z'_\ast & z'_j & z'_k
\end{bmatrix}.
$$

(26)

Where $(x_\ast, y_\ast, z_\ast)$ and $(x'_\ast, y'_\ast, z'_\ast)$ are any points on the curve $C$ such that $(x_\ast, y_\ast, z_\ast) = (\varphi(\theta), \varphi'(\theta), 1)$, $(x'_\ast, y'_\ast, z'_\ast) = (\varphi(\theta'), \varphi'(\theta'), 1)$ with $\theta' = \theta + (\theta_1 + \theta_2 + \theta_3)/3$.

In these coordinates, the translation $T$ (20) acts on the parameters $\theta_i$ as

$$
T(\theta_1, \ldots, \theta_9) = (\theta_1, \ldots, \theta_9) - \frac{1}{3}\theta(2, 2, 2, -1, \ldots, -1),
$$

(27)

where $\theta = \sum_{i=1}^{9} \theta_i$. When $\theta = 0$ (modulo periods), the first 9 points are in special position such that the curve $C$ passing through the 9 points is given by one parameter family (a ‘pencil’ of cubic)

$$
\lambda F(x, y, z) + \mu G(x, y, z) = 0.
$$

(28)

Then the corresponding del Pezzo $B_9$ admits an elliptic fibration $B_9 \rightarrow \mathbb{P}^1 = \{(\lambda : \mu)\}$ and 9 blown-up $\mathbb{P}^1$’s correspond to 9 sections of the fibration. The parameters $\theta_i$ specify the 9 points $(x_i : y_i : z_i) = (\varphi(\theta_i) : \varphi'(\theta_i) : 1)$ where the sections intersect with the marked curve $C$ (at $u = \infty$). These data define the Seiberg–Witten curve for $d = 6$ $E_8$-string as explained in [12,13]. In this special case, by choosing the parameter $(\lambda : \mu)$ suitably $C$ may pass the 10th point $(x : y : z)$ also and we can put $(x : y : z) = (\varphi(\theta_{10}) : \varphi'(\theta_{10}) : 1)$. Then all the action of $W(E_{10})$ are represented by addition and permutation on the variables $\theta_i$ ($i = 1, \ldots, 10$).

4. Painlevé Bäcklund transformations and $U$-duality

4.1. Relation to $M$-theory duality

The del Pezzo surfaces $B_k$ play crucial role in various context of string compactifications. In a recent paper [9], Iqbal, Neitzke and Vafa observed a duality between M-theory on $T^k$ and del Pezzo surfaces $B_k$. In this correspondence, the Weyl group part of the $U$-duality group is identified with the Cremona isometry $W(E_k)$ for del Pezzo $B_k$. As we have seen in the previous section, the Cremona isometry is the origin of the Bäcklund transformation/discrete time evolution of the elliptic Painlevé equation, it is natural to expect some relation between the Painlevé Bäcklund transformations and $U$-duality. In fact, there exist an analogy between these two Weyl group realizations. Namely the permutation part of the duality can be realized as a change of the order of the compactifications [23], correspondingly the Weyl group symmetry of the $E_{10}$ Painlevé equation appears as a change of the blowing down structure [1].

\(^4\)This means that these nine $\theta_i$’s transform under $W(E_8)$ as the $SL(9)$ Cartan subalgebra, and hence correspond to the nine radii in the $T^9$ compactification of M-theory. The extra Weyl reflection $s_0$ (called ‘2/5 transformation’ in [21]) is naturally understood via the $SL(9)$ decomposition of $E_8$ [22].
The Painlevé difference equations reduce to the six Painlevé differential equations in the continuum limit. The latter also possess affine Weyl group symmetries generated by the Bäcklund transformations. Thus it will be interesting to explore whether these differential equations have direct connections with the string equations of motion.

In general relativity, it has been known for some time that some static, axisymmetric solutions of Einstein–Maxwell’s equation(s) obey Painlevé differential equations [24–27]. For example, the Ernst equation, the equation of motion for the scalars in the dimensionally reduced $D = 4$ pure gravity, reduce to the third or the fifth Painlevé equation under certain assumptions. Since $D = 10$, type IIB scalar sigma model is identical to that of the Ernst system, one can exploit the general relativity result to find special IIB scalar solutions that obey Painlevé equations.

Let us consider a consistent truncation of type IIB supergravity

$$
\mathcal{L} = \sqrt{-G^{(10)}} \left( R^{(10)} + \frac{\partial_M \tau \partial^M \bar{\tau}}{2(\text{Im} \tau)^2} \right),
$$

(29)

where $\tau = C + i e^{-\Phi}$ with $C$ and $\Phi$ being the RR scalar and the dilaton, respectively. We further adopt an ansatz that the ten-dimensional Einstein-frame metric $G_{MN}^{(10)}$ is of the form

$$
ds^2_{\text{IIB}} = \lambda^2 (dx^2 + d\rho^2) + \rho^2 d\phi^2 + \left(-dt^2 + \sum_{i=1}^{6} dx_i^2 \right),
$$

(30)

and that $\lambda$ is a real function of $-\infty < x < \infty$, the coordinate parallel to the symmetric axis, and the radial coordinate $\rho \geq 0$. $\phi$ is the angle coordinate. We also assume that the complex potential $\tau$ depends only on $x$ and $\rho$. In this way, we get a two-dimensional system without enlarging the duality symmetry than $\text{SL}(2, \mathbb{R})/\text{U}(1)$.

In fact, this truncation is equivalent to the dimensional reduction of $D = 4$ pure gravity to $D = 2$ with a four-dimensional metric

$$
ds^2_{4D} = e^\Phi (\lambda^2 (dx^2 + d\rho^2) + \rho^2 d\phi^2) - e^{-\Phi} (dt + A_\phi d\phi)^2
$$

(31)

with

$$
\partial_\xi A_\phi = -i \rho e^{2\Phi} \partial_\xi C, \quad \xi \equiv x + i \rho.
$$

(32)

The equation of motion for $\tau$ is given by the Ernst equation

$$
e^{-\Phi} \delta^{\mu\nu} \partial_\mu (\rho \partial_\nu \tau) = -i \rho e^{2\Phi} \partial_\xi \tau \partial_\xi \tau,
$$

(33)

where $x^\mu = (x, \rho)$. If $\tau$ is known, the conformal factor $\lambda$ is consistently determined by integrating the first-order ‘Virasoro constraint’. (See [29] for related technology.)

The metric ansatz (30) is close to that for the D7-brane solutions [30], but the crucial difference is the appearance of $\rho^2$ in $G_{\phi\phi}^{(10)}$. Owing to this explicit coordinate dependence (‘the Weyl canonical coordinate’), $\tau$ cannot be holomorphic, and the solution does not preserve supersymmetry.

4.2. Painlevé III and S-duality

To reduce (33) to a Painlevé equation, we first switch from the $\text{SL}(2, \mathbb{R})$ variable $\tau$ to the $\text{SU}(1, 1)$ variable $F$, defined by [25]

$$
\tau = \frac{1 + F}{1 - F}.
$$

(34)

---

5 One may also trade $\rho$ for the time $t$ to discuss colliding string wave solutions. See, e.g., [28] for recent discussions and further references.
In terms of $F$, the Ernst equation becomes
\begin{equation}
(1 - F \overline{F}) \delta^{\mu \nu} \partial_\mu (\rho \partial_\nu F) = -2 \rho \overline{F} \partial_\mu F \partial_\nu F. \tag{35}
\end{equation}
We further assume the coordinate dependence of $F(x, \rho)$ as
\begin{equation}
F(x, \rho) = f(\rho) e^{i \omega x}, \tag{36}
\end{equation}
where $f(\rho)$ is a real function, and $\omega$ is a real constant. Eq. (35) reduces to
\begin{equation}
(f^2 - 1) \left( f'' + \frac{f'}{\rho} - \omega^2 f \right) = 2 f \left( f' \alpha - \omega^2 f \right). \tag{37}
\end{equation}
Here the prime denotes differentiation with respect to $\rho$. Then by the replacement
\begin{equation}
y = \frac{1 + f}{1 - f}, \tag{38}
\end{equation}
we obtain
\begin{equation}
y'' = \frac{y'^2}{y} - \frac{y'}{\rho} + \frac{\omega^2}{4} \left( y^3 - \frac{1}{y} \right). \tag{39}
\end{equation}
This is Painlevé III
\begin{equation}
y'' = \frac{y'^2}{y} - \frac{y'}{\rho} + \frac{\alpha y^2 + \beta}{\rho} + \gamma y^3 + \frac{\delta}{y}, \tag{40}
\end{equation}
with special parameters
\begin{equation}
\alpha = \beta = 0, \quad \gamma = -\delta = \frac{\omega^2}{4}. \tag{41}
\end{equation}

Painlevé III (with generic parameters) is known to have a symmetry of Bäcklund transformations isomorphic to the Weyl group of type $(A_1 \oplus A_1)^{(1)}$ generated by three independent Weyl reflections [1]. One of them is
\begin{equation}
y \mapsto \frac{1}{y}, \quad (\alpha, \beta, \gamma, \delta) \mapsto (-\beta, -\alpha, -\delta, -\gamma) \tag{42}
\end{equation}
which leaves the condition (41) unchanged. Since this implies $\tau \mapsto -1/\tau$, we see that this Bäcklund transformation of Painlevé III precisely corresponds to $S$-duality of IIB theory. On the other hand, the second Bäcklund transformation is simply
\begin{equation}
y \mapsto -y, \quad \rho \mapsto -\rho, \quad (\alpha, \beta, \gamma, \delta) \mapsto (\alpha, \beta, \gamma, \delta). \tag{43}
\end{equation}
It just flips the sign of $\tau$, and hence is a physically irrelevant transformation. Finally, Painlevé III has yet another independent Bäcklund transformation. It shifts the parameters $\alpha, \beta$ to nonzero values, and therefore the differential equation does not keep its form of what has been reduced from the Ernst equation. Thus it does not correspond to duality, either.

We conclude this subsection with a remark on how the Geroch group [31] is related to the Painlevé Bäcklund transformations. An affine Lie group symmetry of a two-dimensional reduced nonlinear sigma model is a general phenomenon [32], and in the present system (29), (30) the symmetry is $A_1^{(1)}$, the Geroch group. So the natural question is: how does its Weyl group piece act on the Painlevé equation? The answer is as follows: among two independent Weyl reflections of the Geroch group, one is manifestly realized in the sigma model (29) as $S$-duality; this is also a symmetry of the Painlevé equation, as we have seen above. The other is obtained by conjugating
with the Kramer–Neugebauer (KN) involution $^6$ [34,35]; this Weyl reflection is not the symmetry of the Painlevé equation because the KN involution does not preserve the metric ansatz (36).

4.3. Comments on Painlevé V

The Ernst equation is also known to reduce to the fifth Painlevé equation by using a different ansatz [25]. We again start from Eqs. (35), (36), but this time we allow $f(\rho)$ to take complex values. $\omega$ is a real constant, as before. In this case, Eq. (37) is replaced by

$$\left( f \bar{f} - 1 \right) \left( f'' + \frac{f'}{\rho} - \omega^2 f \right) = 2 \bar{f} \left( f'^2 - \omega^2 f^2 \right).$$

(44)

Multiplying $f$ and subtracting the complex conjugate, we find an integral

$$\frac{\rho (\bar{f} f' - f \bar{f}')}{(1 - f \bar{f})^2} = ia,$n

(45)

where $a$ is a real integration constant. Writing

$$f(\rho) = r(\rho) e^{iu(\rho)}$$

(46)

in terms of two real functions $r(\rho)$ and $u(\rho)$, we may express $u'$ as

$$u' = \frac{a(1 - r^2)^2}{2 \rho r^2}.$$n

(47)

Plugging them into (44), we obtain a second order differential equation of a single variable $r(\rho)$. After a short calculation we find

$$Y'' = \left( \frac{1}{2Y} + \frac{1}{Y - 1} \right) Y'^2 - \frac{Y'}{\rho} - \frac{a^2(Y - 1)^2}{2 \rho^2} \left( \frac{1}{Y} \right) + \frac{2 \omega^2 Y(Y + 1)}{Y - 1}.$$n

(48)

for $Y \equiv r^2$. This is the fifth Painlevé equation with parameters

$$\alpha = -\frac{a^2}{2}, \quad \beta = \frac{a^2}{2}, \quad \gamma = 0, \quad \delta = 2 \omega^2$$

(49)

in the standard notation.

The S-duality transformation $\tau \mapsto -1/\tau$ acts on $r$ as $r \mapsto -r$, which leaves $Y$ invariant. Therefore, it does not correspond to any of the Bäcklund transformations of Painlevé V, but is trivially realized.

5. Conclusions and discussions

In this Letter, we studied the Weyl group symmetries from the point of view of Seiberg–Witten theory, the elliptic Painlevé equation and duality symmetry of M/string theory. The results are summarized as follows:

- The configurations of 9 points on $\mathbb{P}^2$ and the cubic curve passing through them play a fundamental role both in the Seiberg–Witten theory for $E_8$-strings and in the elliptic Painlevé equations. The affine Weyl group symmetry $W(E_8^{(1)})$ of these theories is explained by the geometry of the configurations.

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$^6$ The image of (the Euclidean analogue of) (31) under the KN involution is nothing but the four-dimensional piece of the dual M-theory metric [33].
We have given a simple formulation of the elliptic Painlevé equation in which the hidden $W(E_{10})$ symmetry is manifestly realized. The Seiberg–Witten geometry appears as a special case of this, where the solutions reduce to the elliptic functions.

We have studied some Painlevé differential equations arising from dimensionally reduced equations of motion of strings. In some special case, the Bäcklund transformation of the Painlevé equation can be identified with a duality symmetry of M/string theory.

A property of the singularity confinement is proposed as a discrete analog of the Painlevé property [36]. The singularity confinement demands that a singularity depending on the initial data disappears after finite iteration of the mapping and the memory of initial data is recovered. Of course, the $E_{10}$ Painlevé equation has this property. On the other hand, in [21,37] it is argued that in M-theory, the apparent cosmological singularities can be resolved by the duality transformations. This phenomenon may be considered as the Painlevé property. It should be also noted that the hyperbolicity of $W(E_{10})$ is crucial for the chaotic behavior of the cosmological singularity in M-theory [38]. In view of this and the symmetry structure, it is natural to guess that the elliptic $E_{10}$ Painlevé equation, which is chaotic but integrable in some sense, may play some role in certain effective dynamics of M-theory on $T^{10}$.

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References

A new class of matrix models arising from the $W_{1^{+\infty}}$ algebra

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Abstract

We present a new class of Hermitian one-matrix models originated in the $W_{1^{+\infty}}$ algebra: more precisely, the polynomials defining the $W_{1^{+\infty}}$ generators in their fermionic bilinear form are shown to expand the orthogonal basis of a class of random Hermitian matrix models. The corresponding potentials are given, and the thermodynamic limit interpreted in terms of a simple plasma picture. The new matrix models can be successfully applied to the full bosonization of interesting one-dimensional systems, including all the perturbative orders in the inverse size of the system. As a simple application, we present the all-order bosonization of the free fermionic field on the one-dimensional lattice.

The study of one-dimensional fermionic systems has recently attracted much attention due to progress in manufacturing of mesoscopic devices. Although much progress has been made in the theoretical understanding of the dynamics of these strongly correlated systems, many interesting questions remain still unanswered. One theoretical tool that is successfully used is the bosonization of the fermionic degrees of freedom [1]: for a given fermionic system in the thermodynamic limit, one considers the low-lying particle–hole excitations around the Fermi points. Linearization of the dispersion relation around these leads to a relativistic conformal field theory [2] of these gapless effective degrees of freedom. More recently, an extended bosonization procedure incorporating the subsequent non-linear corrections of the actual dispersion relation has been proposed [3,4]. The algebraic bosonization technique is an all-order bosonization approach based on the $W_{1^{+\infty}}$ algebra, a characteristic fermionic symmetry [5] extending the conformal symmetry of the standard bosonization. The main advantage of this method is that it yields a systematic expansion of the full Hamiltonian (including interactions) and all physical observables in powers of the momentum fluctuations around the Fermi points, involving algebraic procedures only.

In this Letter, we present a one-parameter family of Hermitian one-matrix models which originate in the $W_{1^{+\infty}}$ algebra and are intimately connected to the algebraic bosonization of one-dimensional fermionic systems. Consider the Fock space of a $(1+1)$-dimensional relativistic Weyl fermionic field with creation and annihilation operators $a_n^\dagger$ and $a_n$, respectively ($n$ is an integer that labels the momentum), satisfying $[a_n^\dagger, a_m] = \delta_{n,m}$. We define the $W_{1^{+\infty}}$ operators

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\( V_n^k (k = 0, 1, 2, \ldots) \) in this specific basis [6,7]:

\[
V_n^k = \sum_{r \in \mathbb{Z}} P_n^k (r) a_{r-n} a_r,
\]

which satisfy the \( W_{1+\infty} \) algebra [5].

\[
\begin{align*}
[V_n^i, V_m^j] &= (jn - im)V_{n+m}^{i+j-1} + q(i, j, n, m) V_{n+m}^{i+j-3} \\
& \quad + \cdots + \delta^{ij} b_{n+m,0} c d(i, n),
\end{align*}
\]

where the structure constants \( q(i, j, n, m) \) and \( d(i, n) \)
are polynomial in their arguments, \( c \) is the central charge (\( c = 1 \) for the single Weyl fermion), and the dots stand for a finite number of terms involving the operators \( V_{n+m}^{i+j-1-2k} \) with \( k = 0, 1, \ldots, |i + j - 1)/2 | \). Where the brackets denote the integer part function (the complete expression of (2) is given in [6]). The meaning of the operators \( V_n^k \) is as follows: they parametrize particle-hole excitations near the Fermi points, such that the index \( n \) accounts for the fixed momentum transfer, whereas the index \( k \) denotes the geometric moments (i.e., “multipole moments”) of the deformation of the Fermi–Dirac sea [3,4,7].

The explicit expression of the polynomials \( P_n^k (x) \) can be inferred directly from the \( W_{1+\infty} \) algebra [8]:

\[
P_n^k (x) = \frac{(-1)^k}{(2k)} \sum_{j=0}^{n} (-1)^j \binom{n}{j}^2 (n+1-x)_{k-j} (x),
\]

where \( (a)_n = \prod_{\ell=1}^{n} (a + \ell) \) denotes the Pochhammer symbol. Some properties can be deduced from (3): (i) the polynomials have a definite parity with respect to the reflection about \( x = 1/2 \): even \( n \) (respectively, odd) correspond to even (odd) polynomials; (ii) for \( n \) large enough, the zeroes of \( P_n^k (x) \) are located on the line \( x = 1/2 + i \xi \), with \( \xi \) real, and for pure imaginary values of \( n \), all zeroes fall along this line. For \( k \) even, all zeroes come in complex conjugate pairs, and for \( k \) odd the same is true except for a single zero at \( x = 1/2 \). However, the most important consequence of (3) is that the \( P_n^k (x) \) satisfy a three-term recurrence relation:

\[
x P_n^k (x) = P_{n+1}^k (x) + \frac{1}{2} P_n^k (x)
\]

\[
+ \frac{k^2 (n^2 - k^2)}{4(4k^2 - 1)} P_{n-1}^k (x).
\]

This property has the important consequence of leading to a new class of random matrix models associated to the \( W_{1+\infty} \) algebra, as we shall show in the following.

Matrix models constitute another important tool when studying systems displaying universal behavior in the thermodynamic limit [9]. There are many known examples in which the symmetries that characterize the relevant effective degrees of freedom of a given system are simply encoded in a specific random matrix model, which possess the advantage of being a simpler \((0 + 1)\)-dimensional theory. Consider a generic random one-matrix model, defined by its partition function:

\[
Z(N) = \int d^{N^2} \Phi \exp \left[ - \text{Tr} V(\Phi) \right],
\]

where \( \Phi \) is an \( N \times N \) Hermitian matrix, \( \text{Tr} \) is the trace, and the real function \( V(\Phi) \) is the potential. Eventually, we shall be interested in the limit \( N \to \infty \), which is the relevant thermodynamic limit. The matrix \( \Phi \) has real eigenvalues \( \xi_i, i = 1, \ldots, N, \) in terms of which the partition function can be written as [9]:

\[
Z(N) = \Omega(N) \int \cdots \int d_{i=1}^{N} dx_i \Delta_N^2(x) \times \exp \left[ - \sum_{i=1}^{N} V(\xi_i) \right],
\]

where \( \Delta_N^2(x) = \prod_{i < j}(x_j - x_i) \) and \( \Omega(N) = (2\pi)^{N(N-1)/2} / \prod_{k=1}^{N} k! \) is the volume of the “angular” (non-diagonal) entries of \( \Phi \) [9]. A general strategy to compute \( Z(N) \) is to factorize the \( N \) integrals on the r.h.s. of (6), which can be done provided there exists a set of orthogonal polynomials \( P_k(x) \) satisfying:

\[
\int_{-\infty}^{\infty} dx e^{-V(x)} P_k(x) P_l(x) = h_k \delta_{k,l}.
\]

In turn, this condition can be met provided the polynomials \( P_k(x) \) satisfy a three-term recurrence relation given by:

\[
x P_k(x) = P_{k+1}(x) + S_k P_k(x) + R_k P_{k-1}(x),
\]

where \( S_k \) and \( R_k \) are specific to the matrix model [9]. A crucial equality that relates the potential with the recursion relation is \( R_k = h_k / h_{k-1}, \) and the partition
function is given by:

$$Z(N) = \Omega(N) N! \prod_{k=0}^{N-1} \hbar_k$$

$$= \Omega(N) N! h_0^N \prod_{k=0}^{N-1} R_k^{N-k}. \quad (9)$$

In the following, we shall focus in the set of polynomials (3), and show that they are orthogonal with respect to an specific measure which is then used to define the associated matrix model. One anticipates, therefore, an infinite family of models parametrized by the momentum $n$, and defined by the family of potentials $V_n(x)$. Given the location of the zeroes of the polynomials (3), it is natural to perform a “Wick rotation” in $x$ and define $\mathcal{P}_n^k(x) = i^k \mathcal{P}_n^k(-ix + (n + 1)/2)$ as the set of $W_{1+\infty}$ orthogonal polynomials. They possess a definite parity under reflection in $x$; the first few are:

$$\mathcal{P}_0^0(x) = 1, \quad \mathcal{P}_1^1(x) = x,$$

$$\mathcal{P}_2^2(x) = x^2 + (n^2 - 1)/12,$$

$$\mathcal{P}_3^3(x) = x^3 + x(3n^2 - 7)/20.$$  

The recursion relation (4) now takes the form:

$$x \mathcal{P}_n^k(x) = \mathcal{P}_n^{k+1}(x) + \frac{k^2(k^2 - n^2)}{4(k^2 - 1)} \mathcal{P}_n^{k-1}(x), \quad (10)$$

which implies the existence of the set of potentials $V_n(x)$ making them orthogonal. These functions are readily found once one recognizes that the polynomials $\mathcal{P}_n^k(x)$ are Hahn polynomials, for which the orthogonality relations are known [10]. There are, in fact, two classes of Hahn polynomials: discrete and continuous. The first case arises for $n$ integer, such that there is a finite set of polynomials, since the recursion relation (10) stops at $k = n$. However, given (1), we are interested in the case for which the family is infinite: this is the continuous case, that arises for $n$ not an integer. For our purposes, it is convenient to consider the values $n = ip, \ p = 0, 1, 2, \ldots$ (i.e., imaginary momentum) since this yields a positive and non-degenerate inner product. This condition can be viewed as the Wick rotation in momentum space that corresponds to the one previously performed in $x$. Under these conditions, the orthogonality relations are given by [10]:

$$\int_{-\infty}^{\infty} dx \ g_p(x) \mathcal{P}_n^k(x) \mathcal{P}_l^l(x) = \frac{[\Gamma(k + 1 + ip)]^2}{(2k + 1) \left(\frac{2\pi}{k^2}\right)^2} \delta_{k,l},$$

$$g_p(x) = \frac{\pi}{\cosh(2\pi x) + \cosh(p\pi)}. \quad (11)$$

From this expression, one extracts the family of potentials $V_p(x)$ defining the matrix models as in (5):

$$V_p(x) = -\ln g_p(x) = \ln \left[\frac{\cosh(2\pi x) + \cosh(p\pi)}{\pi}\right],$$

$$p = 0, 1, 2, \ldots, \quad (12)$$

which asymptotically behave as $V_p(x) \simeq 2\pi |x| - 2\ln(2\pi)$ for $|x| > \gamma_p/\pi$, $\gamma_p = \ln[2\pi \cosh(p\pi)]$ (see Fig. 1). From (11), one also knows that the norm of the $k$th polynomial in the $p$th matrix model is:

$$\hbar_p^k = \frac{[\Gamma(k + 1 + ip)]^2}{(2k + 1) \left(\frac{2\pi}{k^2}\right)^2}, \quad (13)$$

which is in agreement with the recursion relation (10) through the equality

$$R_p^k = \frac{\hbar_p^k}{\hbar_p^{k-1}} = \frac{k^2(k^2 + p^2)}{4(k^2 - 1)}.$$

From (13), one can therefore define the partition function $Z_p(N)$ and free energy density $f_p(N)$ for the $p$th model as:

$$Z_p(N) = \Omega(N) N! \prod_{k=0}^{N-1} \hbar_p^k = \exp(N^2 f_p(N)). \quad (14)$$
The asymptotic form of \( f_p(N) \) in the thermodynamic limit \( N \to \infty \) is:

\[
f_p(N) = \frac{1}{2} \ln N - \left( \frac{3}{4} + \frac{1}{2} \ln \frac{8}{\pi} \right) + O(\ln N/N),
\]

independent of \( p \) to the given order.

The thermodynamic limit leads also to a simple physical picture in terms of a fictitious one-dimensional plasma: consider the partition function\( \langle ... \rangle/N \), where \( f_p(N) \) is linear, and suggests the definition of a new scaling variable \( y = x/N \), appropriate in the limit \( N \to \infty \). This implies that \( V_0(x) \simeq 2 \pi N|y| - 2 \ln(2\pi) \) for \(|y| > 1/\Lambda \), with \( \Lambda = \pi N/r_0 \). In the scaling limit, \( y \) is defined in the entire real axis, except for a small region of the size of the cutoff \( 1/\Lambda \) around the origin. We can write the partition function approximately as:

\[
Z_0(N) \simeq \frac{\Omega(N)}{N^N} \left( \prod_{i=1}^{N} dy_i \right) \times \exp \left[ -N \left( 2\pi \sum_{i=1}^{N} |y_i| - \frac{1}{N} \sum_{i<j} \ln(y_i - y_j)^2 + f(N) \right) \right],
\]

where \( f(N) = (N - 1) \ln N + 2 \ln(2\pi) \). In the large \( N \) limit, \( Z_0(N) \) is dominated by the saddle-point configurations \( \{y_1, \ldots, y_N\} \) satisfying the equations:

\[
2\pi \text{sgn}(y_i) = \sum_{j \neq i} \frac{1}{y_i - y_j}, \quad i = 1, \ldots, N,
\]

where \( \text{sgn}(y) \) is the sign function. The scaled variable \( y \) is appropriate for taking the continuum limit: \( y_i \to y_i/N, \sum_i(...) \to \int dy \rho(y) (...) \), where \( \rho(y) \) is the density function. The saddle-point equations are:

\[
\pi \text{sgn}(y) = \mathcal{P} \int_{-\infty}^{+\infty} \frac{\rho(y')}{y - y'} dy',
\]

\[
\int_{-\infty}^{+\infty} \rho(y') dy' = 1,
\]

where \( \mathcal{P} \) denotes the Cauchy principal part of the integral, and the second equality is the normalization condition. These equations give a semiclassical physical interpretation to our system: they describe a system of fictitious equally charged "particles", located on a one-dimensional Coulomb force, while they are attracted with constant force to an infinite two-dimensional uniformly charged plane, that is perpendicular to the rod. One can find a solution to Eqs. (19) by assuming that the system is symmetric under \( y \to -y \), and therefore the total number of "particles" is even, with \( N/2 \) of them on each side of the infinite charged plane. Therefore, one seeks for a solution of the form \( \rho(y) = \rho_+(y)E(y) + \rho_-(y)E(-y) \), with \( E(y) \) the Heaviside function. In the following we shall focus on one side only, e.g., \((+)\). As it is customary, one introduces the function

\[
F_+(y) = \int_{-\infty}^{\infty} \frac{\rho(y')}{y - y'} dy',
\]

defined on the complex half-plane \( \text{Re}(y) > 0 \). Assuming that this function has one cut running along the real axis from \( 0 \) to \( \alpha \), one can solve (19) by a standard procedure [9], yielding:

\[
F_+(y) = \pi - \pi \sqrt{\frac{\alpha - y}{y}}.
\]

From (21) one determines the density:

\[
\rho(y) = \sqrt{\frac{\alpha - y}{y}},
\]

where \( \alpha = 1/\pi \) is determined from the normalization condition (19). One can verify that along the cut and close to its extreme \( y \simeq \alpha \), the behavior of \( \rho_+(y) \) is in agreement with the expected one for a linear background potential [9].

Let us now focus on the application of the previous results to one-dimensional condensed matter models. Consider a free non-relativistic fermionic field on a one-dimensional lattice of unit spacing. We shall outline the effective field theory formulation of this problem in terms of the \( W_{1+\infty} \) symmetry generators [4]. While this description does not appear as necessary in the free theory, it becomes rather advantageous when
including the effects of the interactions, as, for example, in the XXZ Heisenberg model. We shall postpone the full discussion of the effects of the interactions for a forthcoming publication, but remark here that the scope of our approach is to develop a bosonization scheme valid to all orders in the expansion parameter, which is the inverse size of the system, given by \( N \).

In the thermodynamic limit, the non-relativistic field can be expanded in the basis of the relativistic Weyl field, and the Fermi sea interpreted as a Dirac sea [4]. Moreover, the free Hamiltonian \( \mathcal{H} \) can be written as the sum of two independent terms, \( \mathcal{H}_+ \) and \( \mathcal{H}_- \), describing the fluctuations around the right (\(+\)) and left (\(\sim\)) Fermi points at momenta \( \pm n_F \), respectively, with \( n_F = (N - 2)/4 \); i.e., we consider the ground state of the system to be at half-filling, for simplicity. Both terms are decoupled, up to global (zero-mode) constraints. Therefore, we shall consider \( \mathcal{H}_+ \) only in the following:

\[
\mathcal{H}_+ = - \sum_{r = -\infty}^{+\infty} f(n_F + r) a^\dagger_r a_r, \tag{23}
\]

where \( a_r \) are the fermionic Fock operators of the Weyl field, obtained from the original ones by shifting the momentum index, and \( r \) is integer. The dispersion relation around the Fermi point given by \( f(n_F + r) \) is non-linear, and can be expanded in terms of the zero-momentum \( W_{1+} \) polynomials (3) as [4]:

\[
\mathcal{H}_+ = \sum_{k = 0}^{+\infty} c_k V_0^k. \tag{24}
\]

where the coefficients \( c_k \) are such that:

\[
\sin \left( \frac{2\pi}{N} \left( r - \frac{1}{2} \right) \right) = \sum_{k = 0}^{+\infty} c_k P_0^k (r). \tag{25}
\]

Making the change of variables \( r \rightarrow (-ir + 1/2) \) and passing to the basis \( P_0^k (r) \) allows us to determine \( c_k \) using the orthogonality relation (11):

\[
c_k = \frac{i^k}{h_0^k} \int_{-\infty}^{+\infty} dx g_0(x) \sin \left( \frac{2\pi}{N} x \right) P_0^k (x), \quad k = 0, 1, 2, \ldots, \tag{26}
\]

where for convenience we have made the redefinition \( 2\pi/N \rightarrow i2\pi/N \), so as to deal with regular expressions; this analytic continuation should be undone at the end of the computations. From (26) one immediately concludes that \( c_{2n} = 0 \) for \( n \) integer, and defines \( C_k = c_{2k+1} (k = 0, 1, 2, \ldots) \) for the other cases, which can be worked out by considering the set of integrals:

\[
I_n(y) = \int_{-\infty}^{+\infty} dx g_0(x) e^{ixy} P_0^n (x),
\]

\[
n = 0, 1, 2, \ldots. \tag{27}
\]

As a consequence of (10), they satisfy the recursion relations:

\[
I_{n+1}(y) = -i \frac{dI_n(y)}{dy} - R_0^n I_{n-1}(y),
\]

\[
I_0(y) = \frac{y}{2 \sinh(y/2)}, \tag{28}
\]

which are formally solved by \( I_n(y) = P_0^n (-i \frac{dy}{\pi}) I_0(y) \). The coefficients (26) are then readily determined yielding:

\[
C_k = \frac{(-1)^k}{h_0^{k+1}} I_{2k+1} \left( \frac{2\pi}{N} \right), \quad k = 0, 1, 2, \ldots. \tag{29}
\]

As and example, after analytically continuing back \( \pi/N \rightarrow -i\pi/N \) as explained above,

\[
C_1 = \frac{6}{\sin(\pi/N)} \left[ 1 - \frac{\pi}{N \tan(\pi/N)} \right]
\]

\[
= \frac{2\pi}{N} + \frac{7}{120} \left( \frac{2\pi}{N} \right)^3 + O \left( \frac{2\pi}{N} \right)^5, \tag{30}
\]

which reproduces the results of [4]. We have also verified that the leading term in \( C_2 = -(2\pi/N)^3/6 + \cdots \) yields the correct result.

Finally, we remark that the matrix models considered in this letter belong to the class of models with non-polynomial potentials, which include also the Penner [12] and Frustrated Spherical [13] models. According to the 't Hooft correspondence, one can associate to each matrix model a tiling of a two-dimensional random surface, such that the number of sides and the ways of gluing the elementary polygons are determined by the algebraic powers of the monomials that define the potential [9,11]. The class of matrix models with non-polynomial potentials possess the characteristic property of having an unbounded set
of polygons and gluing vertices among them. We conclude by remarking that it would be interesting to explore whether other infinite-dimensional Lie algebras could give rise to further new classes of random matrix models.

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Branes in special holonomy backgrounds

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Abstract

We study the flow from the theory of D2-branes in a $G_2$ holonomy background to M2-branes in a Spin$(7)$ holonomy background. We consider in detail the UV and IR regimes, and the effect of topology change of the background on the field theory. We conjecture a non-Abelian $\mathcal{N} = 1$ mirror symmetry.

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1. Introduction

The study of string/M-theory backgrounds with special holonomy groups is receiving much attention lately. One application of special holonomy manifolds is in constructing gravity duals of field theories with reduced supersymmetry, and the analysis of the field theories on brane probes in such backgrounds.

In this Letter we will consider two-brane probes in Spin$(7)$ and $G_2$ holonomy backgrounds. Specifically, we will concentrate on two special manifolds with Spin$(7)$ holonomy that were constructed in [1]; the $A_8$ and $B_8$ manifolds. These non-compact manifolds have two distinct features. While they have different topologies, they admit the same metric up to a sign of a parameter. Asymptotically, this metric is the same in both cases, and is locally conical.

We will analyze the dynamics of a system of M2-branes in these Spin$(7)$ holonomy manifolds. We will describe the flow from the theory of D2-branes on a $G_2$ holonomy background to M2-branes in these Spin$(7)$ holonomy backgrounds. We will consider in detail the UV and IR regimes, and the effect of topology change (from $A_8$ to $B_8$) on the field theory.

The Letter is organized as follows. In the next section we will briefly review the $A_8$ and $B_8$ backgrounds. In Section 3 we will consider M2-branes in these backgrounds. We will take the field theory limit, study the phase structure of the systems, and compute the two-point function of the stress-energy tensor. In Section 4 we will construct the UV field theory and conjecture a non-Abelian $\mathcal{N} = 1$ mirror symmetry. In Section 5 we will discuss some aspects of the IR regime.

2. Spin$(7)$ holonomy manifolds: $A_8$ and $B_8$

Consider the following general ansatz for an eight-dimensional metric

$$ds^2 = h^2(r)dr^2 + a^2(r)\sigma^2 + b^2(r)(D\mu)^2 + c^2(r)d\Omega_4^2.$$  \hspace{2cm} (2.1)
We denote by \((\psi, \theta, \phi)\) the Euler angles, and \(d\Omega_4^2\) is the metric on the unit 4-sphere. Then \(\sigma\) is given by
\[
\sigma \equiv d\phi + A_1 = d\phi + \cos \theta d\psi - \mu_i A_i, \tag{2.2}
\]
where
\[
\begin{align*}
\mu_1 &= \sin \theta \sin \psi, \\
\mu_2 &= \sin \theta \cos \psi, \\
\mu_3 &= \cos \theta.
\end{align*}
\tag{2.3}
\]
and \(A_i\) is the gauge connection of an \(SU(2)\) instanton on \(S^4\). We define the covariant derivative, and instanton field strength as
\[
D\mu_i = d\mu_i + \epsilon_{ijk} A_j \mu_k, \\
F_i = dA_i + \epsilon_{ijk} A_j \wedge A_k. \tag{2.4}
\]

In the following we will consider a special case of the general metric ansatz that has \(Spin(7)\) holonomy [1] (which is a special case of a family of \(Spin(n)\) metrics [1,2])
\[
ds_8^2 = \frac{(r + l)^2 dr^2}{(r + 3l)(r - l)} + \frac{l^2(r + 3l)(r - l)\sigma^2}{(r + l)^2} + \frac{1}{4}(r + 3l)(r - l)(D\mu_i)^2 + \frac{1}{2}(r^2 - l^2) d\Omega_4^2. \tag{2.5}
\]

\(l\) is a real parameter whose significance will be discussed shortly. When \(l > 0\) the radial coordinate \(r\) is in the range \(l \leq r < \infty\). This manifold is denoted as \(A_8\). When \(l < 0\) the radial coordinate is in the range \(-3l \leq r < \infty\). This manifold is denoted as \(B_8\).

In the large \(r\) region, both \(A_8\) and \(B_8\) have the same asymptotic form \(\mathcal{M} \times S^1\), where \(S^1\) is parameterized by \(\sigma\) with \(g_{\sigma\sigma} \rightarrow l^2\), and \(\mathcal{M}\) is a cone over \(CP^3\) with a \(G_2\) holonomy metric [3]
\[
ds_8^2 = dr^2 + r^2 \left(\frac{1}{4}(D\mu_i)^2 + \frac{1}{2} d\Omega_4^2\right). \tag{2.6}
\]

If we dimensionally reduce the asymptotic \(\mathcal{M} \times S^1\) geometry along the \(S^1\) direction, we get the seven-dimensional background (2.6) and a 2-form KK field strength
\[
\mathcal{F}_2 \equiv dA_1 = \frac{1}{2} \epsilon_{ijk} k^k D\mu^i \wedge D\mu^j - \mu_i F_i. \tag{2.7}
\]

The dilaton is asymptotically constant. The geometric symmetries of this background were analyzed in [4]. The base of the cone \(\mathcal{M}\) is the coset space
\[
\frac{Sp(2)}{SU(2) \times U(1)}. \tag{2.8}
\]

Therefore, it is invariant under \(Sp(2)\) action from the left, and \(SU(2) \times U(1) \times Z_2\) from the right. The \(SU(2) \times U(1)\) acts trivially, and \(Z_2\) acts as \(\mu_i \rightarrow -\mu_i\). These symmetries extend to the cone \(\mathcal{M}\). These are not the symmetries of \(\mathcal{F}_2\). It is clear that \(\mathcal{F}_2\) is not invariant under the \(Z_2\) action. An \(SU(2)\) instanton solution on \(S^4\) has an \(SU(2) \times U(1)\) symmetry group, and there is an additional \(SO(3) \approx SU(2)\) symmetry from acting on the \(i\) index. Therefore, the type IIA background is only invariant under \(SU(2) \times SU(2) \times U(1)\) transformations.

The first correction to the asymptotic geometry in the \(l/r\) expansion is linear in \(l\) and distinguishes between the two manifolds
\[
\delta ds^2 = \pm \frac{1}{2} l |r (D\mu_i)|^2, \tag{2.9}
\]
where the ‘+’ sign is for \(A_8\), and the ‘−’ sign for \(B_8\). The correction (2.9) is non-normalizable, \(\|\delta g\| \rightarrow \infty\), which implies that \(l\) is a parameter in the dual field theory. The geometry of the two manifolds in the small \(r\) region is quite different. Near \(r = l\) the \(A_8\) metric reads
\[
ds_8^2 = d\rho^2 + \frac{1}{4} \rho^2 (\sigma^2 + (D\mu_i)^2 + d\Omega_4^2) = d\rho^2 + \rho^2 d\Omega_4^2, \tag{2.10}
\]
where \(\rho^2 = 4l(r - l)\) and \(d\Omega_4^2\) is the metric on the unit 7-sphere. Thus, \(A_8\) is topologically \(R^8\), and near \(\rho = 0\) also geometrically. Near \(r = -3l\) the \(B_8\) metric reads
\[
ds_8^2 = d\rho^2 + \rho^2 d\Omega_4^2 + 4l^2 d\Omega_4^2, \tag{2.11}
\]
where \(\rho^2 = -4l(r + 3l)\), and \(d\Omega_4^2\) and \(d\Omega_4^2\) are the metrics on the unit 3-sphere and 4-sphere, respectively. Thus, the \(B_8\) manifold is topologically an \(R^4\) bundle over \(S^4\). We see that as we vary the parameter \(l\) from positive to negative values we change the topology of the manifold.

3. M2-branes in \(A_8\) and \(B_8\)

In the following we will consider M2-branes in the \(A_8\) and \(B_8\) backgrounds. We will first analyze the phase diagram of the system, as done for D2/M2-branes in flat space in [5]. The background of \(N\) M2-branes takes the usual form [1]
\[
ds^2 = H^{-2/3} d\chi_{012}^2 + H^{1/3} ds_8^2, \tag{2.12}
\]
\[ C_3 = dx_0 \wedge dx_1 \wedge dx_2 \quad H^{-1}, \] (3.1)

where \( H(r) \) obeys the Laplace equation on \( A_8 \). In the \( B_8 \) case, the asymptotics is the same, but near the origin there is a subtlety, which we will address later. \( H(r) \) reads

\[ H = 1 + Q \int_0^\infty \frac{dx}{r} \frac{d(x-l)^3(x+3l)}{x^5} \quad (3.2) \]

where \( Q \approx l_p^6 N/1 \).

We consider the low-energy limit \( I_p \to 0 \), with \( r \) and \( l \) also taken to zero such that

\[ U \equiv \frac{lr}{l_p^2}, \quad LN \equiv \frac{l^2}{l_p^2}N, \quad (3.3) \]

are kept fixed. The ten-dimensional string coupling is

\[ g_s = \left( \frac{l}{l_p} \right)^{2/3} = \frac{l}{l_s}, \quad (3.4) \]

and the three-dimensional gauge coupling is \( g_{YM}^2 \sim g_s l_s^{-1} \). In terms of these parameters, (3.3) reads

\[ U = \frac{r}{l_s^2}, \quad LN = \frac{l}{l_s^2}N, \quad (3.5) \]

which are fixed as \( l_s \to 0 \). Note that \( L \sim g_{YM}^2 \).

Consider the phase diagrams for two-branes in the \( A_8 \) and \( B_8 \) backgrounds. On the supergravity (string) side there are two dimensionless expansion parameters: the effective string coupling \( e^\phi \) and the curvature in string units \( l_s^2 \mathcal{R} \). On the field theory side, the expansion in string coupling and the curvature expansion correspond to the \( 1/N \) expansion and strong coupling expansion in \( (g_{YM}^2 N E^{-1})^{-1/2} \), respectively, with \( E \) being an energy scale. On the supergravity side there is an additional dimensionless parameter \( L/U \). On the field theory side it corresponds to \( g_s^2 E^{-1} \).

### 3.1. The \( A_8 \) case

For large \( U \), it can be seen from (3.2) that the radius of the M-theory circle parametrized by \( \sigma \) vanishes, and we need to dimensionally reduce to the type IIA description. The asymptotic form of \( H \) is given by

\[ H = \frac{g_{YM}^2 N}{l_s^4 U^3} \left( \frac{1}{5} - \frac{L}{3U} + \frac{13L^2}{7U^2} + O \left( \frac{L^3}{U^3} \right) \right). \quad (3.6) \]

The asymptotic (large \( U \)) background is

\[ ds^2 = l_s^2 \left( \frac{U^{5/2}}{\sqrt{g_{YM}^2 N}} d\sigma^2 + \frac{\sqrt{g_{YM}^2 N}}{U^{5/2}} ds^2 \right), \]

\[ e^{2\phi} = \sqrt{\frac{g_{YM}^2 N}{U^5}}, \]

\[ F_2 = \frac{1}{2} \epsilon_{ijk} \mu^i D \mu^j + D \mu^k - \mu^i F_i, \]

\[ F_4 = dx_0 \wedge dx_1 \wedge dx_2 \wedge dH^{-1}. \quad (3.7) \]

This background is the near-horizon limit of \( N \) D2-branes transverse to the \( G_2 \) holonomy background \((2.6)\), with a 2-form KK field strength.

We define the dimensionless expansion parameter \( g_{eff}^2 = g_{YM}^2 N U^{-1} \). The curvature \( l_s^2 \mathcal{R} \) and the effective string coupling of the background (3.7) read

\[ l_s^2 \mathcal{R} \sim \frac{1}{g_{eff}} = \sqrt{\frac{U}{g_{YM}^2 N}}, \]

\[ e^\phi \sim \frac{g_{eff}^{5/2}}{N}. \quad (3.8) \]

In this notation \( L/U \sim g_{eff}^{-1} \).

To leading order in \( L/U \) the phase diagram is the standard one \([5]\). When \( g_{eff} \ll 1 \) we can use the perturbative field theory description. This is the high energy regime and the field theory has \( N = 1 \) supersymmetry. The transition to the type IIA supergravity description is at \( g_{eff} \sim 1 \). This occurs at \( U \sim g_{YM}^2 N \). We can use the type IIA description as long as the effective string coupling is small \( e^\phi \ll 1 \). The metric describes a system of \( N \) D2-branes transverse to a \( G_2 \) holonomy manifold. When \( e^\phi \sim 1 \) we have to use the eleven-dimensional description. This occurs at energy \( U \sim g_{YM}^2 N^{1/5} \). In the extreme IR the uplifted metric approaches the \( AdS_4 \times S^7 \) metric and we have \( N = 8 \) SCFT description.

This picture is not precise when the corrections in \( L/U \) are taken into account. The leading correction to the asymptotic geometry in \( L/U \) is non-normalizable, \( || \delta g || \to \infty \). This means that in the dual field theory description \( L \) should be understood as a parameter rather than as a vev of a field. This is indeed the case, \( L \sim g_{YM}^2 \). Near \( U = L \) the harmonic function
has the following expansion in \((U - L)/L\)

\[
H = \frac{N}{l_p^2 U^3} \left( 1 - \frac{3}{4} \frac{U}{L} + \frac{9}{16} \frac{U^2}{L^2} + O \left( \frac{U}{L} \right)^3 \right), \quad (3.9)
\]

where we redefined \(U - L\) as \(U\). At \(U = 0\) the metric (3.1) becomes \(AdS_4 \times S^7\), with \(R_{\text{sphere}} = 2R_{\text{AdS}} \sim l_p N^{1/6}\). This is the same background as the near horizon limit of M2-branes in flat space with corrections in \(U/L\). Note, however, that the expansion in \(U/L\) is not really useful since it requires \(U \ll L\) in order to be valid.

As we noted before, changing the sign of \(L\) (i.e., taking the field theory limit) implies a topology change in the supergravity background. On the field theory side it corresponds to \(g_{\text{SYM}} \rightarrow -g_{\text{SYM}}\), which is a sign of a phase transition. In particular, the singularity of the supergravity background at \(L \rightarrow 0\) corresponds to a free field theory \(g_{\text{SYM}} \rightarrow 0\). Note in comparison that M-theory on \(G_2\) holonomy background yields a theory with four supercharges where holomorphy can be used to argue for smooth interpolation between different geometrical backgrounds [4]. Here we consider theories with two supercharges where holomorphy cannot be used, and indeed the process of moving from one background to the other is not smooth.

3.2. The \(B_8\) case

Consider the phase diagram of M2-branes on the \(B_8\). The asymptotic form of \(B_8\) is the same as that of \(A_8\) to leading order in the \(L/U\) expansion. Thus, we expect the supergravity approximation to hold up to \(U \sim g_{\text{SYM}}^2 N\) as in the \(A_8\) case. The IR is quite different. If we change the sign of \(L\) in (3.2) the point \(U = 3L\) is a singularity. This singularity is of the “good” kind [6] in the sense that the time component of the metric, \(g_{00}\), is decreasing as we approach the singular point.

The harmonic function near \(U = 3L\) reads

\[
H = \frac{N}{l_p^2 L^2 U} \left( 1 - \left( \frac{13}{12} - \ln 4 + \ln \frac{U}{L} \right) + \cdots \right), \quad (3.10)
\]

where we shifted \(U - 3L\) to \(U\). From the type IIA point of view the dilaton is not monotonic. It peaks at some intermediate point and then decreases as we approach the singularity, where it vanishes. Therefore, near the singular point we should use the ten-dimensional description.

The dimensional reduction of the \(B_8\) manifold to ten dimensions describes a D6-brane wrapped on an \(S^4\) coassciative cycle in the seven-dimensional \(G_2\) holonomy manifold (2.6). Adding D2-branes implies that we should describe this regime using the D2–D6 system wrapping \(S^4\). Note that the type IIA background is that of D2-branes smeared on the \(S^4\). This system does not have a decoupling limit. Technically, this is because the power of \(U\) in \(H\) is not the same as in the \(A_8\) case. Following [7], we should use the solution of localized D2-branes, which does have a decoupling limit, in which we also hold fixed \(N_2/N_6\). For small \(r\) the \(B_8\) is geometrically \(R^4 \times S^4\). The volume of the \(S^4\) is proportional to \(N_2/N_6\). Therefore, we expect that for \(N_2 \gg N_6\) we can treat the \(B_8\) manifold as \(R^8\) with good accuracy in this small \(r\) region. This will reproduce the phase diagram of the flat D2–D6 system in [7]. Specifically, we will have a background of the form \(AdS_4 \times X^7\), where \(X^7\) is the \(Z_{N_6}\) orbifold of \(B_8\). We can also reduce this background to ten dimensions, which will result in a fibered AdS background. In Section 5 we will outline some features of the field theory that governs the IR of this D2–D6 system.

3.3. Two-point function

In the following we will use the supergravity description in order to compute the two-point function of the stress-energy tensor. We will be interested, in particular, in the \(L\) dependent corrections. The computation requires the solution of the graviton equation. However, for certain polarization and momentum vector, the graviton equation reduces to the minimally coupled scalar equation. A similar computation in the supergravity background of D2-branes has been done in [8]. The minimally coupled scalar equation is

\[
\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu \nu} \partial_\nu) \phi(U,x) = 0, \quad (3.11)
\]

We take

\[
\phi(U,x) = f(U) \exp(ikx)
\]

and we impose the boundary condition \(f(U_\epsilon) = 1\) at the UV cut-off \(U_\epsilon\). The two-point function is given by

\[
\langle T(k)T(-k) \rangle = \frac{N^2}{k^2} \mathcal{F}_T
\]
\[ f(z) = \frac{N^2}{\lambda^2} f(U) \sqrt{g} g^{UU} \partial_U f(U) \bigg|_{U_0}, \quad (3.12) \]

where \( \lambda = g_{YM}^2 N \) is the 't Hooft parameter, and \( U_0 = L \) for the \( A_8 \) manifold and \( U_0 = 3L \) for the \( B_8 \) manifold. It is convenient to change variables to \( z = (\lambda k^3)^{1/3} U^{-1} \), and rescale \( L \) as \( L \rightarrow (\lambda k^3)^{1/3} L \). Since we will only need to know the small \( z \) behavior of \( f(z) \) we can solve (3.11) order by order. After expanding the coefficients in (3.11) we get

\[ f''(z) + \frac{1}{z} \left( -4 + 2(Lz) - 22(Lz)^2 + \cdots \right) f'(z) \]

\[ + z \left( \frac{1}{5} - \frac{(Lz)}{3} + \frac{93(Lz)^2}{35} + \cdots \right) f(z) = 0. \tag{3.13} \]

The solution of (3.13) after imposing the boundary condition is given by

\[ f(z) = C z^5 \left( 1 - \frac{5}{3} L z + O(z^2) \right) \]

\[ - \frac{4}{105} L^2 z^4 \ln(z) \left( 1 + O(z) \right) + \frac{1 + \frac{1}{30} z^3}{105} + O(z^5), \tag{3.14} \]

where \( C \) is an integration constant. It is fixed by the condition that at \( U = U_0 \) the solution is smooth, \( f'(U_0) = 0 \).

We use (3.12) to calculate the flux factor, and extract the cut-off independent piece

\[ \mathcal{F}_\mathcal{T} = \frac{N^2}{\lambda^2} \left( 5C \lambda^{5/3} k^{10/3} - \frac{8}{63} L^2 k^2 \lambda \ln(k) \right) \]

\[ \quad + \text{analytic in } k. \tag{3.15} \]

In coordinate space the two-point function that follows from (3.12) reads

\[ \langle T(0)T(x) \rangle \sim \frac{N^2}{\lambda^2} \left( \frac{\lambda^{5/3}}{|x|^{19/3}} + \frac{\lambda L^2}{|x|^8} \right). \tag{3.16} \]

The first term in (3.16) has been obtained in \[8\] for the D2-branes in flat space. The second term is the \( L/U \) correction.

At the crossover regime between the supergravity description and the field theory description \( g_{YM}^2 N |x| \ll 1 \) we get

\[ \langle T(0)T(x) \rangle \sim \frac{N^2}{|x|^6}, \tag{3.17} \]

which matches to a field theory with \( N^2 \) degrees of freedom.

Note that in this computation we did not find an odd power of \( L \) dependence that will be different in the \( A_8 \) and \( B_8 \) cases. However, the computation is valid only in the UV regime. As we noted and will also discuss later, the IR dynamics of M2-branes on the two manifolds is quite different.

4. The UV field theory

The field theory in the UV is a three-dimensional \( \mathcal{N} = 1 \) supersymmetric gauge theory. Let us first briefly review some aspects of \( \mathcal{N} = 1 \) supersymmetry in three dimensions. The supersymmetry algebra reads

\[ [Q_\alpha, \bar{Q}_\beta] = 2\gamma_{\alpha\beta}^\mu P_\mu, \quad \mu = 0, 1, 2, \quad \alpha = 1, 2. \] \tag{4.1} \]

\( Q_\alpha \) are two real spinor supercharges, and \( \gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_3, \quad \gamma^2 = i\sigma_1 \). Superspace is described by three space–times coordinates, and by two real anti-commuting variables \( \theta_\alpha \). A scalar superfield \( \Phi \) takes the form

\[ \Phi = \phi + \bar{\theta} \chi + \frac{1}{2} \bar{\theta} \bar{\theta} F, \tag{4.2} \]

where \( \phi \) is a real scalar, \( \chi_\alpha \) is a real spinor and \( F \) is an auxiliary field. A vector multiplet can be written as a real spinor superfield \( V_\alpha \) (in the Wess–Zumino gauge)

\[ V_\alpha = i (\gamma^\mu A_\mu \theta)_\alpha + \frac{1}{2} \bar{\theta} \bar{\theta} \lambda_\alpha. \tag{4.3} \]

The field strength tensor is

\[ F_\alpha = \lambda_\alpha - i (\gamma^\mu B_\mu \theta)_\alpha - \frac{1}{2} \bar{\theta} \bar{\theta} (\gamma^\mu \partial_\mu \lambda)_\alpha. \tag{4.4} \]

where \( B_\mu = \frac{1}{2} \varepsilon_{\mu\nu\lambda} F^{\nu\lambda} \). The supersymmetric action is

\[ S = \int d^3 x d^2 \theta \]

\[ \times \left( -\frac{1}{2} F_\alpha F_\alpha + \frac{1}{2} (D\Phi)(D\Phi) + W(\Phi) \right), \tag{4.5} \]

where \( D_\alpha = D_\alpha - i e V_\alpha \) is the supercovariant derivative. The action (4.5) contains kinetic terms for the scalar, vector and spinor fields, Yukawa couplings and a scalar potential

\[ V = \frac{1}{2} g^{ij} \partial_i W \partial_j W, \tag{4.6} \]
where $g^{ij}$ is the kinetic term metric.

As we discussed in the previous section, at large $r$ the metrics of $A_8$ and $B_8$ have the asymptotic form $\mathcal{M} \times S^1$, where $\mathcal{M}$ is a cone over $CP^3$. Let us look for an $\mathcal{N} = 1$ supersymmetric gauge theory that has this space as its classical moduli space of vacua. Consider a supersymmetric $U(1) \times U(1)$ gauge theory with an $\mathcal{N} = 1$ vector multiplet and four $\mathcal{N} = 2$ chiral (eight $\mathcal{N} = 1$ scalar) superfields $A_i, B_i, i = 1, 2$ with $U(1) \times U(1)$ charges $(1, -1)$ for $A_i$ and $(-1, 1)$ for $B_i$. As reviewed above, there is no D-term for $\mathcal{N} = 1$ supersymmetry in three dimensions. Thus, the classical moduli space of vacua is given by the space parametrized by the scalar components of $A_i, B_i$ modulo the $U(1) \times U(1)$ action.

With the above charges, $A_i$ and $B_i$ are neutral under the diagonal $U(1)$. We define

$$w_1 = a_1, \quad w_2 = a_2, \quad w_3 = b_1^*, \quad w_4 = b_2^*, \quad (4.7)$$

where $a_i, b_i$ are the scalar components of $A_i, B_i$. The classical moduli space of the theory is $\mathcal{M} \times S^1$.

A generalization to $N$ D2-branes requires that as we separate the branes we get $N$ copies of the above moduli space. Consider a $U(N) \times U(N)$ $\mathcal{N} = 1$ vector multiplets with $A_i, B_i, i = 1, 2, \mathcal{N} = 2$ chiral superfields in the $\mathcal{N}, \bar{N})$ representations of the gauge groups, respectively. When $A_i, B_i$ are diagonal with distinct eigenvalues, the gauge group is broken to $U(1)^N$. The scalars together with the duals to the $N$ photons parameterize a 7N-dimensional space, made of $N$ copies of a seven-dimensional space that admits a metric of $G_2$ structure.

It is tempting to conjecture a non-Abelian version of the $\mathcal{N} = 1$ mirror symmetry, and suggest that the non-Abelian theories $A$ and $B$ are mirror in the sense that the Coulomb branch of $A$ matches the Higgs branch of $B$.

5. The IR regime

In this section we discuss some aspects of the IR regime.
5.1. The $A_8$ case

As we saw, in the $A_8$ case there is a single IR fixed point where supersymmetry is enhanced back to $N = 8$. The irrelevant operators that deform the $S^7$ are dual to scalars in $AdS_4$. The question is how many irrelevant operators are turned on, and what are their conformal dimensions and representation under $SO(8)$. Fortunately, some of the work was already carried out in [10], where a more symmetric deformation of $S^7$ was analyzed.

Our starting point will be the one-dimensional Lagrangian, from which one can derive the equations that define the $A_8$ manifold [1].

\[
L = 2\alpha'^2 + 12\gamma'^2 + 4\alpha'\beta' + 8\beta'\gamma' + 16\alpha^2 \gamma' + \frac{1}{2} b^2 c^4 (4a^6 + 2a^4 b^2 - 2a^4 c^2 - 4a^2 e^2 - 4a^2 c^4 + b^2 c^4),
\]

(5.1)

where $a$, $b$, $c$ are as in (2.1), $h(r) = 1$, and $\alpha' = \partial_r \ln(a)$ and the same for $\beta$ and $\gamma$. We shall now modify this Lagrangian to include the $N$ $M$2-branes. Note that the original Lagrangian was derived for an eight-dimensional space, and in order to describe the supergravity solution of the $M$2-branes we should first add three flat directions, and then multiply by the appropriate factors of $H(\rho)$. The potential term $V(a,b,c)$ must be added with the contribution of the 4-form field strength

\[
\Delta V = a^8 b^4 c^{12} Q^2.
\]

(5.2)

We can change to a more familiar set of variables [10], which will make the geometric picture more clear. (For $a = b$ we should get the Lagrangian of [11].) We define:

\[
a = 2^{1/6} \exp\left(-\frac{3}{4} u - 2v + \frac{1}{3} w\right),
\]

\[
b = 2^{1/6} \exp\left(-\frac{3}{4} u - 2v - \frac{2}{3} w\right),
\]

\[
c = 2^{1/6} \exp\left(-\frac{3}{4} u + \frac{3}{2} v\right),
\]

(5.3)

and denote the four-dimensional metric by $g_{ij}$. With the new variables the new Lagrangian is

\[
L = \sqrt{g} \left( R_g - \frac{63}{2} (\partial u)^2 - 21 (\partial v)^2 + \frac{4}{3} (\partial w)^2 - 2Q^2 e^{-2uv} - e^{-9u - 10v} \left( 8e^{2w/3} + 4e^{-4w/3} \right) - 48e^{-9u - 3v} + e^{-9u + 4v} \left( 2e^{-8w/3} - 8e^{-2w/3} \right) \right).
\]

(5.4)

The geometric meaning of each of the three functions in now clear. Let us for the moment set $w = 0$ ($a = b$) identically (not just at $r = 0$). In this case (5.2) is the action that governs deformations of $S^7$ that preserve an $SO(5) \times SO(3)$ isometry [10]. In particular, the potential has two critical points, that correspond to Einstein manifolds. One is the round $S^7$ and the other is the squashed $S^7$. The round $S^7$ is the solution given by

\[
v_0 = w_0 = 0, \quad u_0 = \frac{\ln(Q^2/9)}{12},
\]

(5.5)

and corresponds to an IR fixed point. The squashed $S^7$ solution, that will not be of interest here, has $v \neq 0$, and corresponds to a UV fixed point.

In our case the symmetry group is smaller, since $w \neq 0$ ($a \neq b$) except at $r = 0$. We consider fluctuations of $u, v, w$ around the round $S^7$ solution.

In eleven-dimensional supergravity compactified on $AdS_5 \times S^7$ there are 3 scalar KK towers denoted by $S_1$, $S_2$, $S_3$. Their $SO(8)$ representations and 4-dimensional masses are given by [13]

\[
S_1 \quad (n + 2, 0, 0, 0) \quad m^2 = (n - 2)^2 - 9,
\]

\[
S_2 \quad (n - 2, 0, 0, 0) \quad m^2 = (n + 7)^2 - 9,
\]

\[
S_3 \quad (n - 2, 2, 0, 0) \quad m^2 = (n + 3)^2 - 9.
\]

(5.6)

There are also 2 pseudo-scalar KK towers denoted by $P_1$ and $P_2$

\[
P_1 \quad (n, 0, 2, 0) \quad m^2 = (n + 1)^2 - 9,
\]

\[
P_2 \quad (n - 2, 0, 0, 2) \quad m^2 = (n + 5)^2 - 9.
\]

(5.7)

Following the calculations in [10], the masses of the 3 scalars in (5.4) are given by (in $AdS_4$ mass units)

\[
M_{ij}^2 = \frac{1}{2} \partial^2_{\rho_{ij}} V(u, v, w) \bigg|_{u_0, v_0, w_0},
\]

(5.8)

after rescaling the fields so that the kinetic terms are canonically normalized. Specifically we get that

\[
M_{uu}^2 = M_{vv}^2 = M_{ww}^2 = +16.
\]

(5.9)
By the AdS/CFT correspondence these 3 scalars are dual to irrelevant operators of dimensions $\Delta = 4$ [12]. The first scalar $u$ is just the overall volume of $S^4$, and is identified with $n = 7$ of $S_1$. The second scalar $v$ can be identified as $n = 2$, of $S_1$ transforming as the $300$ of $SO(8)$ [10]. The third scalar $w$ is actually a pseudoscalar $n = 4$, of $P_1$, and is responsible for deforming the $S^3$ fibers. It transforms as the $5775$ of $SO(8)$. With all three irrelevant perturbations turned on there is no second non-trivial fixed point, which means that in the UV the system should be described by perturbative field theory as is indeed the case.

5.2. The $B_8$ case

As discussed before, the $B_8$ manifold reduced to ten dimensions describes the background of a D6-brane wrapped on an $S^4$ supersymmetric 4-cycle ($\text{Vol}(S^4) \sim 4^4$) in a $G_2$ holonomy manifold [1]. The manifold is the bundle of anti-self-dual two-forms over $S^4$. Note that to get $N_6$ D6-branes one should consider a $Z_{N_6}$ orbifold of the $S^4$ on which we reduce.

Let us consider the D-branes field theory when adding D2-branes in this background. At energies $E \ll l^{-1}$ the effective theory on the D6-brane world-volume is a $U(1)$ $N = 1$ supersymmetric gauge theory with a Chern–Simons term [15]

$$L = \frac{1}{4\kappa_5^2} \int d^3 x \left( F^2 + i \bar{\psi} \Gamma \cdot D\psi \right) + \frac{i(k+\frac{1}{2})}{4\pi} \int (A \wedge dA + \bar{\psi} \psi). \quad (5.10)$$

The gauge coupling $\kappa_5^2 \sim g_0^2 l_5^{d-4}$. The Chern–Simons term arises from the WZ term, when taking into account the half integral $G_4$ flux required for the consistency of the compactification on $B_8$ [15]. Since the $S^4$ is rigid there are no massless scalars to parameterize its embedding in the seven-dimensional space.

Adding $N_2$ D2-branes to the system means an additional $N = 1$ $U(N)$ vector multiplet with gauge coupling $g_2^2 \sim g_0^2 l_5^{-1}$. There is one $N = 2$ hypermultiplet that parameterizes the motion of the D2-branes in $S^4$, and three scalar superfields that parameterize the three directions normal to the $S^4$. We need to study the system at low energies $E \ll g_2^2, g_0^2$. At these energies the kinetic terms are irrelevant. Since a D2-brane in D6-branes wrapping $S^4$ can be viewed as an instanton on $S^4$, it is plausible to expect the IR theory will be a conformal field theory on the Higgs branch, which is the moduli space of $N_2$-instantons of $U(N_6)$ on $S^4$. This is consistent with the fact that the supergravity background for small $U$ has an $AdS_4$ factor, as noted in Section 3.2.

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References


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Footnote 1: For a somewhat similar discussion in another context see [16].
Three-graviton amplitude in Berkovits–Vafa–Witten variables

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Abstract

We compute the three-graviton tree amplitude in Type IIB superstring theory compactified to six dimensions using the manifestly (6d) supersymmetric Berkovits–Vafa–Witten worldsheet variables. We consider two cases of background geometry: the flat space example $R^6 \times K3$, and the curved example $AdS_3 \times S^3 \times K3$ with Ramond flux, and compute the correlation functions in the bulk. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

We compute string tree correlation functions using the manifestly supersymmetric covariant formulation of Berkovits–Vafa–Witten (BVW) for Type IIB superstrings compactified to six dimensions [1–6]. Unlike the ten-dimensional covariant pure spinor quantization [7–13], the six-dimensional version [1–3] we use here incorporates an $N = 4$ topological string formulation to compute tree level scattering amplitudes.

For IIB superstrings either on flat six-dimensional space times K3, or on $AdS_3 \times S^3 \times K3$, the massless degrees of freedom correspond to a $D = 6, N = (2, 0)$ supergravity multiplet and 21 tensor multiplets. In this Letter, we consider the string theory tree level scattering of three gravitons both in the flat case and the $AdS_3 \times S^3$ case, where the latter has background Ramond flux. For these amplitudes, the relevant massless compactification independent vertex operator, in the BVW worldsheet formalism, contains the graviton, dilaton and two-form field which contribute to the supergravity and one of the tensor multiplets.

In this formalism [1–3], the Type IIB superstring compactified on K3, which has 16 supercharges corresponding to 16 unbroken supersymmetries, has 8 which are manifest in that they act geometrically on the target space. These are given by $F_\alpha, \tilde{F}_\alpha$ described below and are related to the presence of 8 theta world sheet variables $\theta^\alpha, \tilde{\theta}^\alpha$. The other 8 supersymmetries $E_\alpha, \tilde{E}_\alpha$ are not manifest, but can still be expressed in terms of ordinary world sheet fields, i.e., not spin operators. Therefore, in addition to making some of the supersymmetry geometric (either in $R^6$ or $AdS_3 \times S^3$), this formalism is also advantageous to describing background fields belonging to the Ramond–Ramond (RR) sector, since the worldsheet fields which couple to the RR background fields are not spin fields. Thus for strings on $AdS$ that require RR backgrounds, the purpose of using BVW variables is that RR background fields can be added to the worldsheet action without adding spin fields to the worldsheet action [3].
On flat space (R^6), the BVW variables describe a free worldsheet conformal field theory. Their operator products (OPEs) are reviewed in Section 2, together with the N = 4 topological method for computing string correlation functions. In Section 3 we use these OPEs to evaluate the flat space three graviton correlation function in position space, and show that it reduces to the conventional answer.

On curved space, the BVW variables do not satisfy free operator product relations. Nonetheless, we proceed to evaluate the three graviton correlation function on AdS_3 × S^3 by assuming a form for the OPEs. It is motivated by [4], where the vertex operator constraint equations were derived for AdS_3 × S^3 by requiring them to be invariant under the AdS supersymmetry transformations. It can be shown that our assumed OPEs result in the same constraint equations. In Section 4, we compute the curved space three-graviton tree amplitude using these OPEs.

2. Review of components

The N = 4 topological prescription [1–3] for calculating superstring tree-level amplitudes is

\[ \left\{ V_1(z_1)(G_0^+ V_2(z_2))(G_0^+ V_3(z_3)) \prod_{r=1}^n \int dz_r \ G_0^- V(z_r) \right\}, \tag{2.1} \]

where G_0^± are elements of the topological N = 4 super Virasoro algebra, and the notation O_r V(z) denotes the pole of order d + n in the OPE of O(ξ) with V(z), when O is an operator of conformal dimension d.

2.1. N = 4 superconformal algebra in flat space

For the IIB superstring there is both a holomorphic N = 4 superconformal algebra and another anti-holomorphic one. The holomorphic generators, specialized for the IIB string compactified to six dimensions, and in terms of BVW worldsheet variables, are [3]

\[ \tilde{T}(z) = -\frac{1}{2} \partial x^m \partial x_m - p_a \partial \theta^a - \frac{1}{2} \partial \rho \partial \rho - \frac{1}{2} \partial \sigma \partial \sigma + \frac{3}{2} \partial^2 (\rho + i \sigma) + \tilde{T}_C, \]

\[ G^+(z) = -e^{-2\rho-i\sigma}(p)^4 + i e^{-\rho} p_a p_b \partial x^{ab} + e^{i\sigma} \left( -\frac{1}{2} \partial x^m \partial x_m - p_a \partial \theta^a - \frac{1}{2} \partial (\rho + i \sigma) \partial (\rho + i \sigma) + \frac{1}{2} \partial^2 (\rho + i \sigma) \right) + G^+_C, \]

\[ G^-(z) = e^{-i\sigma} + G^-_C, \quad J(z) = \partial (\rho + i \sigma) + J_C, \]

\[ J^+(z) = e^{i\sigma} J^+_C = -e^{\rho+i\sigma+iH_C}, \quad J^-(z) = -e^{-\rho-i\sigma} J^-_C = -e^{-\rho-i\sigma-iH_C}, \]

\[ \tilde{G}^+(z) = e^{iH_C+\rho} + e^{\rho+i\sigma} \tilde{G}^+_C, \]

\[ \tilde{G}^-(z) = e^{-iH_C} \left[ -e^{-3\rho-2i\sigma}(p)^4 - i e^{-2\rho-i\sigma} p_a p_b \partial x^{ab} + e^{-\rho} \left( -\frac{1}{2} \partial x^m \partial x_m - p_a \partial \theta^a - \frac{1}{2} \partial (\rho + i \sigma) \partial (\rho + i \sigma) + \frac{1}{2} \partial^2 (\rho + i \sigma) \right) \right] - e^{-\rho-i\sigma} \tilde{G}^-_C. \tag{2.2} \]

These currents are given in terms of the left-moving bosons \partial x^m, \rho, \sigma, and the left-moving fermionic worldsheet fields \( p_a, \theta^a \), where 0 ≤ m ≤ 5, 1 ≤ a ≤ 4. The conformal weights of \( p_a, \theta^a \) are 1 and 0, respectively. The BVW variables no longer exhibit the matter times ghost sector structure familiar from the conventional Ramond–Neveu–Schwarz formalism, with cancelling contributions of ±15 to the central charge. In BVW, the residual ghost fields are \( \rho, \sigma \). (In the ten-dimensional version, these are promoted to complex worldsheet boson fields...
\( \lambda^\alpha \) with a spacetime Majorana spinor index \( \alpha \), which are parameterized by eleven complex fields [7–13]. We define \( p^4 \equiv \frac{1}{16} \epsilon^{abcd} p_a p_b p_c p_d = p_1 p_2 p_3 p_4 \), and \( \partial x^{ab} = \partial x^m \sigma^m_{ab} \), where the sigma matrices in flat space satisfy \( \sigma^m_{ab} \sigma^m_{ac} + \sigma^n_{ab} \sigma^m_{ac} = \eta_{mn} \delta^m_{bc} \). Here lowered indices mean \( \sigma_{ab} \equiv \frac{1}{2} \epsilon^{abcd} \sigma_{cd} \). Note that \( \epsilon^\sigma \) and \( \epsilon^{\sigma} \) are worldsheet fermions. Also \( \epsilon^{\rho + i \sigma} \equiv \epsilon^\sigma \epsilon^\rho = - \epsilon^{\rho} \epsilon^\sigma \). Here \( J C \equiv i \partial H C, J C^+ \equiv - e^{-i H C}, J C^- \equiv e^{-i H C} \). Both \( \tilde{T} \), \( \tilde{G}^\pm \), \( J \) decompose into a \( c = 0 \) six-dimensional part and a \( c = 6 \) compactification-dependent piece.

\( \tilde{T}(z) = - \frac{1}{8} \epsilon^{abcd} K_{ab} K_{cd} - F^a E^a - \frac{1}{2} \partial \rho \partial \rho - \frac{1}{2} \partial \sigma \partial \sigma + \frac{3}{2} \partial^2 (\rho + i \sigma) + \tilde{T}_C, \\
G^+(z) = - \frac{1}{6} e^{-2\rho - i \sigma} e^{abcd} U_{ab} U_{cd} + i e^{-\rho} K^{ab} U_{ab} \\
+ e^{i \sigma} \left( - \frac{1}{8} \epsilon^{abcd} K_{ab} K_{cd} - F^a E^a - \frac{1}{2} \partial (\rho + i \sigma) \partial (\rho + i \sigma) + \frac{1}{2} \partial^2 (\rho + i \sigma) \right) + G^+_C, \\
G^-(z) = e^{-i \sigma} + G^-_C, \\
J(z) = \partial (\rho + i \sigma) + J_C, \\
\tilde{G}^+(z) = e^{i H C + \rho} + e^{\rho + i \sigma} \tilde{G}^+_C, \\
(2.3)
\]
where
\[ U_{ab} = \left( 1 - \frac{1}{4} e^{\tilde{\phi}} \right)^{-2} \left[ \frac{1}{4} F_a F_b - \frac{1}{2} e^{2\tilde{\phi}} E_a E_b - i \frac{1}{4} e^{-\rho} (E_a E_b + F_a F_b) \right], \\
(2.4)
\]
and the remaining generators \( J^\pm, \tilde{G}^- \) can be constructed from \( \tilde{T}, \tilde{G}^\pm, J \) [3]. Here \( e^\phi \equiv e^{-\rho - i \sigma}, \) and \( F_a, E_a, K_{ab} \) are the fermionic and bosonic \( z \)-components of the right-invariant \( PSU(2|2) \) currents. There is a corresponding anti-holomorphic \( N = 4 \) algebra in terms of barred worldsheet fields. Although the \( N = 4 \) generators have definite holomorphicity, the worldsheet fields \( F_a, E_a, K_{ab}, \bar{F}_a, \bar{E}_a, \bar{K}_{ab} \) do not. They are each functions of \( z \) and \( \bar{z} \) and have non-free operator products, which are not known in closed form. They reduce to the free conformal fields \( \rho_a, \partial \theta_a, \partial x_{ab}, \bar{p}_a, \tilde{\partial} \bar{\theta}_a, \tilde{\partial} x_{ab} \) only in the flat space.

3. Three-graviton tree amplitude in flat space

We first compute the six-dimensional three-graviton tree level amplitude in (6d) flat space, for Type IIB superstrings on \( \mathbb{R}^6 \times K3 \) in the BVW formalism. It is contained in the closed string three-point function
\[ \langle V(z_1, \bar{z}_1) (\tilde{G}_0^+ \tilde{G}_0^+ + V(z_2, \bar{z}_2)) (\tilde{G}_0^+ \tilde{G}_0^+ + V(z_3, \bar{z}_3)) \rangle, \]
where the \( N = 4 \) supercurrents are found in (2.2), and the vertex operators are given by
\[ V(z, \bar{z}) = e^{i \sigma(z) + \bar{\rho}(\bar{z})} e^{i \tilde{\rho}(\bar{z}) + \bar{\rho}(\bar{z})} \theta^a(\bar{z}) \bar{\theta}^a(\bar{z}) \bar{G}^a(\bar{z}) \sigma^m_{ab} \sigma_{mn} \phi_{mn} (X(z, \bar{z})), \]
(3.1)
when the field
\[ \phi_{mn} = g_{mn} + b_{mn} + \tilde{g}_{mn} \]
satisfies the constraint equations [3.4] \( \partial^m \phi_{mn} = 0 \), and \( \Box \phi_{mn} = 0 \). These constraints imply the gauge conditions \( \partial^m b_{mn} = 0 \) for the two-form, and \( \partial^m g_{mn} = -\partial_n \phi \) for the traceless graviton \( g_{mn} \) and dilaton \( \phi \). The constraints follow from the physical state conditions which in this formalism are implemented by the \( N = 4 \) generators, as shown in [3]. (Since the \( N = 4 \) algebra is twisted, i.e., topological, the nilpotent generators \( G^+, \tilde{G}^+, \hat{G}^+ \) are dimension one, and their cohomology essentially determines the physical states.) There is a residual gauge symmetry
\[ g_{mn} \rightarrow g_{mn} + \partial_n \xi_n + \partial_m \xi_m, \quad \phi \rightarrow \phi, \quad b_{mn} \rightarrow b_{mn}, \quad (3.3) \]
with \( \Box \xi_n = 0 \), \( \partial \cdot \xi = 0 \). To evaluate (3.1), we first extract the simple poles
\[ G^+_0 \tilde{G}^+_0 V(z, \bar{z}) = e^{i\sigma \cdot \epsilon} (-4) \times \left[ \phi_{mn}(X) \partial X^m \bar{\partial} X^n - p_a \partial^b \sigma_{cb}^m \rho^c \bar{\partial} X^n \partial_p \phi_{mn}(X) \right. \]
\[ + p_a \partial^b \bar{p}_d \sigma_{cb}^m \rho^c \sigma_{ab}^n \bar{\partial} \partial_p \phi_{mn}(X) \].
\[ (3.4) \]
\[ G^+_0 \tilde{G}^+_0 V(z, \bar{z}) = e^{iH_c \cdot \rho + i\sigma} e^{iH_c + 2\tilde{\sigma}} \sigma^a \partial^b \bar{\partial} \partial_p \phi_{mn}(X). \]
Then, using the OPEs for the ghost fields and \( H_c \), we partially compute (3.1) by exhibiting their contribution as [14–17]
\[ \{ V_1(z_1, \bar{z}_1) \left( G^+_0 \tilde{G}^+_0 V_2(z_2, \bar{z}_2) \right) \} \left( G^+_0 \tilde{G}^+_0 V_3(z_3, \bar{z}_3) \right) \]
\[ = (z_1 - z_2)(z_2 - z_3)(z_1 - z_3)^{-1}(z_1 - \bar{z}_2)(z_2 - \bar{z}_3)(z_1 - \bar{z}_3)^{-1} \times 4 \left[ e^{iH_c \cdot \rho (z_1 + 2\bar{z}(z_1))} e^{iH_c \cdot \rho \bar{z}(z_1 + 2\bar{z}(z_1))} e^{i\tilde{\sigma}(z_1)} \times \partial \phi_{mn}(X(z_1, \bar{z}_1)) \right. \]
\[ \times \phi_{jk}(X(z_2, \bar{z}_2)) \partial X^k \bar{\partial} X^l (z_2) - p_c \partial(z_2) \theta^f(z_2) \sigma_{af}^j \sigma_{cp}^e \bar{\partial} X^k (z_2) \partial_p \phi_{jk}(X(z_2, \bar{z}_2)) \]
\[ + p_c \partial(z_2) \theta^f(z_2) \sigma_{af}^j \sigma_{cp}^e \bar{\partial} X^k (z_2) \partial_p \phi_{jk}(X(z_2, \bar{z}_2)) \]
\[ \times \left. \left( \theta^d(z_3) \partial^e(z_3) \bar{\partial} (z_3) \sigma_{cd}^e \sigma_{cd}^e \phi_{jh}(X(z_3, \bar{z}_3)) \right) \right]. \]
Computing the remaining \( z_2, z_3 \) operators products, and using the \( SL(2, \mathbb{C}) \) invariance of the amplitude to take the three points to constants \( z_1 \rightarrow \infty, \bar{z}_1 \rightarrow \infty, z_2 \rightarrow 1, \bar{z}_2 \rightarrow 1, z_3 \rightarrow 0, \bar{z}_3 \rightarrow 0 \), we have
\[ \{ V_1(z_1, \bar{z}_1) \left( G^+_0 \tilde{G}^+_0 V_2(z_2, \bar{z}_2) \right) \} \left( G^+_0 \tilde{G}^+_0 V_3(z_3, \bar{z}_3) \right) \]
\[ = (z_2 - z_3)(\bar{z}_2 - \bar{z}_3)(z_2 - z_3)^{-1}(\bar{z}_2 - \bar{z}_3)^{-1} \times 4 \left[ e^{iH_c \cdot \rho (0) + 3\tilde{\sigma}(0) + e^{iH_c \cdot \rho (0) + 3\tilde{\sigma}(0) + 3i\tilde{\sigma}(0)} \theta^d \bar{\partial} \partial p \partial \phi_{jk}(X(0)) \right. \]
\[ \times \sigma_{ab}^m \sigma_{ab}^m \sigma_{ab}^m \phi_{mn}(X(0)) \theta^d \bar{\partial} \partial p \phi_{jk}(X(1)) \]
\[ + 2\sigma_{ab}^m \sigma_{ab}^m \phi_{mn}(X(0)) \theta^d \bar{\partial} \partial p \phi_{jk}(X(1)) \theta^d \bar{\partial} \partial p \phi_{jk}(X(0)) \]
\[ + 2\sigma_{ab}^m \sigma_{ab}^m \phi_{mn}(X(0)) \theta^d \bar{\partial} \partial p \phi_{jk}(X(1)) \theta^d \bar{\partial} \partial p \phi_{jk}(X(0)) \]
\[ + 4\sigma_{ab}^m \sigma_{ab}^m \phi_{mn}(X(0)) \theta^d \bar{\partial} \partial p \phi_{jk}(X(1)) \theta^d \bar{\partial} \partial p \phi_{jk}(X(0)) \].
\[ I = \int d^d x \frac{1}{|g|} \left( -\frac{R}{2K^2} \right). \]

Expanding to third order in \( K \) using \( g_{\mu\nu} = \eta_{\mu\nu} + 2K h_{\mu\nu} \), we find the three-point interaction \( I_3 \). In harmonic gauge, i.e., when \( \partial^\mu h_{\mu\nu} = 0 \), and on shell \( \Box h_{\mu\nu} = 0 \), it is given by
\[
I_3 = -K \int d^d x \left[ h^{\mu\nu} h^{\rho\sigma} \partial_\mu \partial_\rho h_{\nu\sigma} + 2h^{\mu\nu} \partial_\mu h^{\rho\sigma} \partial_\rho h_{\nu\sigma} \right].
\]
The gauge transformations
\[ h_{\mu \nu} \rightarrow h_{\mu \nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \] (3.8)
leave invariant the harmonic gauge condition and I₃, given in (3.7), when \( \Box \xi_\mu = 0 \). Using this gauge symmetry, we could further choose \( h^\rho_\rho = 0 \), \( \partial^\rho h_{\mu \nu} = 0 \). Then I₃ represents the three-graviton amplitude, and is invariant under residual gauge transformations that have \( \partial \cdot \xi = 0 \).

To extract the string theory three-graviton amplitude from (3.6), we set \( b_{mn} \) to zero, and use the field identifications [4] that relate the string fields \( g_{mn} \), \( \phi \) to the supergravity field \( h_{mn} \) via \( \phi \equiv -\frac{1}{2} h^\rho_\rho \) and \( g_{mn} \equiv h_{mn} - \frac{1}{2} \xi_{[m} \xi_{n]} h^\rho_\rho \), where \( h_{mn} \) is in harmonic gauge. Then \( \phi_{mn} = h_{mn} - \frac{1}{2} \xi_{[m} \xi_{n]} h^\rho_\rho \), and from (3.6) the on shell string tree amplitude is

\[
\frac{-K}{12} \left\{ V_1(z_1, \bar{z}_1) \left( G^+_0 G^+_1 V_2(z_2, \bar{z}_2) \right) \left( \tilde{G}^+_0 G^+_1 V_3(z_3, \bar{z}_3) \right) \right\} = -K \int d^dx \left[ \phi_{mn}(x) \phi_{jk}(x) \partial_m \partial_k \phi_{jk}(x) + 2 \phi_{mn}(x) \partial_m \partial_k \phi_{jk}(x) \right] = -K \int d^dx \left[ h_{mn}(x) \partial_m h_{jk}(x) + 2 h_{mn}(x) \partial_m \partial_k h_{jk}(x) \right] + K \int d^dx \left[ h_{mn}(x) \partial_m h_{jk}(x) \partial_j h_{nk}(x) \right] = I_3 + I'_3, \quad (3.9)
\]

where \( I'_3 \) is the one graviton – two dilaton amplitude, \( I_3 \) is the three graviton interaction in harmonic gauge, and \( d = 6 \). We note that \( I_3 \) and \( I'_3 \) separately are invariant under the gauge transformation (3.8) with \( \Box \xi_\mu = 0 \) and \( \partial \cdot \xi = 0 \), which corresponds to the gauge symmetry of the string field \( \phi_{mn} \rightarrow \phi_{mn} + \partial_m \xi_n + \partial_n \xi_m \). Furthermore, \( I_3 \) is also invariant under gauge transformations for which \( \partial \cdot \xi \neq 0 \), and these can be used to eliminate the trace of \( h_{mn} \) in \( I_3 \). In the string gauge, the trace of \( \phi_{mn} \) is related to the dilaton \( \phi^\mu_\mu = 6 \phi \), so even when \( h_{mn} = 0 \), (3.6) contains both the three graviton amplitude and the one graviton – two dilaton interaction. From (3.9) we see that we could have extracted \( I_3 \) from (3.6) merely by setting both \( b_{mn} = 0 \) and \( \phi = 0 \), since then \( \phi_{mn} = g_{mn} \) and \( \partial_m g_{mn} = 0 \).

4. Three-graviton tree amplitude in \( AdS_3 \times S^3 \)

In this section, we compute the six-dimensional three-graviton amplitude in the Type IIB superstring on \( AdS_3 \times S^3 \times K3 \) with background Ramond flux. Consider

\[
\left\{ V_1(z_1, \bar{z}_1) \left( G^+_0 G^+_1 V_2(z_2, \bar{z}_2) \right) \left( \tilde{G}^+_0 G^+_1 V_3(z_3, \bar{z}_3) \right) \right\} \quad (4.1)
\]

where the \( N = 4 \) supercurrents are reviewed in (2.3), (2.4). They are expressed in terms of the left action generators defined in [4] as

\[
F_a = \frac{d}{d \theta^a}, \quad K_{ab} = -\theta_a \frac{d}{d \theta^b} + \theta_b \frac{d}{d \theta^a} + h_{ab},
\]

\[
E_a = \frac{1}{2} \varepsilon_{abcd} \theta^b \left( r^{cd}_{L} - \theta^c \frac{d}{d \theta^d} \right) + h_{ab} \frac{d}{d \theta^b}, \quad (4.2)
\]

and corresponding right action generators \( \tilde{F}_a, \tilde{E}_a, \tilde{K}_{ab} \). The vertex operators are

\[
V(z, \bar{z}) = e^{i \sigma(z) + \rho(z)} e^{i \bar{\theta}(\bar{z}) + \bar{\rho}(\bar{z})} \theta^a(z, \bar{z}) \bar{\theta}^a(z, \bar{z}) \theta^b(z, \bar{z}) \bar{\theta}^b(z, \bar{z}) \sigma^a_{\alpha \beta} \sigma^b_{\gamma \delta} \phi_{mn}(X(z, \bar{z})).
\]

On \( AdS_3 \), in addition to the equation of motion, the string field \( \phi_{mn} = g_{mn} + b_{mn} + \xi_{mn} \phi \) satisfies constraints given by \( r^{ab} h^c \bar{h}^d h^e a_{bc} \alpha\beta \gamma \delta \phi_{mn} = 0 \), \( r^a h^b h^c \alpha_{ab} \gamma \delta \phi_{mn} = 0 \), which were derived in [4], where \( r^a L, r^a R \) describe invariant derivatives on the \( SO(4) \) group manifold. These can be related to covariant derivatives \( T^{cd}_L \equiv -\sigma^{cd} D_p \).
In deriving (4.3), we keep only the contribution carry vector or spinor indices, they differ so that for example on spinor indices with \( \delta b \equiv V_0 + 0 \). These are the AdS \( \times S \) analog of the flat space constraints \( \partial^m \phi_{mn} = 0 \). On AdS\( \times S \),

\[
G^+_0 \tilde{G}^+_0 V(z, \bar{z}) = e^{i\sigma} e^{i\delta} (-4)
\]

\[
\times \left[ \frac{1}{4} K^{ab}(z, \bar{z}) \tilde{K}^{\tilde{a}b}(z, \bar{z}) \sigma_{ab}^{\alpha} \sigma^{\alpha}_{\tilde{a}b} \phi_{mn}(X) 
- \frac{1}{2} F_a(z, \bar{z}) \theta^b(z, \bar{z}) \tilde{K}^{\tilde{a}b}(z, \bar{z}) (t^{\alpha c} - \delta^{ac}) \sigma_{\tilde{a}b}^{\alpha} \sigma^{\alpha}_{ab} \phi_{mn}(X) 
- \frac{1}{2} \tilde{F}_a(z, \bar{z}) \tilde{\theta}^b(z, \bar{z}) K^{ab}(z, \bar{z}) (t^{\tilde{\alpha}c} - \delta^{\tilde{\alpha}c}) \sigma_{ab}^{\tilde{\alpha}} \sigma^{\tilde{\alpha}}_{\tilde{a}b} \phi_{mn}(X) 
+ F_a(z, \bar{z}) \theta^b(z, \bar{z}) \tilde{F}_b(z, \bar{z}) \tilde{\theta}^b(z, \bar{z}) (\tilde{t}^{\tilde{\alpha}c} - \delta^{\tilde{\alpha}c}) (t^{\alpha c} - \delta^{ac}) \sigma_{ab}^{\tilde{\alpha}} \sigma^{\tilde{\alpha}}_{\tilde{a}b} \phi_{mn}(X) \right] ,
\]

(4.3)

\[
G^+_0 \tilde{G}^+_0 V(z, \bar{z}) = e^{Hc + 2\phi + \phi h} e^{Hc + 2\phi + \phi h} \phi^a \tilde{\phi}^a \sigma_{ab}^{\alpha} \sigma^{\alpha}_{ab} \phi_{mn}(X). \]

(4.4)

In deriving (4.3), we keep only the contribution \( i e^{-\rho} K^{ab} U_{ab} \) to \( G^+ \) with \( U_{ab} \sim \frac{1}{4} F_a F_b \), since other terms in \( G^+ \) do not survive the ghost measure in the vacuum expectation value. We have also assumed that the OPE of \( F_a(z, \bar{z}) \) with \( \theta^a(z, \bar{z}) \) can be replaced with \( F_a(z, \bar{z}) \theta^a(z, \bar{z}) \sim (z - \bar{z})^{-1} \delta^a_0 \), in accordance with (4.2). This is motivated by the observation that evaluating the OPEs in this manner leads to the constraint equations found in [4], where those equations were derived solely by requiring supersymmetric invariance (and not from the action of the \( N = 4 \) generators).

\[
\left\{ V_1(z_1, \bar{z}_1) \left( G^+_0 \tilde{G}^+_0 V_2(z_2, \bar{z}_2) \right) \left( \tilde{G}^+_0 \bar{G}^+_0 V_3(z_3, \bar{z}_3) \right) \right\} 
= (z_1 - z_2) (z_2 - z_3) (z_1 - z_3)^{-1} (\tilde{z}_1 - \tilde{z}_2) (z_2 - \tilde{z}_3) (z_1 - \tilde{z}_3)^{-1} 
\times 4 \left( e^{-Hc(z_1)} e^{Hc(z_2)} e^{3i\sigma(z_1)} e^{Hc(z)} e^{2\phi + \phi h} e^{3i\tilde{\delta}(z_2)} z \right) 
\times \sigma^a(z_1) \theta^b(z_1) \theta^a(z_1) \sigma_{ab}^{\alpha} \sigma^{\alpha}_{ab} \phi_{mn}(X(z_1, \tilde{z}_1)) 
\times \left[ \frac{1}{4} \phi_{jk}(X(z_2, \tilde{z}_2)) \sigma^j \sigma^k K^{ef}(z_2, \tilde{z}_2) \tilde{K}^{\tilde{e}f}(z_2, \tilde{z}_2) 
- \frac{1}{2} F_e(z_2, \tilde{z}_2) \theta^f(z_2, \tilde{z}_2) \tilde{K}^{\tilde{e}f}(z_2, \tilde{z}_2) (t^{\alpha e} - \delta^{\alpha e}) \sigma_{\tilde{a}b}^{\alpha} \sigma^{\alpha}_{ab} \phi_{mn}(X) 
- \frac{1}{2} \tilde{F}_e(z_2, \tilde{z}_2) \tilde{\theta}^{\tilde{f}}(z_2, \tilde{z}_2) K^{ab}(z_2, \bar{z}_2) (t^{\tilde{\alpha}e} - \delta^{\tilde{\alpha}e}) \sigma_{ab}^{\tilde{\alpha}} \sigma^{\tilde{\alpha}}_{\tilde{a}b} \phi_{mn}(X) 
+ F_e(z_2, \tilde{z}_2) \theta^f(z_2, \tilde{z}_2) \tilde{F}_e(z_2, \tilde{z}_2) \tilde{\theta}^{\tilde{f}}(z_2, \tilde{z}_2) (\tilde{t}^{\tilde{\alpha}e} - \delta^{\tilde{\alpha}e}) (t^{\alpha e} - \delta^{\alpha e}) \sigma_{ab}^{\tilde{\alpha}} \sigma^{\tilde{\alpha}}_{\tilde{a}b} \phi_{mn}(X) \right] 
\times \theta^d(z_3) \theta^d(z_3) \tilde{\theta}^d(z_3) \tilde{\theta}^d(z_3) \sigma^{ef} \sigma_{ef} \phi_{gh}(X(z_3, \tilde{z}_3))) . \]

(4.5)

Evaluating at \( z_1 \to \infty, \tilde{z}_1 \to \infty, \) and restricting \( \phi_{mn}, \phi_{jk}, \phi_{gh} \) to be symmetric, we have

\[
\left\{ V_1(z_1, \bar{z}_1) \left( G^+_0 \tilde{G}^+_0 V_2(z_2, \bar{z}_2) \right) \left( \tilde{G}^+_0 \bar{G}^+_0 V_3(z_3, \bar{z}_3) \right) \right\} 
= (z_2 - z_3) (\bar{z}_2 - \bar{z}_3) (z_2 - z_3)^{-1} (\tilde{z}_2 - \tilde{z}_3)^{-1} \cdot 4 
\times \left( e^{-Hc(0) + 3i\sigma(0)} e^{-Hc(0) + 3i\sigma(0)} e^{3i\tilde{\delta}(0)} e^{3i\tilde{\delta}(0)} \phi_{gh} \phi_{gh} \phi_{gh} \phi_{gh} \right) \left. \right|_{e^{abcd} a^0 b^0 c^0 d^0} \leq 1 \}
\]
where the $z_2, z_3$ OPEs have been evaluated in similar fashion, the $\text{SL}(2, \mathbb{C})$ invariance sets $z_2 \to 1, z_3 \to 0$, and cancellations occur among the contributions to the OPEs from the four terms in the sum in (4.5).

Then

\[
\langle V_1(z_1, \bar{z}_1) (G^+_0 \tilde{G}^+_0 V_2(z_2, \bar{z}_2)) (\bar{G}^+_0 \tilde{G}^+_0 V_3(z_3, \bar{z}_3)) \rangle
= 4 \left[ \frac{1}{16} \sigma_{\mu \nu} \sigma_{\mu \nu} \sigma^{j \ell} \sigma_{j \ell} \langle \phi_{mn} (X(\infty)) \rangle \phi_{jk}(X(1)) \left[ \tilde{e}^f_{R} \tilde{e}^f_{R} \sigma^g_{cd} \sigma_{gh}(X(0)) \right] \right.
\]

\[
+ \frac{1}{4} \sigma_{\mu \nu} \sigma_{\mu \nu} \sigma^{j \ell} \sigma_{j \ell} \langle \phi_{mn} (X(\infty)) \rangle \left[ \tilde{e}^f_{R} \tilde{e}^f_{R} \sigma^g_{cd} \sigma_{gh}(X(0)) \right] \]

\[
+ \frac{1}{4} \sigma_{\mu \nu} \sigma_{\mu \nu} \sigma^{j \ell} \sigma_{j \ell} \langle \phi_{mn} (X(\infty)) \rangle \left[ \tilde{e}^f_{R} \tilde{e}^f_{R} \sigma^g_{cd} \sigma_{gh}(X(0)) \right] \]

\[
\left. + 4 \sigma_{\mu \nu} \sigma_{\mu \nu} \sigma^{j \ell} \sigma_{j \ell} \langle \phi_{mn} (X(\infty)) \rangle \left[ \tilde{e}^f_{R} \tilde{e}^f_{R} \sigma^g_{cd} \sigma_{gh}(X(0)) \right] \right],
\]
\[ g_{mn} = -\bar{g}_{mn} \]

To interpret (4.9) on shell, we recall [4] that the first order linearized duality equation of motion for one of the supergravity fields related to \( b_{mn} \) is

\[ g^{1} \nu g^{1} \rho g^{1} \mu g^{1} \lambda g^{1} \sigma g^{1} \mu g^{1} \lambda g^{1} \sigma = -\bar{H}_{m n} g^{1} \nu g^{1} \rho g^{1} \mu g^{1} \lambda g^{1} \sigma g^{1} \mu g^{1} \lambda g^{1} \sigma. \]

Part of (4.9) is then identified as

\[ -4 \bar{H}_{m n} g^{1} \nu g^{1} \rho g^{1} \mu g^{1} \lambda g^{1} \sigma g^{1} \mu g^{1} \lambda g^{1} \sigma = -4 \bar{H}_{m n} g^{1} \nu g^{1} \rho g^{1} \mu g^{1} \lambda g^{1} \sigma g^{1} \mu g^{1} \lambda g^{1} \sigma. \]  

Assuming that the only non-vanishing string field fluctuation is \( g_{mn} \), we have \( \phi_{mn} = g_{mn} \) and the gauge condition becomes \( D_{m} g_{mn} = 0 \).

Then on \( \text{AdS}_3 \times S^3 \), the string theory three graviton amplitude is

\[ \left\langle V_1(z_1, \bar{z}_1) G_0 \bar{G}_0 V_2(z_2, \bar{z}_2) \bar{G}_0 \bar{G}_0 V_3(z_3, \bar{z}_3) \right\rangle = -12 \int d^3 x \sqrt{\bar{g}} \left[ g^{mn} g^{jk} D_{m} D_{n} g_{jk} + 2g^{mn} g^{jk} D_{m} g_{jk} + 2g^{mn} g^{jk} D_{m} g_{jk} \right], \]

where \( D_{m} \) is the covariant derivative on \( \text{AdS}_3 \times S^3 \). Eq. (4.12) is the curved space analog of (3.7). In three dimensions, the graviton has no propagating degrees of freedom, which means the graviton field can be gauged to zero. Nonetheless, expanding (4.12) in spherical harmonics on \( S^3 \), and noting that \( \bar{g}_{mn} = \bar{g}_{\mu \nu}, \bar{g}_{\alpha \beta} \), the background metric of \( \text{AdS}_3 \times S^3 \) for \( 1 \leq \mu, \nu \leq 3 \) and \( 1 \leq \alpha, \beta \leq 3 \), we find that the covariant derivatives factorize and the string theory calculation retains the familiar structure

\[ \sim \int d^3 x \sqrt{\bar{g}} \left[ g^{\mu \nu} g^{\rho \sigma} D_{\mu} D_{\nu} g_{\rho \sigma} + 2g^{\mu \nu} D_{\mu} g^{\rho \sigma} D_{\rho} g_{\sigma \mu} \right]. \]

In general, \( \alpha’ \) corrections are expected to occur in four or higher \( n \)-point string tree amplitudes, but will be calculable only as an expansion in \( \alpha’ \) since the worldsheet theory is not free. To study the AdS/CFT correspondence, the bulk correlations functions on shell can be related to correlations on the boundary. We note that \( \alpha’ \) is related the coupling constant of the spacetime conformal field theory. To investigate this correspondence systematically it would be of interest to attain tree-level expressions that are exact in \( \alpha’ \), perhaps by adapting integrable methods for sigma models which have a supergroup manifold target space [18] such as this \( \text{AdS}_3 \times S^3 \) theory.

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References

Low-energy dynamics of noncommutative $CP^1$ solitons in $2+1$ dimensions

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Abstract

We investigate the low-energy dynamics of the BPS solitons of the noncommutative $CP^1$ model in $2+1$ dimensions using the moduli space metric of the BPS solitons. We show that the dynamics of a single soliton coincides with that in the commutative model. We find that the singularity in the two-soliton moduli space, which exists in the commutative $CP^1$ model, disappears in the noncommutative model. We also show that the two-soliton metric has the smooth commutative limit.

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1. Introduction

Noncommutative geometry appears in M-theory, string theory and condensed matter physics [1]. Noncommutative field theories are known to describe the low-energy effective theory of D-branes in a background $B$-field [2–4]. $(2+1)$-dimensional noncommutative theories have applications to the quantum Hall effect [5]. In spite of the nonlocality and entangled UV/IR mixing in perturbation theory, the appearance of these theories in string theory suggests that some class of noncommutative field theories is a sensible deformation of ordinary field theories.

In view of understanding nonperturbative effects in noncommutative field theories, noncommutative solitons and instantons have been investigated. In four-dimensional noncommutative Yang–Mills theory, there exists a $U(1)$ instanton which is nonsingular due to the position-space uncertainty [6]. The moduli space of instantons are also smooth [7,8]. In noncommutative scalar field theories, there exist solitons (GMS solitons) [9] at the limit of large noncommutativity parameter $\theta$; they cannot exist in the commutative counterpart.

Low-energy dynamics of solitons can be approximated by geodesic motion on the moduli space of static solutions [10]. In commutative theories, scattering properties of solitons in gauge theories and nonlinear sigma models were studied extensively using this approximation [11–13]. A common feature is that the scattering of two solitons occurs at right angle for the head-on collision. In noncommutative theories, the scatterings of solitons in the GMS model, Yang–Mills theories and integrable models were investigated [14–16].
In this Letter we investigate the low-energy dynamics of BPS solitons of the noncommutative \( CP^1 \) model in \( 2+1 \) dimensions. It was shown that the noncommutative \( CP^N \) model has the BPS solutions as the commutative model does [17]. In the commutative \( CP^N \) model, the moduli space of solitons is known to be a Kähler manifold [18]. We show that the moduli space of the noncommutative \( CP^N \) model is a Kähler manifold too. We calculate the \( \theta \)-dependence of the moduli space metric in the \( CP^1 \) case.

This Letter is organized as follows. In Section 2, we review the BPS solutions of the noncommutative \( CP^N \) model. In Section 3, we give the compact form of the Kähler potential of the moduli space of these solutions. In Section 4, we investigate the one-soliton metric of the noncommutative \( CP^1 \) model and show that the motion of a single soliton is the same as that of the commutative model. In Section 5, we study the two-soliton metric of the model and find that the singularity which exists in the commutative model disappears in the noncommutative model. Furthermore, we show that the two-soliton metric has the smooth commutative limit.

2. BPS solution of the noncommutative \( CP^N \) model

We recapitulate the \( (2+1) \)-dimensional noncommutative \( CP^N \) model, closely following Lee, Lee and Yang [17].

We consider the \( (2+1) \)-dimensional field theory on the noncommutative space. The noncommutativity is introduced by

\[
[x, y] = i\theta, \quad \theta > 0.
\]

We set

\[
z = \frac{1}{\sqrt{2}}(x + iy), \quad \bar{z} = \frac{1}{\sqrt{2}}(x - iy).
\]

Then Eq. (1) becomes

\[
[z, \bar{z}] = \theta,
\]

or

\[
[a, a^\dagger] = 1, \quad a = \frac{z}{\sqrt{\theta}}, \quad a^\dagger = \frac{\bar{z}}{\sqrt{\theta}}.
\]

This is the algebra of the creation and annihilation operators. We use the Fock space of the quantum harmonic oscillator for the representation of the algebra (1).

The derivative and the integral on the noncommutative space are given by

\[
\partial_s \Phi = i\theta^{-1} [y, \Phi], \quad \partial_s \Phi = -i\theta^{-1} [x, \Phi],
\]

\[
\int d^2 x \mathcal{O} \rightarrow \text{Tr} \mathcal{O} = 2\pi \theta \sum_{n \geq 0} \langle n | \mathcal{O} | n \rangle,
\]

where \( |n \rangle \ (n = 0, 1, 2, \ldots) \) are the basis of the Fock space.

The Lagrangian of the noncommutative \( CP^N \) model [17] is defined by

\[
L = \text{Tr} \left[ D_\mu \Phi^\dagger D^\mu \Phi + \lambda (\Phi^\dagger \Phi - 1) \right],
\]

\[
D_\mu = \partial_\mu \Phi - i \Phi A_\mu, \quad A_\mu = -i \Phi^\dagger \partial_\mu \Phi,
\]

where the field \( \Phi = (\phi_1, \phi_2, \ldots, \phi_{N+1}) \) is the complex \( (N+1) \)-component vector, and \( \lambda \) is the Lagrange multiplier field which gives the constraint \( \Phi^\dagger \Phi = 1 \). This theory has the global \( SU(N+1) \) symmetry and \( U(1) \) gauge symmetry \( \Phi(x) \rightarrow \Phi(x) g(x) \), where \( g(x) \in U(1) \).
The energy functional is
\[ E = \text{Tr}(|D_0 \Phi|^2 + |D_z \Phi|^2 + |D_{\bar{z}} \Phi|^2). \] (9)

We have the Bogomolnyi bound
\[ E \geq \text{Tr}(|D_0 \Phi|^2) + 2\pi|Q|, \] (10)
where \( Q \) is the topological charge,
\[ Q = \frac{1}{2\pi} \text{Tr}(|D_z \Phi|^2 - |D_{\bar{z}} \Phi|^2). \] (11)

The BPS equation of the static solution is
\[ D_{\bar{z}} \Phi = 0 \quad \text{for self-dual solution}, \] (12)
\[ D_z \Phi = 0 \quad \text{for anti-self-dual solution}. \] (13)

We parametrize \( \Phi \) as
\[ \Phi = W (W^\dagger W)^{-1/2}. \] (14)
where \( W \) is the complex \((N + 1)\)-component vector. We define the projection operator \( P \) by
\[ P = 1 - W (W^\dagger W)^{-1} W^\dagger. \] (15)

Then (7) and (11) are
\[ L = \text{Tr} \left[ \frac{1}{\sqrt{W^\dagger W}} \partial_\mu W^\dagger P \partial^\mu W \frac{1}{\sqrt{W^\dagger W}} \right], \] (16)
\[ Q = \frac{1}{2\pi} \text{Tr} \left[ \frac{1}{\sqrt{W^\dagger W}} (\partial_{\bar{z}} W^\dagger P \partial_z W - \partial_z W^\dagger P \partial_{\bar{z}} W) \frac{1}{\sqrt{W^\dagger W}} \right]. \] (17)

The BPS equation (12) becomes
\[ D_{\bar{z}} \Phi = P (\partial_{\bar{z}} W) (W^\dagger W)^{-1/2} = 0. \] (18)

This equation is equivalent to \( \partial_{\bar{z}} W = WV \), where \( V \) is an arbitrary scalar. In the commutative case, using the \( N \)-component vector \( w \), we set \( W = t(w, 1) \) by the gauge transformation
\[ W \to W' = W \Delta(z, \bar{z}), \] (19)
where \( \Delta(z, \bar{z}) \) is an arbitrary scalar. Then (18) becomes \( \partial_{\bar{z}} w = 0 \), namely, \( w \) is holomorphic. In the noncommutative case, the Lagrangian (16) is invariant under the transformation (19) when \( \Delta \) is invertible.

We will be mainly concerned with the one- and two-soliton solutions of the noncommutative \( CP^1 \) model. The one- and two-soliton solutions of the commutative \( CP^1 \) model are respectively given by \[ w = \lambda + \frac{\mu}{z - \nu}, \] (20)
\[ w = \alpha + \frac{2\beta z + \gamma}{z^2 + \delta z + \epsilon}, \] (21)
where \( \alpha, \beta, \ldots \in C \) are the moduli parameters. We may set the moduli parameters \( \lambda \) and \( \alpha \) to zero, using the global \( SU(2) \) symmetry.
In the noncommutative case, \( W = (z, 1) \) and \( W' = W_{\bar{z}^{-1}} = (1, \bar{z}^{-1}) \) are gauge inequivalent, where \( z^{-1} \) is defined to be \( z^{-1} = (\bar{z}z)^{-1} \bar{z} \equiv \bar{z}(\bar{z} + \theta)^{-1} \). \( z^{-1} \) satisfies \( z\bar{z}^{-1} = 1 \) and \( \bar{z}^{-1}z = 1 - |0\rangle\langle 0| \). \( W \) satisfies the BPS equation (18) but \( W' \) does not. In general, \( W \) satisfies the BPS equation (18) if the components of \( W \) are polynomials of \( z \). The BPS one- and two-soliton solutions in the noncommutative \( CP^1 \) model corresponding to (20) and (21) are respectively given by

\[
W = \begin{pmatrix} z - \nu \\ \mu \end{pmatrix},
\]
(22)

\[
W = \begin{pmatrix} z^2 + \delta z + \epsilon \\ 2 \beta z + \gamma \end{pmatrix}.
\]
(23)

3. Moduli space metric

In the following section, we consider the scattering of BPS solutions of the noncommutative \( CP^N \) model. In the low-energy limit (near the Bogomolnyi bound), it is a good approximation that only the moduli parameters depend on the time [10]. Their time evolution is determined by minimizing the action \( S = \int dt L \). It amounts to dealing with the kinetic energy \( T \) (the term in the Lagrangian including the time derivative only), since the rest of the Lagrangian gives the topological charge. The kinetic energy is given by

\[
T = \text{Tr} \left( \frac{1}{\sqrt{W^\dagger W}} \partial_t W^\dagger P \partial_t W - \frac{1}{\sqrt{W^\dagger W}} \right) = \frac{1}{2} \text{Tr}(\partial_t P')^2,
\]
(24)

where \( P' \) is defined by

\[
P' = 1 - P = W(W^\dagger W)^{-1} W^\dagger.
\]
(25)

\( T \) is written as \( T = \frac{1}{2}(ds/dt)^2 \), where \( ds^2 \) is the line element of the moduli space. The dynamics of solitons is given by the geodesic line in the moduli space. We denote generically the moduli parameters by \( \zeta^a \). We then have

\[
T = \frac{1}{2} g_{ab} \frac{d\zeta^a}{dt} \frac{d\zeta^b}{dt},
\]
(26)

where \( g_{ab} \) is the moduli space metric.

It is convenient to express \( P' \) in the Fock space language,

\[
P' = \sum_{n,m} |\psi_n\rangle h_{nm} \langle \psi_m|, \quad |\psi_n\rangle = W|n\rangle,
\]
(27)

\[
h_{nm} = \langle \psi_n|\psi_m\rangle, \quad h_{nm} = (h_{nm})^{-1}.
\]
(28)

The BPS solution \( W \) (or \(|\psi_n\rangle\)) is a holomorphic function of the moduli parameters. It was shown that the moduli space in the case where \(|\psi_n\rangle \) is holomorphic is a Kähler manifold [9]. Hence, we write

\[
T = \frac{1}{2} g_{\dot{a}\dot{b}} \frac{d\zeta^{\dot{a}}}{dt} \frac{d\zeta^{\dot{b}}}{dt}, \quad g_{\dot{a}\dot{b}} = \partial_{\dot{a}} \partial_{\dot{b}} K,
\]
(29)

where the Kähler potential \( K \) is given by

\[
K = \text{Tr} \ln(h_{nm}) = \text{Tr} \ln(W^\dagger W).
\]
(30)
4. One-soliton metric

The BPS one-soliton solution of the noncommutative $CP^1$ model is given by (22). We substitute (22) into (24), then we have

$$
T = \text{Tr} \left[ \frac{1}{\sqrt{(\bar{z} - \bar{v})(z - v) + |\mu|^2}} \partial_t \left( \frac{z - v}{\mu} \right) \frac{1}{(\bar{z} - \bar{v})(z - v) + |\mu|^2} \left( \bar{z} - \bar{v} \right) \mu \right] \right] (z - v) \sqrt{(\bar{z} - \bar{v})(z - v) + |\mu|^2}.
$$

The $\dot{\bar{v}}\dot{\mu}$ term in $T$ is

$$
2\pi \theta \hat{\beta} \hat{\mu} \sum_{n \geq 0} \frac{1}{\theta n + |\mu|^2} \left[ \frac{\theta n}{\theta n + |\mu|^2} + \frac{|\nu|^2}{\theta (n + 1) + |\mu|^2} \right].
$$

The first term in (32) diverges. We set $\mu$ to a constant in the low-energy approximation. Calculating the trace in (31) with $\dot{\bar{v}} = 0$, we obtain

$$
T = 2\pi \frac{d\bar{v}}{dt} \frac{dv}{dt}, \quad \text{or} \quad ds^2 = 4\pi \frac{d\bar{v}}{dt} dv.
$$

This is $\theta$-independent and coincides with the commutative case. A single soliton moves straight without changing the size.

The result (33) can be explained in a more general framework. In the low-energy limit, the action $S = \int dt L$ is invariant under the Galilean transformation

$$
t \to t, \quad x \to x + vt, \quad y \to y + vt,
$$

since this transformation does not change the commutation relation (1). Under the Galilean transformation, the kinetic energy in the center-of-mass frame $T_{cm}$ is transformed as

$$
T_{cm} \to \frac{1}{2} Mv^2 + T_{cm},
$$

where $M$ is the total mass of the solitons; $M = 2\pi |Q|$. From (35), it follows that the contribution of the center-of-mass coordinates to the kinetic energy is the same as in the commutative case. From now on, we restrict the moduli parameters to the center-of-mass frame.

5. Two-soliton metric

The BPS two-soliton solution of the noncommutative $CP^1$ model in the center-of-mass frame (i.e., $\delta = 0$) is

$$
W = \begin{pmatrix}
\frac{z^2 + \epsilon}{2\beta z + \gamma} \\
\bar{z}^2 + \bar{\epsilon}
\end{pmatrix}.
$$

Computing the kinetic energy in the low-energy limit, we can see that the contribution of the $\dot{\bar{v}}\dot{\beta}$ term to the kinetic energy diverges. In the low-energy approximation, we set $\beta$ to a constant. We consider the case of $\beta = 0$ for simplicity. In the commutative model, the moduli space metric is calculated by Ward [12]. There exists a singularity at $(\epsilon, \gamma) = (0, 0)$. In the next subsection, we will see the disappearance of this singularity in the noncommutative model.

The Kähler potential corresponding to (36) is

$$
K = \text{Tr} \ln \left[ (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon) + \gamma \gamma \right].
$$
This is a formal expression since the trace in (37) diverges. A finite Kähler potential is obtained by subtracting the divergent terms using the Kähler transformation

\[ K(\gamma, \epsilon; \tilde{\gamma}, \tilde{\epsilon}) \rightarrow K(\gamma, \epsilon; \tilde{\gamma}, \tilde{\epsilon}) - f(\gamma, \epsilon) - \tilde{f}(\tilde{\gamma}, \tilde{\epsilon}). \]  

(38)

In (37), only the terms with the same number of \( z^2 \) and \( \bar{z}^2 \) contribute to the trace. Hence, \( K \) is the function of \( \epsilon \epsilon \) and \( \tilde{\gamma} \gamma \) only; \( K = K(\epsilon \epsilon, \tilde{\gamma} \gamma) \). It then follows that the moduli space is manifestly invariant under the following three kinds of transformations: (i) \( (\epsilon, \gamma) \leftrightarrow (\tilde{\epsilon}, \tilde{\gamma}) \), (ii) \( \epsilon \rightarrow e^{i \theta} \epsilon \), and (iii) \( \gamma \rightarrow e^{i \chi} \gamma \).

The moduli space metric is

\[ g_{\gamma \gamma} = \text{Tr} \left[ \frac{1}{\bar{\gamma} \gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)} \left( 1 - \frac{\bar{\gamma} \gamma}{\bar{\gamma} \gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)} \right) \right]. \]  

(39a)

\[ g_{\epsilon \epsilon} = -\text{Tr} \left[ \bar{\gamma} \gamma \left( \bar{\gamma} \gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon) \right)^2 \right]. \]  

(39b)

\[ g_{\gamma \epsilon} = -\text{Tr} \left[ \gamma \left( \bar{\gamma} \gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon) \right)^2 \left( \bar{\gamma} \gamma + (z^2 + \epsilon)(\bar{z}^2 + \bar{\epsilon}) \right) \right]. \]  

(39c)

\[ g_{\epsilon \gamma} = \text{Tr} \left[ \gamma \left( \bar{\gamma} \gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon) \right)^2 \left( \bar{\gamma} \gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon) + 4 \theta \bar{z} z + 2 \theta^2 \right) \right]. \]  

(39d)

It is difficult to compute the trace of (39) exactly, but we can investigate the moduli space metric of the two solitons in the case of \(|\gamma|, |\epsilon| \ll \theta\) or \(|\gamma|, |\epsilon| \gg \theta\).

5.1. The case of \(|\gamma|, |\epsilon| \ll \theta\)

In the case of \(|\gamma|, |\epsilon| \ll \theta\), we write (39a) as

\[ g_{\gamma \gamma} = \frac{1}{\theta^2} \text{Tr} \left[ \frac{1}{\bar{\gamma} \gamma / \theta^2 + (a^2 + \epsilon / \theta) (a^2 + \epsilon / \theta)} \left( 1 - \frac{\bar{\gamma} \gamma / \theta^2}{\bar{\gamma} \gamma / \theta^2 + (a^2 + \epsilon / \theta) (a^2 + \epsilon / \theta)} \right) \right]. \]  

(40)

The operator \( a^2 + \epsilon / \theta \) has two zero eigenstates \( e^{\pm \sqrt{a^2 / \theta}} |0\rangle \). These states do not contribute to the trace in (40). Then, we can easily compute the trace in the lowest order of \( \theta^{-1} \). For (39b), (39c) and (39d), we can calculate the trace similarly. Then we obtain

\[ ds^2 = \frac{2 \pi}{\theta} \left( d\bar{\gamma} d\gamma + 2 \frac{d\epsilon}{3} d\bar{\epsilon} \right) + O(\theta^{-2}). \]  

(41)

Therefore, the metric is flat. The singularity which exists in the commutative model at \((\epsilon, \gamma) = (0, 0)\) disappears in the noncommutative model. The same phenomena are known in noncommutative Yang–Mills theories [7,8]. Since the relative coordinate of the solitons is \( ie^{1/2} \), it seems that the geodesic which connects \( \epsilon = \epsilon_0 \) and \( \epsilon = -\epsilon_0 \) represents the right angle scattering. However, (41) is valid only in the region \(|\epsilon| \ll \theta\). To investigate the scattering, a further study of the moduli space is needed.

5.2. The case of \(|\gamma|, |\epsilon| \gg \theta\)

In the case of \(|\gamma|, |\epsilon| \gg \theta\), it is convenient to use the \( \star \)-product formalism to compute (39) rather than the operator formalism. The \( \star \)-product is defined by

\[ f(z, \bar{z}) \star g(z, \bar{z}) = f(z, \bar{z}) \exp \left[ \frac{\theta}{2} (\bar{\delta}_z \partial_{\bar{z}} - \partial_z \bar{\delta}_{\bar{z}}) \right] g(z, \bar{z}). \]  

(42)
We use the formulae
\[ \frac{1}{\tilde{g} \gamma + (\tilde{z}^2 + \tilde{\epsilon})(\tilde{z}^2 + \epsilon)} = \int_0^\infty du \exp\left[-u\{\tilde{g} \gamma + (\tilde{z}^2 + \tilde{\epsilon})(\tilde{z}^2 + \epsilon)\}\right] \]
\[ \to \int_0^\infty du \exp\left[-u\{\tilde{g} \gamma + (\tilde{z}^2 + \tilde{\epsilon}) \ast (\tilde{z}^2 + \epsilon)\}\right]. \tag{43} \]
\[ \frac{1}{[\tilde{g} \gamma + (\tilde{z}^2 + \tilde{\epsilon})(\tilde{z}^2 + \epsilon)]^2} = \int_0^\infty du u \exp\left[-u\{\tilde{g} \gamma + (\tilde{z}^2 + \tilde{\epsilon})(\tilde{z}^2 + \epsilon)\}\right] \]
\[ \to \int_0^\infty du u \exp\left[-u\{\tilde{g} \gamma + (\tilde{z}^2 + \tilde{\epsilon}) \ast (\tilde{z}^2 + \epsilon)\}\right]. \tag{44} \]

where \( \exp_\ast \) is defined by
\[ \exp_\ast(A) = 1 + A + \frac{1}{2!} A \ast A + \frac{1}{3!} A \ast A \ast A + \cdots. \tag{45} \]

For small \( \theta \), we have the following relation
\[ \exp_\ast(A) = \exp(A) + O(\theta^2). \tag{46} \]

Using the formulae (43)–(46), the moduli space metric up to the order \( \theta \) is
\[ g_{\tilde{g} \gamma} = \int d^2x \left[ \frac{1}{\tilde{g} \gamma + (\tilde{z}^2 + \tilde{\epsilon})(\tilde{z}^2 + \epsilon)} \left( 1 - \frac{\tilde{g} \gamma}{\tilde{g} \gamma + (\tilde{z}^2 + \tilde{\epsilon})(\tilde{z}^2 + \epsilon)} \right) \right] + O(\theta^2). \tag{47a} \]
\[ g_{\epsilon \gamma} = -\int d^2x \left[ \frac{\tilde{g} \gamma}{[\tilde{g} \gamma + (\tilde{z}^2 + \tilde{\epsilon})(\tilde{z}^2 + \epsilon)]} + \frac{4 \tilde{g} \gamma z(\tilde{z}^2 + \epsilon)}{[\tilde{g} \gamma + (\tilde{z}^2 + \tilde{\epsilon})(\tilde{z}^2 + \epsilon)]^3} \right] + O(\theta^2). \tag{47b} \]
\[ g_{\gamma \epsilon} = \int d^2x \left[ \frac{\epsilon(z^2 + \epsilon)}{[\tilde{g} \gamma + (\tilde{z}^2 + \tilde{\epsilon})(\tilde{z}^2 + \epsilon)]} + \frac{4 \epsilon \gamma z(\tilde{z}^2 + \epsilon)}{[\tilde{g} \gamma + (\tilde{z}^2 + \tilde{\epsilon})(\tilde{z}^2 + \epsilon)]^3} \right] + O(\theta^2). \tag{47c} \]
\[ g_{\epsilon \epsilon} = \int d^2x \left[ \frac{\gamma(\tilde{z}^2 + \tilde{\epsilon})}{[\tilde{g} \gamma + (\tilde{z}^2 + \tilde{\epsilon})(\tilde{z}^2 + \epsilon)]} + \frac{4 \gamma \epsilon (\tilde{z} + \epsilon)}{[\tilde{g} \gamma + (\tilde{z}^2 + \tilde{\epsilon})(\tilde{z}^2 + \epsilon)]^3} \right] + O(\theta^2). \tag{47d} \]
The integrals appearing in the coefficients of \( \theta \) in (47) converge. Hence, the moduli space metric has the smooth commutative limit \( \theta \to 0 \) with \( \gamma \) and \( \epsilon \) fixed.

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On the dependence of the gauge-invariant field-strength correlators in QCD on the shape of the Schwinger string✩

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Abstract

We study, by numerical simulations on a lattice, the dependence of the gauge-invariant two-point field-strength correlators in QCD on the path used to perform the color parallel transport between the two points. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The gauge-invariant two-point correlators of the field strengths in the QCD vacuum are defined as

$$ D_{\mu\nu,\sigma\tau}^{(C)}(x) = g^2 \langle 0 | \text{Tr} \{ G_{\mu\rho}(0) S(0, x|C) G_{\sigma\varphi}(x) S^\dagger(0, x|C) \} | 0 \rangle, $$

(1.1)

where $G_{\mu\rho} = T^a G^a_{\mu\rho}$ and $T^a$ are the matrices of the algebra of the color group SU(3) in the fundamental representation. The trace in (1.1) is taken on the color indices

$$ S(0, x|C) \equiv \text{Pexp} \left( i g \int_0^x dz^\mu A_\mu(z) \right), $$

(1.2)

with $A_\mu = T^a A^a_\mu$, is the Schwinger phase operator needed to parallel-transport the tensor $G_{\rho\varphi}(x)$ to the point 0. “P” stands for “path ordering” and $C$ is any path from 0 to $x$.

Field-strength correlators play an important role in hadron physics. In the spectrum of heavy $Q\bar{Q}$ bound states, they govern the effect of the gluon condensate on the level splittings [1–3]. They are the basic quantities in models of stochastic confinement of color [4–6] and in the description of high-energy hadron scattering [7–13]. In some recent works [14,15], these correlators have been semi-classically evaluated in the single-instanton approximation and in the instanton dilute-gas model, so providing useful information about the role of the semiclassical modes in the QCD vacuum.

The correlators (1.1) in principle depend on the choice of the path $C$. Usually, in developing the “Stochastic Vacuum Model” (SVM), the path $C$ appearing in Eqs. (1.1) and (1.2) is assumed to be the straight line connecting the points 0 and $x$. Lattice data [16–19],
which are the input for these SVM calculations, also refer to the choice of the straight-line parallel transport. With this choice the Schwinger string (1.2) reads:

\[
S(0, x) = \exp \left( ig \int_0^1 dt \, x^\mu A_\mu(xt) \right). \tag{1.3}
\]

Then, in the Euclidean region, translational, O(4) and parity invariance require the correlator (1.1) to be of the following form [4–6]:

\[
D_{\mu\rho,\nu\sigma}(x) = (\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\sigma}\delta_{\rho\nu}) \left[ D(x^2) + D_1(x^2) \right] + (x_\mu x_\nu \delta_{\rho\sigma} - x_\mu x_\sigma \delta_{\rho\nu} + x_\rho x_\sigma \delta_{\mu\nu} - x_\rho x_\nu \delta_{\mu\sigma}) \frac{\partial D_1(x^2)}{\partial x^2}, \tag{1.4}
\]

where \(D\) and \(D_1\) are invariant functions of \(x^2\).

The functions \(D(x^2)\) and \(D_1(x^2)\) have been directly determined by numerical simulations on a lattice in the quenched (i.e., pure-gauge) theory, with gauge group \(SU(2)\) [16], in the quenched \(SU(3)\) theory in the range of physical distances between 0.1 and 1 fm [17,18] and also in full QCD, i.e., including the effects of dynamical fermions [19].

No analysis exists of what happens for a different choice of the path \(C\) than the straight line, except for qualitative statements that the physics should not strongly depend on the deformations of the path [4–6]. In this Letter we compute the gauge-invariant field-strength correlators for deformed paths (see Fig. 1), in the quenched \(SU(3)\) lattice gauge theory, for the benefit of the developments of the SVM.

2. Computations and results

In principle, every choice of a path \(C_{\{0,x\}}\) connecting the two points 0 and \(x\) in the expression (1.2) for the Schwinger string operator will generate a different field-strength correlator, that we have denoted in Eq. (1.1) as \(D_{\mu\rho,\nu\sigma}^{C}(x)\).

On the lattice we can define a lattice operator \(D_{\mu\rho,\nu\sigma}^{C,L}\), which is proportional to \(D_{\mu\rho,\nu\sigma}^{C}\) in the continuum limit, when the lattice spacing \(a \to 0\). Since the lattice analogue of the field strength is the open plaquette \(\Pi_{\mu\rho}(n)\) (the parallel transport along an elementary square of the lattice lying on the \(\mu\rho\)-plane, starting from the lattice site \(n\) and coming back to \(n\)), \(D_{\mu\rho,\nu\sigma}^{C,L}\) will be defined as [17–19]

\[
D_{\mu\rho,\nu\sigma}^{C,L}(\hat{d}a) = \frac{1}{2} \Re \{ \langle \text{Tr} \left[ \Pi_{\mu\rho}(n) S(n, n + \hat{d}a|C) \times \Omega_{\nu\sigma}(n + \hat{d}a) S^\dagger(n, n + \hat{d}a|C) \right] \rangle \}, \tag{2.1}
\]

where \(\Re\) stands for real part and the lattice operator \(\Omega_{\nu\sigma}(n)\) is given by [17,18]

\[
\Omega_{\nu\sigma}(n) = \Pi_{\nu\sigma}^\dagger(n) - \Pi_{\nu\sigma}(n) - \frac{1}{3} \text{Tr} \left[ \Pi^\dagger_{\nu\sigma}(n) - \Pi_{\nu\sigma}(n) \right]. \tag{2.2}
\]

As explained in Refs. [17,18], the inclusion of the operator \(\frac{1}{2} \Omega_{\nu\sigma}\) (in place of \(\Pi^\dagger_{\nu\sigma}\)) on the right-hand side of Eq. (2.2) ensures that the disconnected part and the singlet part of the correlator are left out, so that we are indeed taking the correlation of two operators with the exchange of the quantum numbers of a color octet.

The lattice site \(n + \hat{d}a\) is the site at a distance \(d\) lattice spacings from \(n\), in the direction of a coordinate axis. The Schwinger string \(S(n, n + \hat{d}a|C)\) is a parallel transport connecting the point \(n\) with the point \(n + \hat{d}a\) along the path \(C\).

![Fig. 1. The different paths for the Schwinger string S that we have considered for our analysis.](image)
We have explored the dependence on the path by measuring the correlators for the various paths shown in Fig. 1. Fig. 1(a) shows the straight-line path (corresponding to the straight-line Schwinger operator in Eq. (1.3)), which, as explained in the introduction, has already been widely analyzed. Fig. 1(b) shows our second choice: the transverse size $\delta$, in units of the lattice spacing $a$, parametrizes the deviation from the straight-line case. We shall label by $b_0$ the paths in Fig. 1(b), for a given value of the transverse size $\delta$. If an average is made over the orientation of the staple in the plane orthogonal to $a$, the correlator will have the same symmetries (translation, $O(4)$ and parity) as the one in Fig. 1(a) and the parametrization (1.4) will still apply, with, of course, different functions $D^{(b_0)}$ and $D^{(b_0)}$. The same considerations also apply to the collection $c_1$ of paths shown in Fig. 1(c), for a given value of the transverse size $\delta$, if we average over the orientation of the plane containing the path. We have chosen the inversion point $p$ in Fig. 1(c) just in the middle of the line $(0, x)$, i.e., $p = x/2$, in order to preserve parity invariance. For other choices, i.e., $p = \alpha x$ with $\alpha \in [0, 1]$ and $\alpha \neq 1/2$, one should average between the path with $p = \alpha x$ and the path with $p = (1 - \alpha)x$ in order to preserve parity.

In both cases, we define [17] the two independent functions $D^{(C)}_\parallel(x^2)$ and $D^{(C)}_\perp(x^2)$ as follows. We go to a reference frame in which $x^\mu$ is parallel to one of the coordinate axes, say $\mu = 0$. Then

$$D^{(C)}_\parallel \equiv \frac{1}{3} \sum_{i=1}^{3} D^{(C)}_{0i,0i}(x) = D^{(C)} + D^{(C)}_1 + x^2 \frac{\partial D^{(C)}_1}{\partial x},$$

$$D^{(C)}_\perp \equiv \frac{1}{3} \sum_{i<j=1}^{3} D^{(C)}_{ij,ij}(x) = D^{(C)} + D^{(C)}_1,$$  

(2.3)

where $D^{(C)} = D^{(C)}(x^2)$ and $D^{(C)}_1 = D^{(C)}(x^2)$ are the two invariant functions entering in the parametrization (1.4) for the given collection of paths $C$.

In the naive continuum limit ($a \to 0$)

$$D^{(C)L}_{\mu\nu,\nu\sigma}(\delta a) \sim a^4 D^{(C)}_{\mu\nu,\nu\sigma}(\delta a) + O(a^6).$$

(2.4)

Making use of the definition (2.3) we can also write, in the same limit,

$$D^{(C)L}_\parallel(d^2a^2) \sim a^4 D^{(C)}_\parallel(d^2a^2) + O(a^6),$$

$$D^{(C)L}_\perp(d^2a^2) \sim a^4 D^{(C)}_\perp(d^2a^2) + O(a^6).$$

(2.5)

In order to remove the lattice artefacts, i.e., the terms $O(a^6)$ in Eqs. (2.4) and (2.5), we shall make use of the cooling technique, described in previous papers (see Refs. [20,21] and [17–19]). Cooling is a local procedure, which affects correlations at distances that grow as the square root of the number of cooling steps, as in a diffusion process. If the distance at which we observe the correlation is sufficiently large, we then expect that lattice artefacts are frozen by cooling long before the correlation is affected: this will produce a plateau in the dependence on the cooling step. Our data are the values of the correlators at the plateau; the error is the typical statistical error at the plateau, plus a systematic error which is estimated as the difference between neighbouring points at the plateau.\footnote{In Figs. 2–5 we have plotted only the points corresponding to a clear plateau in the cooling process. For some points at large distances, our cooling proved to be not long enough to reach the plateau.}

We have measured the correlations on a $16^4$ lattice at distances ranging from 3 to 8 lattice spacings and $\beta = 6$. The value of the lattice spacing $a$ in physical units can be extracted from the value of the string tension [22,23] and it is $\alpha \approx 0.1$ fm at our value of $\beta$. At this value of $\beta$ scaling has been already tested for the case of the straight-line paths in Refs. [17,18]. The data of Refs. [17,18] came from several different values of $\beta$ (including also $\beta = 6$) and two lattice sizes, $16^4$ and $32^4$. They showed no visible dependence neither on the ultraviolet, nor on the infrared scale. We therefore do not expect significant scaling violations.

The results are shown in Figs. 2–5, versus the distance $d$ in units of the lattice spacing $a$. The lines are drawn as an eye-guide. Figs. 2 and 3 refer to the paths in Fig. 1(b), for various values of the transverse size $\delta$ of the staple. Figs. 4 and 5 give the analogous results for the paths shown in Fig. 1(c).

The very result of this Letter is the unexpectedly strong dependence of the correlators on the shape of the path. The correlator with the straight-line path (1.3) (see Fig. 1(a)), has the largest signal (for every distance $d$), compared with the two other choices for the path. Every deformation from the straight-line path seems to produce a sharp decrease of the value of the correlator. What seems to be
Fig. 2. The function $a^4D_{\text{para}}^{(b\eta)}$ versus the distance $d$ in units of the lattice spacing $a$. Circles correspond to the straight-line path of Fig. 1(a); triangles to Fig. 1(b) with $\delta = 1$; squares to Fig. 1(b) with $\delta = 4$; diamonds to Fig. 1(b) with $\delta = 6$. Lines are drawn as eye-guides. The thick continuum line has been obtained using the parameters of the best fit obtained in Ref. [18], Eqs. (2.10) and (2.11).

Fig. 3. The function $a^4D_{\text{perp}}^{(b\eta)}$ versus the distance $d$ in units of the lattice spacing $a$. The symbols are the same as in Fig. 2.
Fig. 4. The function $a^4D_{\parallel}^{(c_3)}$ versus the distance $d$ in units of the lattice spacing $a$. The symbols are the same as in Fig. 2, except that they now refer to the path of Fig. 1(c).

Fig. 5. The function $a^4D_{\perp}^{(c_3)}$ versus the distance $d$ in units of the lattice spacing $a$. The symbols are the same as in Fig. 2, except that they now refer to the path of Fig. 1(c).
path-independent is the slope of the curves, which look parallel to one another in the linear–log plots of Figs. 2–5. This means that the correlation length $\lambda_A$ defined in Refs. [17–19] is approximately path-independent, at least for the classes of paths that we have considered.

In Figs. 2–5 the best fit to the data for straight-line Schwinger string of Ref. [18], Eqs. (2.10) and (2.11), is displayed for comparison (see also Ref. [24], where a critical comparison among different best fits is performed).

In order to quantitatively test the independence of the correlation length $\lambda_A$ on the path $C$, we have tried a best fit to the data for $D_\perp = D + D_1$, at intermediate distances $5 \leq d \leq 8$, i.e., about $0.5 \text{ fm} \leq s \leq 0.8 \text{ fm}$, with the same function used in Ref. [18], Eq. (2.10):

$$D_\perp(x^2) = A_\perp e^{-|x|/\lambda_A} + \frac{A_\perp}{|x|^4} e^{-|x|/\lambda_\perp}. \quad (2.6)$$

Keeping $\lambda_A$ fixed to the value of Ref. [18], Eq. (2.11), i.e., $\lambda_A = 1/182 A_L$, where $A_L \simeq 4.9 \text{ MeV}$ is the lattice scale [22,23], a good fit results with a reasonable $\chi^2/N_{\text{dof}}$. The path-dependence of the correlator reflects in a rather strong path-dependence of the coefficient $A_\perp$ of the exponential term. In the range of distances chosen, the second term of Eq. (2.6) is compatible with zero, within the errors. The coefficient $A_\perp$ changes up to a factor of four, going from $\delta = 0$ to $\delta = 6$ for the collection of paths $b_6$ (Fig. 1(b)); and it changes up to a factor of two, going from $\delta = 0$ to $\delta = 6$ for the collection of paths $c_6$ (Fig. 1(c)).

In Ref. [18] Eq. (2.6) was considered as a split-point regulator of the gluon condensate, which was extracted from the coefficient $A_\perp$ of the exponential term (see also Ref. [24]). It is not clear to us how this determination depends on the shape of the path.

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References

Imprint of SNO neutral current data on the solar neutrino problem

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Abstract

We perform a global analysis in the framework of two active neutrino oscillations of all solar neutrino data, including the recent SNO day and night spectra (comprised of the charged current (CC), elastic scattering (ES) and neutral current (NC) events), the Super-Kamiokande (SK) day and night spectra (from 1496 days) and the updated SAGE results. We find that the Large Mixing Angle (LMA) solution is selected at the 99\% C.L.; the best-fit parameters are $\Delta m^2 = 5.6 \times 10^{-5}$ eV\(^2\) and $\theta = 32^\circ$. No solutions with $\theta \geq \pi/4$ are allowed at the 5\,$\sigma$ C.L. Oscillations to a pure sterile state are excluded at 5.3\,$\sigma$, but a sizeable sterile neutrino component could still be present in the solar flux. © 2002 Elsevier Science B.V. All rights reserved.

The end of the era of the solar neutrino problem is on the horizon. Neutrino oscillations are a compelling explanation of the solar neutrino deficit relative to the Standard Solar Model [1] found by several experiments [2–8]. The SNO experiment has provided cogent evidence in favor of this hypothesis by separately measuring the incident $\nu_e$ flux via the CC reaction and the total active neutrino flux via the NC reaction in the same energy range [7]. With the continued accumulation of solar neutrino data, the LMA solution has become the most favored. If LMA is in fact the solution, the ongoing KamLAND experiment [10] will provide a precise and SSM-independent measurement of the oscillation parameters by measuring the suppression and distortions of the $\bar{\nu}_e$ flux emerging from the surrounding nuclear reactors [11]. Aside from a possible complication of partial oscillations to sterile neutrinos, the thirty year old solar neutrino problem will be solved.

Here we assess the extent to which the recent SNO results home in on an unique oscillation solution by performing a global analysis of all solar neutrino data in a two neutrino framework. There is little difference between a three neutrino analysis and an effective two neutrino analysis because the mixing between the solar and atmospheric scales is known to be small [12]. We first show through a model-independent analysis that pure active to sterile oscillations are excluded at high confidence and then proceed with a global analysis involving only active flavors.

A universal sum rule that holds for arbitrary active–sterile oscillation admixtures has been derived [13] which relates the NC flux at SNO to the CC flux at...
SNO and the neutrino flux measured at SK with elastic scattering:
\[
\Phi_{NC} = \frac{(\Phi_{SK} - (1 - r)\Phi_{CC})}{r},
\]
where \( r \equiv \sigma_{\nu_{\mu},\nu_{\tau}}/\sigma_{\nu_e} \) is the ratio of the \( \nu_{\mu,\tau} \) to \( \nu_e \)
elastic scattering cross-sections on electrons. Above the 5 MeV threshold of the two experiments,
\( r = 1/6.48 \) and the relation becomes
\[
\Phi_{NC} = 6.48 \Phi_{SK} - 5.48 \Phi_{CC}.
\]
The values from the data (assuming an undistorted \(^8\)B spectrum [7–9]), for the left- and right-hand sides of
Eq. (2) are, respectively,
\[
5.09 \pm 0.62, \quad 5.58 \pm 0.71
\]
showing agreement between the data and the sum rule. The fact that the two values have similar percentage
uncertainties (\( \sim 12\% \)) shows that the accuracy of the SNO NC measurement is already comparable to
that which can be inferred from the SNO CC and SK data.

The SNO NC rate is a measure of the flux of active neutrinos in the high energy part of the solar neutrino
spectrum. If an active–sterile admixture is responsible for the solar neutrino deficit,
\[
\frac{\Phi_{NC}}{\Phi_{SSM}} = \beta \frac{\Phi_{SSM} - \Phi_{v_s}}{\Phi_{SSM}},
\]
with measured value (see Table 1),
\[
\frac{\Phi_{NC}}{\Phi_{SSM}} = 1.01 \pm 0.12.
\]

Here \( \beta \) is a normalization of the \(^8\)B flux with respect to the central value of the SSM prediction
\( \Phi_{SSM} = 5.05 \times 10^6 \text{ cm}^{-2} \text{ s}^{-1} \) and \( \beta \Phi_{v_s} \) is the total sterile neutrino flux to which the electron neutrinos
oscillate.\(^1\) Since the measured \( \Phi_{NC} \) is consistent with the SSM prediction, we impose the SSM constraint
\( \beta = 1 \pm 0.18 \), and obtain
\[
\frac{\Phi_{v_s}}{\Phi_{SSM}} = P(v_\nu \rightarrow v_\tau) = -0.01 \pm 0.22.
\]

While the central value of the SNO NC rate suggests a solution with oscillations only to active flavors, the

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\(^1\) As in the SNO analysis [7] we conservatively adopt the SSM \(^8\)B flux of BPB2000 [1] since the \( S_{17}(0) \) determination [14] used in
Ref. [15] is being reanalyzed [16].

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Table 1

<table>
<thead>
<tr>
<th>Measurement</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Homestake</td>
<td>2.56 \pm 0.23 SNU</td>
</tr>
<tr>
<td>GALLEX+GNO</td>
<td>73.3 \pm 4.7 \pm 4.0 SNU</td>
</tr>
<tr>
<td>SAGE</td>
<td>70.8 \pm 5.3 \pm 3.5 SNU</td>
</tr>
<tr>
<td>SK</td>
<td>2.35 \pm 0.02 \pm 0.06 \times 10^6 cm^{-2} s^{-1}</td>
</tr>
<tr>
<td>SNO CC</td>
<td>1.76 \pm 0.06 \pm 0.09 \times 10^6 cm^{-2} s^{-1}</td>
</tr>
<tr>
<td>SNO NC</td>
<td>5.09 \pm 0.44 \pm 0.45 \times 10^6 cm^{-2} s^{-1}</td>
</tr>
<tr>
<td>SNO ADN</td>
<td>0.07 \pm 0.049 \pm 0.013</td>
</tr>
<tr>
<td>( \Phi_{SSM} )</td>
<td>5.05(1 \pm 0.18) \times 10^6 cm^{-2} s^{-1}</td>
</tr>
</tbody>
</table>

uncertainty is too large to rule out a substantial sterile neutrino flux on this basis alone.\(^2\)

To see this freedom from another angle, we return to the approach of Ref. [13]. Of the neutrinos that
oscillate, the fraction that oscillate to active neutrinos is
\[
\sin^2 \alpha = \frac{\Phi_{NC} - \Phi_{CC}}{\beta \Phi_{SSM} - \Phi_{CC}} = 1.01 \pm 0.34.
\]

Thus, the evidence for transitions to active flavors is
at the 3\( \sigma \) C.L. However, large sterile fractions are
allowed even at the 2\( \sigma \) C.L. See Fig. 1 for illustration.

The dark-shaded and light-shaded regions enclose the values of \( \Phi_{NC} \) and \( \Phi_{CC} \) allowed by the SSM at 1\( \sigma \)
for \( \sin^2 \alpha = 1 \) and \( \sin^2 \alpha = 0.5 \), respectively. The lines
through the center of these bands correspond to the central value of the SSM \(^8\)B flux prediction.
The region above the diagonal, \( \Phi_{NC} > \Phi_{CC} \), is forbidden because \( \Phi_{CC} > \Phi_{NC} \) is impossible. The measured
SNO CC and NC rates are marked by a cross. Even a doubling of the widths of the SSM bands and the
SNO rate uncertainties (effectively 2\( \sigma \)) allows large sterile fractions.

We emphasize that a large sterile neutrino flux is
viable not only for an analysis of the total rates, but
also when the SK day and night spectra are included.

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\(^2\) Assuming an LMA solution and a small sterile admixture, a combination of KamLAND and SNO CC data can determine \( \beta \)
to 10\% [17]. This uncertainty is still too large to eliminate the possibility of a significant sterile component in the solar flux.
Fig. 1. Graphical representation of Eq. (7). The diagonal ($\Phi_{NC} = \Phi_{CC}$) corresponds to $\sin^2 \alpha = 0$ or pure sterile oscillations. The dark-shaded band encloses the values of $\Phi_{NC}$ and $\Phi_{CC}$ allowed by the SSM at 1 $\sigma$ for $\sin^2 \alpha = 1$ or pure active oscillations. The light-shaded band is the region allowed by the SSM at 1 $\sigma$ if $\sin^2 \alpha = 0.5$. The SNO NC and CC measurements assuming an undistorted $^8$B spectrum (with 1 $\sigma$ uncertainties) are marked with a cross.

(contrary to the assertion in Ref. [18]). For the issue in question, the effect of imposing the SSM $^8$B flux constraint is equivalent to including the day and night spectra: large values of $\beta$ are not allowed and the best-fit value of $\sin^2 \alpha$ is close to pure active oscillations. That this is the case can be seen from Fig. 4 of Ref. [13] ($^8$B flux constraint applied) and Fig. 2 of Ref. [17] (day and night spectra used). The reason for this correlation is that the day and night spectra rule out a large day–night effect, which is the same region of the LMA solution (low $\Delta m^2$) that favors large $\beta$ for small $\sin^2 \alpha$. Therefore, in our rate analysis, imposing the $^8$B flux constraint should yield the same effect as including the SK day and night spectra, and we see from Eq. (7) that a large sterile flux is still allowed.

In the above analysis, pure active and pure sterile oscillation solutions are treated on an equal footing. If instead we use the a priori $\beta$-independent criterion that for a pure sterile oscillation solution,

$$\Phi_{NC} = \Phi_{CC}$$

(contrary to the assertion in Ref. [18]), then such a solution is allowed only at the 5.3 $\sigma$ C.L. In light of the strong evidence from SNO that oscillations to a pure sterile state are not responsible for the solar anomaly, we hereafter only consider oscillations between active neutrinos. However, oscillations to an active–sterile admixture remains viable and are worthy of investigation.

For a pure active oscillation solution, the NC rate measurement at SNO provides a direct determination of the $^8$B flux produced in the Sun. This frees us from relying on the Standard Solar Model (SSM) prediction

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3 However, the NC rate derived from SNO's day and night spectra is dependent on the shape of the CC energy spectrum, and
of this flux which has a large uncertainty. That is particularly significant because both SK and SNO are mainly sensitive to $^8$B neutrinos. These experiments are also sensitive to hep neutrinos, which according to the SSM constitute a very tiny fraction compared to the $^8$B neutrinos. According to Ref. [19], the hep flux is in fact the same as that of the SSM with an uncertainty of 20%. To avoid any dependence on the SSM $^8$B flux, previous authors have performed $^8$B flux free analyses so as to extract the oscillation parameters [9]. As a result, the CC and NC flux, we adopt the SSM predictions from the $^8$B flux free analyses so as to extract the oscillation parameters directly from the data [20]. With the recent SNO results, this is no longer necessary; the NC rate is itself a $^8$B flux measurement and thus a probe of the oscillation dynamics. We take this approach. Also, we fix the unoscillated hep flux at the SSM value.

Global analyses of solar data are by now quite standard other than the $\chi^2$ function employed for the statistical treatment [21]. We briefly describe the salient features of our analysis. We work in the framework of oscillations between two active neutrino flavors and focus on the Mikheyev–Smirnov–Wolfenstein (MSW) solutions since the vacuum solution is tenuous at best; for completeness we also analyze the data in the limit of a pure vacuum solution. To evaluate the survival probability of solar neutrinos, we consider neutrino production points that are nonradial and consider the possibility of double resonances. The production point region of the different neutrino fluxes is as given by the SSM. We use semi-analytic expressions of the survival probability that have been derived for the almost exponential matter density of the Sun [22], but numerically integrate the evolution equations for the passage of neutrinos through the earth. For the earth-matter density, we make use of the Preliminary Reference Earth Model [23]. The time spent by a detector at a particular zenith angle is given by the exposure function [24], which determines the extent to which earth matter affects the survival probability.

For the unoscillated neutrino fluxes other than the $^8$B flux, we adopt the SSM predictions from the $pp$ chain and CNO cycle. We treat the $^8$B flux normalization $\beta$ as a parameter that is constrained by the SNO NC measurement. The undistorted spectrum shape of the $^8$B neutrinos is given in Ref. [25]. To determine the expected signal at each detector, the fluxes are convoluted with the survival probability at the detector, the neutrino cross-sections and the detector response functions (for SK and SNO). We use the neutrino–electron elastic scattering cross-sections of Ref. [26], and the CC and NC cross-sections of neutrinos on deuterium of Ref. [27]. The response functions are given in Ref. [28] for SK and in Ref. [9] for SNO.

We analyze the event rate from the Homestake experiment [2], the combined rate from GALLEX and GNO [3,4], the latest SAGE event rate [5], the SK day and night spectra corresponding to 1496 days of running [6], and the SNO day and night spectra (which include the CC, ES and NC fluxes) [7–9]. We do not use the SK rate since the SK day and night spectra already include the normalization information [20].

The statistical significance of an oscillation solution is determined by evaluating a suitably chosen $\chi^2$ function. We define a $\chi^2$ that depends sensitively on the SNO NC flux and is completely independent of the SSM $^8$B prediction. It is

$$\chi^2 = \sum_{i,j=1,3} (R^\text{th}_{ij}(\beta) - R^\text{exp}_{ij}) (\sigma^2_{RI})^{-1} (R^\text{th}_{ij}(\beta) - R^\text{exp}_{ij})$$

$$+ \sum_{i,j=1,38} (R^\text{th}_{ij}(\beta) - R^\text{exp}_{ij}) (\sigma^2_{SK})^{-1} (R^\text{th}_{ij}(\beta) - R^\text{exp}_{ij})$$

$$\times (R^\text{th}_{ij}(\beta) - R^\text{exp}_{ij})$$

$$+ \sum_{i,j=1,34} (R^\text{th}_{ij}(\beta) - R^\text{exp}_{ij}) (\sigma^2_{SNO})^{-1} (R^\text{th}_{ij}(\beta) - R^\text{exp}_{ij})$$

$$\times (R^\text{th}_{ij}(\beta) - R^\text{exp}_{ij}).$$

In Eq. (9), $R^\text{th}_{ij}$ and $R^\text{exp}_{ij}$ denote the theoretical and experimental value of the event rate or flux measurement (depending on whether $i$ is an experiment or a spectrum bin) normalized to the expectation for no oscillations. The first term in $\chi^2$ is the contribution of the rate measurements to the analysis. The sum runs from 1 to 3 because the Homestake, GALLEX+GNO and SAGE rates are included. The 3 × 3 matrix $\sigma^2_R$ contains the experimental (statistical and systematic) and theoretical uncertainties. This matrix involves strong correlations arising from solar model parameters [29]. Note that $\sigma^2_R$ does not include a theoretical uncertainty for the $^8$B neutrino flux.

The second term in Eq. (9) is the contribution of the distortions of the SK day and night spectra and implicitly, that of the SK rate. $\sigma^2_{SK}$ is a 38 × 38 matrix that

---

hence on the oscillation parameters [9]. As a result, the CC and NC fluxes are strongly statistically correlated.
contains the statistical and systematic uncertainties of the 19 + 19 spectral bins. The systematic uncertainties include the energy correlated, energy uncorrelated and energy independent contributions.

The third term encapsulates the contributions of the SNO CC, ES and NC rates and distortions of the SNO day and night spectra. The neutron and low energy backgrounds [9] are included in $R_{th}$. The $^8\text{B}$ flux contribution to $R_{th}$ is multiplied by the normalization factor $\beta$ (relative to the central value of the SSM $^8\text{B}$ value), which is constrained by the oscillation parameter dependent NC and CC flux components of the SNO day and night spectra.

In Fig. 2, we show the results of an analysis in which $\Delta m^2$, $\tan^2 \theta$ and $\beta$ have been varied. We only show those regions that are allowed at 3$\sigma$. Then the Small Mixing Angle (SMA) and VAC solutions are absent. We are left with only the LMA and LOW solutions. The contours represent the 95.4% C.L. (2$\sigma$), 99% and 99.73% C.L. (3$\sigma$) allowed regions which correspond to $\Delta \chi^2 = 6.17, 9.21$ and 11.83, respectively. Values of $\theta$ larger than $\pi/4$ are not allowed at the 5$\sigma$ C.L. The best-fit parameter values from the analysis are presented in Table 2.

![Fig. 2. The $2\sigma$, 99% C.L. and 3$\sigma$ allowed regions from a fit to the Homestake, GALLEX+GNO and SAGE rates, and the SK and SNO day and night spectra.](image)

<table>
<thead>
<tr>
<th>Solution</th>
<th>$\Delta m^2$ (eV$^2$)</th>
<th>$\tan^2 \theta$</th>
<th>$\beta$</th>
<th>NC/CC</th>
<th>$\chi^2_{\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LMA</td>
<td>$5.6 \times 10^{-5}$</td>
<td>0.39</td>
<td>1.09</td>
<td>3.19</td>
<td>50.7</td>
</tr>
<tr>
<td>LOW</td>
<td>$1.1 \times 10^{-7}$</td>
<td>0.46</td>
<td>1.03</td>
<td>2.92</td>
<td>59.9</td>
</tr>
<tr>
<td>SMA</td>
<td>$7.9 \times 10^{-6}$</td>
<td>$2.0 \times 10^{-3}$</td>
<td>1.46</td>
<td>4.85</td>
<td>108</td>
</tr>
<tr>
<td>VAC</td>
<td>$1.6 \times 10^{-10}$</td>
<td>0.25(3.98)</td>
<td>0.89</td>
<td>2.46</td>
<td>76.3</td>
</tr>
</tbody>
</table>
LOW at the 99% C.L. The correlation between the CC and NC rates extractable from the SNO day and night spectra goes a long way in choosing the LMA over the LOW. Table 2 shows that the data choose large NC/CC ratios in all MSW regions. An NC/CC ratio $\sim 3$ in the LOW region is at the upper end of the range possible in this region.

The best-fit, minimum and maximum values of the CC day–night asymmetry, $A_{DN}(\beta)$, at 2$\sigma$ for the LMA region are displayed in Table 3.

We next illustrate how well the best-fits of the two contending solutions, LMA and LOW, do in relation to the average survival probabilities of the high energy ($^8$B and $hep$), intermediate energy ($^7$Be, $pep$, $^{15}$O and $^{13}$N) and low energy ($pp$) neutrinos extracted from the experimental rates. For a description of how these probabilities are obtained see Refs. [13,31]. In Fig. 3, we plot each model-independently extracted survival probability at the mean energy of the high, intermediate and low energy neutrinos relevant to the experiments. The vertical error bars result from the experimental uncertainties in the rate measurements and the theoretical uncertainties in the SSM flux predictions. The horizontal error bars span the energy ranges of high, intermediate and low energy neutrinos. The solid and dashed lines superimposed on the plot are flux-weighted survival probabilities at the SK  

Fig. 3. The flux-weighted survival probabilities of the best-fit LMA (solid) and LOW (dashed) solutions from Table 2 in relation to the model-independently extracted values from the data. 

Table 3

<table>
<thead>
<tr>
<th>$A_{DN}(\beta)$</th>
<th>$\Delta m^2$ (eV$^2$)</th>
<th>$\tan^2 \theta$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>At best-fit</td>
<td>0.043 (2$\sigma$)</td>
<td>5.6 x 10$^{-5}$</td>
<td>0.39</td>
</tr>
<tr>
<td>Minimum at 2$\sigma$</td>
<td>0.18 (3$\sigma$)</td>
<td>1.8 x 10$^{-4}$</td>
<td>0.35</td>
</tr>
<tr>
<td>Maximum at 2$\sigma$</td>
<td>0.122 (4$\sigma$)</td>
<td>2.8 x 10$^{-5}$</td>
<td>0.35</td>
</tr>
</tbody>
</table>

$A_{DN} = \frac{N - D}{N + D}$, 

where $D$ and $N$ are the total CC fluxes detected during the days and nights, respectively. Discussions of earth regeneration effects can be found in Ref. [30].
detector corresponding to the best-fit LMA and LOW points of Table 2, respectively. The monoenergetic $^7$Be and pep fluxes are not included in the averaging. The wiggles in the survival probabilities at high energies is a result of earth-matter effects. Aside from the averaging, a similar plot was made in Ref. [32] with pre-SNO NC data.

As a glimpse of the precision in the LMA region that KamLAND data may provide us with in three years, we have simulated data (with an antineutrino energy threshold of 3.3 MeV) at the best-fit LMA point and overlayed the expected 2σ, 99% C.L. and 3σ regions on the currently allowed LMA region in Fig. 4.

We conclude that the LMA solution with nonmaximal mixing and a large active neutrino component in the solar flux is favored at the 99% C.L.; the 2σ allowed region spans $(2.7 \times 10^{-5})-(1.8 \times 10^{-4})$ eV$^2$ in $\Delta m^2$ and $0.27-0.55$ in $\tan^2 \theta$. KamLAND is an ideal experiment to precisely measure oscillation parameters in this range. No solution with $\theta \geq \pi/4$ is allowed at 5σ. Since the mixing is found to be nonmaximal, a determination of the neutrino mass scale via neutrinoless double beta decay is an exciting possibility if neutrinos are Majorana [33]. A large sterile neutrino component in the solar flux remains viable.

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References

Entropy of Horowitz–Strominger black holes due to arbitrary spin fields

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Abstract

Using the membrane model which is based on brick-wall model, we calculated the free energy and entropy of Horowitz–Strominger black holes due to arbitrary spin fields. The result shows that the entropy of scalar field and fermions field has similar formulas. There is only a coefficient difference between them. © 2002 Elsevier Science B.V. All rights reserved.

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Keywords: Horowitz–Strominger black hole; Entropy; Spin

1. Introduction

In theoretical physics, the thermodynamics of black holes remains an enigma. It turns out to be a junction of general relativity, quantum mechanics and statistical physics. ’t Hooft proposed brick wall model in 1985 \cite{1}. By using of this model, he investigated the statistical properties of a free scalar field in the Schwarzschild black hole background and obtained an expression of entropy in terms of the area of the event horizon. Furthermore, the entropy can be written as $S = A_h/4$ when the cut-off satisfies a certain condition. But when the cut-off tends to zero, the entropy would be divergent. However, ’t Hooft deemed that it attributed to the infinite density of states at the vicinity of the event horizon. In quantum mechanics, geometric entropy satisfies the following assumptions: if particles are bosons and obey Bose–Einstein statistics, the entropy obtained is conventionally called the bosons entropy; if the geometric entropy is calculated by counting the fermions particle states, the corresponding entropy is called fermions entropy.

The statistical origin of black holes entropy has aroused much interest among many researchers \cite{2–10} since the mid of 1990s. But up to now, few people have made investigation on the entropy of $P$-brane black holes.

In this Letter, by using membrane model \cite{11} which is based on brick wall model, we obtained the free energy and entropy of gravitational field (spin $s = 2$), electromagnetic field (spin $s = 1$) and neutrino field (spin $s = \frac{1}{2}$). The result shows that the entropy of
scalar field and fermions field have similar formulas. There is only a coefficient difference between them.

The Letter is organized as follows: in Section 2 we deduced the field equation (spin $s = 1, 2, 3$) in Horowitz–Strominger black hole background. In Section 3 we calculated the entropy of Horowitz–Strominger black hole. In Section 4 the entropy of extremal Horowitz–Strominger black hole was obtained. Section 5 comprises the concluding remarks.

2. Field equation

The metric for Horowitz–Strominger black hole is given by [12, 13]

$$dS^2 = \left[1 - \left(\frac{r_+}{r}\right)^2\right]\left[1 - \left(\frac{r_-}{r}\right)^2\right]^{-1} dt^2$$
$$- \left[1 - \left(\frac{r_+}{r}\right)^2\right]\left[1 - \left(\frac{r_-}{r}\right)^2\right]^{-1} dr^2$$
$$- r^2 \left[1 - \left(\frac{r_-}{r}\right)^2\right] \left(\sin^2 \theta + \sin^2 \theta d\phi^2\right), \quad (1)$$

where $r_+, r_-$ are the event horizon and the Cauchy horizon, $b$ is a constant,

$$b = \frac{P}{(P + 1)}, \quad (2)$$

$P$ is the $P$-brane.

Now we deduce the corresponding field equations. Choose the null tetrad as follows

$$l^\mu = \left[1 - \left(\frac{r_+}{r}\right)^2\right]^{-1} \left[1 - \left(\frac{r_-}{r}\right)^2\right]^{-1} \left(1 - \frac{r_+}{r}\right), 1, 0, 0, 0, \quad (3)$$

$$n^\mu = \frac{1}{2} \left[1 - \left(\frac{r_+}{r}\right)^2\right]^{-1} \left[1 - \left(\frac{r_-}{r}\right)^2\right]^{-1} \left(1 - \frac{r_-}{r}\right), 0, 0, 0, \quad (4)$$

$$m^\mu = \frac{1}{\sqrt{2} r(1 - \frac{r_+}{r})^{1/2}} \left(0, 0, 1, \frac{i}{\sin \theta}\right), \quad (5)$$

$$\bar{m}^\mu = \frac{1}{\sqrt{2} r(1 - \frac{r_-}{r})^{1/2}} \left(0, 0, 1, -\frac{i}{\sin \theta}\right)$$

The non-vanishing spin coefficients are [14]

$$\rho = -\frac{1}{r} - \frac{P}{2(P + 1) r^2} \left(1 - \frac{r_-}{r}\right)^{-1},$$

$$\alpha = -\beta = -\frac{1}{2 \sqrt{2} r} \left(1 - \frac{r_-}{r}\right)^{-1} \frac{\partial R}{\partial \Phi},$$

$$\mu = -\frac{1}{2r} \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{1/2},$$

$$\gamma = \frac{r_+}{4 r^2} \left(1 - \frac{r_+}{r}\right)^{1/2} \left(1 - \frac{r_-}{r}\right)^{-1/2},$$

$$\Delta = \frac{P}{4(P + 1) r^2} \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{1/2}, \quad (6)$$

Equations (4), (5) show that the HS (Horowitz–Strominger) metric is of Petrov-type D. Using the result of Teukolsky [15, 16], the field equation of spin $s = \frac{1}{2}, 1, 2$ for the source free case can be combined into

$$[D - (2s + 1) \rho] [\Delta - 2s \gamma + \mu] \Phi_{+s}$$

$$- \left[\delta + (2s - 2) \alpha\right][\bar{\delta} - 2s \alpha]$$

$$+ (2s - 1)(s - 1) \Psi_2] \Phi_{+s} = 0,$$

$$[\Delta + (2s - 2) \gamma + (2s + 1) \mu][D - \rho] \Phi_{-s}$$

$$- [\bar{\delta} + (2s - 2) \alpha][\delta - 2s \alpha]$$

$$+ (2s - 1)(s - 1) \Psi_2] \Phi_{-s} = 0,$$
where $A(r), B(r)$ and the angular equation

$$\left[ \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) \right] pY^m_\ell (\theta, \phi) + \frac{1}{\sin^2 \theta} \partial_\phi^2 pY^m_\ell (\theta, \phi) - \left[ p^2 \cot^2 \theta + p - \lambda^2 \right] pY^m_\ell (\theta, \phi) = 0.$$  

(12)

Eq. (12) shows that $pY^m_\ell$ is the spin-weighted spherical harmonic [17,18], and the separation constant $\lambda$ satisfies

$$\lambda = \sqrt{(l-p)(l+p+1)}.$$  

(13)

Here $l$ and $m$ are integers satisfying the inequalities

$$l \geq |p|, \quad -l \leq m \leq l.$$  

(14)

3. Free energy and entropy

In this section we will calculate the entropy via membrane model. According to membrane model, the entropy of black hole mainly comes from a layer of quantum fields in the vicinity of event horizon. So the boundary condition of the wave function can be written as

$$\Phi(r) = 0,$$  

(15)

when $r < r_+ + \varepsilon$, and

$$\Phi(r) = 0,$$  

(16)

when $r \geq r_+ + \epsilon + \delta$. Where $\epsilon$ and $\delta$ are positive and satisfy the relation $\epsilon \ll r_+$ and $\delta \ll r_+ r_+$ is the radius of event horizon. Set

$$pR_{lE}(r) = \varphi Z,$$  

(17)

and use WKB approximation, we obtain

$$K^2 = (\partial_\rho Z)^2 = \left( \frac{1 - r_+}{r} \right)^2 \left( \frac{1 - r_+}{r} \right)^{-1} E^2 + r^{-2} \left( \frac{1 - r_+}{r} \right)^{-1} \left( \frac{1 - r_+}{r} \right)^{-1} A(r) - \frac{(l-p)(l+p+1)}{r^2} \times \left( \frac{1 - r_+}{r} \right)^{-1} \left( \frac{1 - r_+}{r} \right)^{-1},$$  

(18)

where $K$ is the radial wave number.
The constraint of semi-classical quantum condition imposed on $K$ reads

$$n\pi = \int_{r_+}^{r_+ + \epsilon + \delta} K \, dr.$$  \hspace{1cm} (19)

Note that $n$ is assumed to be a non-negative integer. Energy $E$ is always positive and wave number $K$ is real.

According to the ensemble theory, the free energy is given by

$$\beta F = \langle H \rangle + \sum \ln (1 \pm e^{-\beta \omega}).$$ \hspace{1cm} (20)

where $\beta$ is the inverse of Hawking temperature, i.e.,

$$\frac{k}{2\pi} = \frac{1}{4\pi} (r_+ - r_-) r_+^{-\frac{p+2}{p-1}}.$$ \hspace{1cm} (21)

Take the density of states as continuous and transform summation into integration, we obtain

$$\sum \rightarrow \int_0^{\infty} dE \, g(E),$$ \hspace{1cm} (22)

where $g(E)$ is the density of states, i.e.,

$$g(E) = \frac{d\Gamma(E)}{dE}.$$ \hspace{1cm} (23)

$\Gamma(E)$ is the number of the microscopic states, that is

$$\Gamma(E) = \sum_p \sum_l (2l + 1)n.$$ \hspace{1cm} (24)

Transform the summation of $l$ into integration and require $K \geq 0$, then we obtain

$$\Gamma(E) = \sum_p \int_{0}^{\infty} dE \, g(E) = \frac{1}{\pi} \sum_p \Gamma_r \left( 1 - \frac{r_+}{r} \right)^{-\frac{1}{2}} \left( 1 - \frac{r_-}{r} \right)^{-\frac{1}{2}}$$

$$\times \sum_l \left[ \int_{l_{max}} 1 \, dl \right] (2l + 1) \frac{r_+ + \epsilon + \delta}{r_+ + \epsilon}$$

$$\times \left[ 1 - \frac{r_+}{r} \right]^{-1} \left( 1 - \frac{r_-}{r} \right)^{\frac{p-1}{p+1}} E^2$$

$$+ r^{-2} A(r) - \frac{l(p + 1)}{r^2}.$$

The free energy becomes

$$F = \frac{2}{3\pi} \sum_p \int_{r_+ + \epsilon}^{\infty} dE \left( 1 - \frac{r_+}{r} \right)^{-\frac{1}{2}} \left( 1 - \frac{r_-}{r} \right)^{-\frac{1}{2}} r^2$$

$$\times \left[ \left( 1 - \frac{r_+}{r} \right)^{-1} \left( 1 - \frac{r_-}{r} \right)^{\frac{p-1}{p+1}} E^2$$

$$+ r^{-2} A(r) - \frac{l(p + 1)}{r^2} \right].$$ \hspace{1cm} (25)

In the case of $r_+ \neq r_-$, the free energy is

$$F\text{bosons} = \frac{4\omega \pi^3}{90 \beta^4} \int_{r_+ + \epsilon}^{\infty} dE \left( 1 - \frac{r_+}{r} \right)^{-2}$$

$$\times \left( 1 - \frac{r_-}{r} \right)^{\frac{p-1}{p+1}} r^2.$$ \hspace{1cm} (27)

$$F\text{fermions} = \frac{74\omega \pi^3}{8 \beta^4} \int_{r_+ + \epsilon}^{\infty} dE \left( 1 - \frac{r_+}{r} \right)^{-2}$$

$$\times \left( 1 - \frac{r_-}{r} \right)^{\frac{p-1}{p+1}} r^2.$$ \hspace{1cm} (28)

where $\omega$ is the degeneracy due to spin. For the gravitational and electromagnetic fields we have $\omega = 2$; for the neutrino and scalar fields we have $\omega = 1$.

Considering the relation between entropy and free energy

$$S = \beta^2 \frac{\partial F}{\partial \beta},$$ \hspace{1cm} (29)

we obtain the entropy (when $r_+ \neq r_-)$

$$S\text{bosons} = \omega \frac{7\pi^3}{45 \beta^3 \epsilon} (r_+ - r_-) \left( \frac{r_+ - 1}{r_+} \right)^{\frac{p-1}{p+1}} \frac{\delta}{\epsilon + \delta},$$ \hspace{1cm} (30)

$$S\text{fermions} = \omega \frac{8\pi^3}{45 \beta^3 \epsilon} (r_+ - r_-) \left( \frac{r_+ - 1}{r_+} \right)^{\frac{p-1}{p+1}} \frac{\delta}{\epsilon + \delta}.$$ \hspace{1cm} (31)
Substitute the Hawking temperature

\[ T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi} (r_+ - r_-) \]  

(32)

and the area of outer event horizon

\[ A_+ = 4\pi r_+^2 (r_+ - r_-) \]  

(33)

into Eqs. (30), (31), we obtain

\[ S_{\text{bosons}} = \frac{7}{8} \omega \frac{T_+}{\beta} \frac{1}{4^4} A_+ \frac{\delta}{\epsilon + \delta} \]  

(34)

\[ S_{\text{fermions}} = \frac{7}{8} \omega \frac{T_+}{\beta} \frac{1}{4^4} A_+ \frac{\delta}{\epsilon + \delta} \]  

(35)

When we choose

\[ \frac{\delta}{\epsilon (\delta + \epsilon)} = \frac{\omega T_+}{90} \]  

(36)

the entropy can be written as

\[ S_{\text{bosons}} = \frac{1}{8} A_+ \]  

(37)

\[ S_{\text{fermions}} = \frac{7}{8} \frac{1}{4} A_+ \]  

(38)

4. The extremal case

In the end let us consider the entropy of extremal black hole. For the extreme black hole \((r_+ = r_-)\), the area of the event horizon is zero. However, the entropy is not zero,

\[ s_{\text{bosons}}^{\text{ext}} = \omega \frac{7(P + 1)\pi^3}{135} \left( \frac{r_+}{\beta} \right)^3 \times \left( \frac{r_+}{\epsilon} \right)^{\frac{3}{2}} \left( \frac{\delta + \epsilon}{\epsilon} \right)^{\frac{1}{2}} \]  

(39)

\[ s_{\text{fermions}}^{\text{ext}} = \omega \frac{7(P + 1)\pi^3}{135} \left( \frac{r_+}{\beta} \right)^3 \times \left( \frac{r_+}{\epsilon} \right)^{\frac{3}{2}} \left( \frac{\delta + \epsilon}{\epsilon} \right)^{\frac{1}{2}} \]  

(40)

5. Discussion

In conclusion, we have studied the quantum entropy in the background of Horowitz–Strominger black hole. Using the membrane model which is based on brick-wall model and the WKB approximation, we calculated the free energy and entropy of Horowitz–Strominger black hole due to arbitrary spin fields (spin \(s = 1, 2, \frac{1}{2}\)). Both Eqs. (37), (38) and Eqs. (39), (40) show that whether the black hole is extremal or not its entropy due to scalar field and fermions field has similar formulas. There is only a coefficient \(\frac{\delta}{\epsilon}\) difference between them. It should be noted that fermions entropy is \(\frac{\delta}{\epsilon}\) of the bosons in Ref. [10] where the degeneracy of the fields is not considered. The case of \(P = 0\) in our result is exactly for Reissner–Nordström black hole.

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References

New measurements of the $D^0$ and $D^+$ lifetimes

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Abstract

A high statistics sample of photoproduced charm particles from the FOCUS (E831) experiment at Fermilab has been used to measure the $D^0$ and $D^+$ lifetimes. Using about 210,000 $D^0$ and 110,000 $D^+$ events we obtained the following values: $409.6 \pm 1.1$ (statistical) $\pm 1.5$ (systematic) fs for $D^0$ and $1039.4 \pm 4.3$ (statistical) $\pm 7.0$ (systematic) fs for $D^+$. © 2002 Elsevier Science B.V. All rights reserved.

The study of the charm hadron lifetimes has been fundamental for our understanding of the heavy quark decays. The most important contribution is the spectator diagram which contributes equally to the widths of all hadrons of a given flavour [1]. In the early days of charm physics it was quite a surprise when the experiments measured a large value for $\tau_{D^+}/\tau_{D^0}$. It is generally believed that this large ratio ($\sim 2.5$) is mainly due to the destructive interferences between different quark diagrams that contribute only to $D^+$ decays. The increasingly precise measurements of the heavy quark lifetimes have stimulated the development of theoretical models, like the Heavy Quark Theory [2], which are able to predict successfully the rich pattern of charm hadron lifetimes, that span one order of magnitude from the longest lived ($D^+$) to the shortest lived ($\Omega_c^0$).

In this Letter we present the most accurate measurement to date of the lifetimes of the $D^+$ and $D^0$. Although the accuracy reached by the previous experiments is remarkable, for example the $D^0$ lifetime is known with an uncertainty of $\sim 1\%$ [3], we think that a more precise determination of the $D^0$ lifetime would be needed. For example it would allow a more accurate check for the determination of the lifetime difference in the neutral $D$-meson system (to evaluate the parameter $\gamma = \Delta \Gamma/2 \Gamma$ of the $D^0-\bar{D}^0$ mixing [4]).

Charmed particles were produced by the interaction of high energy photons, obtained by means of bremsstrahlung of electron and positron beams (with typically 300 GeV endpoint energy), with a beryllium oxide target. The mean energy of the photon beam was approximately 180 GeV. The data were collected at Fermilab during the 1996–1997 fixed-target run. More than $6.3 \times 10^9$ triggers were collected from which more than 1 million charmed particles have been reconstructed.

The particles from the interaction are detected in a large-aperture magnetic spectrometer with excellent vertex measurement, particle identification and calorimetric capabilities. The vertex detector consists of two systems of silicon microvertex detectors. The upstream system consists of 4 planes interleaved with the experimental target [5] (2 target slab upstream then 2 silicon planes and the replica of this setting), while the downstream system consists of 12 planes of microstrips arranged in three views. These detectors provide high resolution separation of primary (production) and secondary (decay) vertices with an average proper time resolution of $\sim 35$ fs. The momentum of the charged particles is determined by measuring their deflections in two analysis magnets of opposite polarity with five stations of multiwire proportional chambers. Kaons and pions in the $D$-meson final states are well separated up to 60 GeV/c of momentum using three multicell threshold Čerenkov counters.

The final states are selected using a candidate driven vertex algorithm [6]. A secondary vertex is formed from the reconstructed tracks and the momentum vector of the $D$ candidate is used as a seed to intersect the other tracks in the event to find the primary vertex. Once the production and decay vertices
are determined, the distance $\ell$ between them and the relative error $\sigma_\ell$ are computed. Cuts on the $\ell/\sigma_\ell$ ratio are applied to extract the $D$ signals from the prompt background. The primary and secondary vertex are required to have a confidence level greater than 1%.

The vertices (primary and secondary) have to lie inside a fiducial volume and the primary vertex must be formed with at least two reconstructed tracks in addition to the seed track. The Čerenkov particle identification cuts used in FOCUS are based on likelihood ratios between the various stable particle identification hypotheses. These likelihoods are computed for a given track from the observed firing response (on or off) of all cells within the track’s $\beta=1$ Čerenkov cone for each of our three Čerenkov counters. The product of all firing probabilities for all cells within the three Čerenkov cones produces a $\chi^2$-like variable $W_i = -2\ln(\text{Likelihood})$ where $i$ ranges over the electron, pion, kaon and proton hypotheses (see Ref. [7] for more details). We require $\Delta K = W_\pi - W_K > 1$, called kaonicity, for the tracks reconstructed as a kaon. Analogously the tracks reconstructed as pions have a pionicity, $\Delta_\pi = W_K - W_\pi$, exceeding 1.

In Fig. 1(a) and (b) we show the invariant mass plots obtained with this set of cuts and with $\ell/\sigma_\ell > 9$ for the decay modes $D^0 \to K^-\pi^+$ and $D^0 \to K^-\pi^+\pi^+\pi^-$, respectively (throughout this Letter the charge conjugate state is implied). Fig. 1(c) is the invariant mass plot for the decay mode $D^+ \to K^-\pi^+\pi^+$ with $\ell/\sigma_\ell > 14$. This set of cuts is chosen to optimize the yield and the background underneath the signal ($S/N$ ratio).

From a binned maximum likelihood fit we find $139433 \pm 520\, D^0 \to K^-\pi^+$, $68274 \pm 360\, D^0 \to K^-\pi^+\pi^+\pi^-$ and $109877 \pm 385\, D^+ \to K^-\pi^+\pi^+$ candidates. The fits are fit with two Gaussians with the same mean but different widths to take into account the different resolution in momentum of the tracks passing through one or two magnets (see Ref. [6] for more details) of our spectrometer plus a 2nd order polynomial. The low mass region is excluded in the fit to avoid possible contamination due to other hadronic charm decays involving an additional $\pi^0$.

The lifetime is measured using a binned maximum likelihood fitting technique [8]. A fit is made to the reduced proper time distribution in the signal region. The reduced proper time is defined by $t' = (\ell - N\sigma_\ell)/(\beta\gamma c)$ where $\ell$ is the distance between the primary and the secondary vertex, $\sigma_\ell$ is the resolution.

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2 The reason for this cut is the presence of a trigger counter just upstream of the second microstrip device, therefore we define the fiducial volume as the region between the first slab of the experimental target and this trigger counter.
on $\ell$ and $N$ is the minimum “detachment” cut required to extract the signal. Our vertexing algorithm provides very uniform reduced proper time acceptance even at very low reduced proper times. If absorption and acceptance corrections are small enough that they can be neglected, and if $\sigma_\ell$ is independent of $\ell$, one can show that the $t'$ distribution for decaying charged particles will follow an exponential distribution. These assumptions are very nearly true in FOCUS [6].

The signal region reduced proper time distributions (indicated with dashed lines in Fig. 1) are formed from events with invariant mass within $\pm 2\sigma$ of the mean $D$ mass; $\sigma$ stands for weighted sigma because the invariant mass plots were fitted with two Gaussians. The dependence of the lifetime measurement on the choice of the signal and background region is discussed later in the text. The binned maximum likelihood method allows direct use of the proper time distribution of the data above and below the $D$ mass peak to represent the background underneath the signal instead of using a background parametrization. We have chosen two sidebands starting $4\sigma$ above and below the mean $D$ mass, each half as wide as the signal region (indicated with dotted lines in Fig. 1). The signal and background reduced proper time distributions are binned in proper time wide bins (200 fs) spanning about 10 nominal lifetimes.

The observed numbers of events in a reduced proper time bin $i$ (centered at $t'_i$) in the signal and sideband histograms are labeled $s_i$ and $b_i$, respectively. The predicted number of events $n_i$ in a reduced proper time bin is given by

$$n_i = (N_i - B) \frac{f(t'_i) \exp(-t'_i/\tau)}{\sum_j f(t'_j) \exp(-t'_j/\tau)} + B \frac{b_i}{\sum_j b_j},$$

(1)

where $N_i$ is the total number of events in the signal region, $B$ is the total number of background events in the signal region and $f(t'_i)$ is a correction function. The fit parameters are $B$ and $\tau$. The $f(t'_i)$ correction function, derived from a Monte Carlo simulation, corrects the reduced proper time evolution of the signal for the effects of geometric acceptance, reconstruction efficiency, analysis cuts, hadronic absorption and decay of charm secondaries. The use of a multiplicative $f(t'_i)$ correction, rather than an integral over a resolution factor, is justified since our reduced proper time resolution ($\sim 35$ fs) is much less than the $D^0$ or $D^+$ lifetime.

A separate $f(t')$ correction function is used for each of the three decay modes. Our Monte Carlo simulation includes the Pythia [9] model for photon–gluon fusion and incorporates a complete simulation at the digitization level of all detector and trigger systems and includes all known multiple scattering and particle absorption effects. The Monte Carlo was run with $\sim 15$ times the statistics of the experiment.

The plots (a) of Figs. 2, 3 and 4 show the correction function $f(t')$ for the three decay modes in bins of reduced proper time. The $f(t')$ function is obtained by dividing the simulated reconstructed charm yield in each bin by the input decay exponential. The fall off in $f(t')$ for the $D^+$ case is due to the exclusion of long lived events with vertices downstream of the fiducial volume.

A factor $L_{bg}$ is included in the likelihood function in order to relate $B$ to the number of background events expected from the side band population. The background level is thereby jointly determined from the invariant mass distribution and from the reduced proper time evolution in the side bands. The likelihood function is then given by:

$$L = L_{signal} L_{bg},$$

(2)

where

$$L_{signal} = \prod_{i=1}^{\text{bins}} \frac{n_i \exp(-n_i)}{s_i !},$$

(3)

and

$$L_{bg} = \frac{B N_{bg}}{N_{bg} !} \exp(-B),$$

(4)

with $N_{bg} = \sum_i b_i$ (we assume a linear background because the 2nd order term of the polynomial is negligible).

The plots (b) of Figs. 2, 3 and 4 show the predicted events (histogram) superimposed on the observed events, the background events $b_i$ are also superimposed. In plots (c) of Figs. 2, 3 and 4 a pure exponential function with the fitted lifetime is superimposed on the background subtracted and $f(t')$ corrected $t'$ distribution.

The measured lifetimes are: $408.75 \pm 1.42$ fs for $D^0 \rightarrow K^- \pi^+$, $411.25 \pm 1.95$ fs for $D^0 \rightarrow K^- \pi^+ \pi^+ \pi^-$ and $1039.42 \pm 4.28$ fs for $D^+ \rightarrow K^- \pi^+ \pi^+$. 
Our lifetime measurements have been tested by modifying each of the vertex and Čerenkov cuts individually. For example, in Fig. 5 one can see the measured lifetimes versus the $\ell/\sigma_\ell$ detachment cut. The measured lifetimes of the three decay modes are stable with respect to $\ell/\sigma_\ell$. Our measured lifetimes show no significant variation with the cuts employed to extract the signal.

We check our $D^0$ lifetime evaluation partitioning the total sample into $D^*-tag$ and $no-tag$ according to their origin. The obtained lifetimes are in very good agreement with the reported values.
To further check our lifetime measurements we have used tight cuts in order to extract a signal with virtually no background. Fig. 6 shows the invariant mass plots of the three decay modes for this set of cuts. The lifetime measurements from these samples, $411.26 \pm 3.11$ fs for $D^0 \rightarrow K^- \pi^+$, $413.10 \pm 4.80$ fs for $D^0 \rightarrow K^- \pi^+ \pi^-$, and $1036.66 \pm 8.00$ fs for $D^+ \rightarrow K^- \pi^+ \pi^-$, are in very good agreement with our previous determination.

Systematic uncertainties for these lifetime measurements can arise from several sources. We performed a detailed study to analyze these sources.

There is an uncertainty due to the absolute time scale which was determined by studying the absolute length and momentum scale in the experiment (see...
Fig. 6. Invariant mass distributions obtained using tight cuts for:

(a) $K^-\pi^+$, (b) $K^-\pi^+\pi^+\pi^-$ and (c) $K^-\pi^+\pi^+\pi^-$. The functions used to fit the data (solid curves) are similar to those of Fig. 1 and the numbers quoted are the yields.

Ref. [10] for more details. We estimate an uncertainty of $\pm0.11\%$ for this source.

Another source of systematic uncertainty is linked to the detector and reconstruction efficiency. The $f(t')$ corrects the reduced proper time distribution for these effects, but an uncertainty could originate if there is a mismatch between the Monte Carlo simulation and the data. We have verified that our Monte Carlo accurately reproduces the distributions of several relevant variables, such as the longitudinal and transverse momenta, the multiplicity of the production vertex, the measured decay length and the estimated error on the reconstructed proper time. In order to estimate this uncertainty we split our total sample into independent subsamples depending on $D$ momentum, particle versus antiparticle and the different periods in which the data were collected. The splits into $D$ momentum and charge conjugation are the natural tests to reveal a possible mismatch between data and Monte Carlo because they probe the response of the detector. The main reason for the period dependence is the insertion of the upstream silicon system (which improved the resolution) in the target region during the 1997 fixed-target run period. A technique, employed in FOCUS and in the predecessor experiment E687, modeled after the S-factor method from the Particle Data Group [3], was used to try to separate true systematic variations from statistical fluctuations. The lifetime is evaluated for each of the 8 ($=2^3$) statistically independent subsamples and a **scaled variance** is calculated; the **split sample** variance is defined as the difference between the reported statistical variance and the scaled variance if the scaled variance exceeds the statistical variance. This contribution to the systematic error is reported as **split sample** in Table 1.

The reported lifetimes are obtained with a particular set of fitting conditions. For example the width of the bins or the range of the $t'$ distribution. This is a particular choice and the lifetime should be independent of it. We investigated if this could be a possible source of uncertainty by varying the width of the bins, the upper limit of the $t'$ distribution, the location and the width of the sidebands, and the width of the signal region. In addition we studied the effect of using only the low or only the high mass sideband as well as the effect of eliminating the background term in the likelihood (the second term in Eq. (2)). For all these **fit variants** the sample variance is used as an estimate of this uncertainty because the various measurements are all taken as a priori likely.

A further source of systematic error can be due to uncertainties in the target absorption corrections. Two effects are present: hadronic absorption of decay daughters which would increase the fitted lifetime if neglected and absorption of the $D$ in the target which would tend to decrease the fitted lifetime if not taken into account. In our Monte Carlo all known
Table 1
Contributions in percent to the systematic uncertainty

<table>
<thead>
<tr>
<th>Source</th>
<th>$D^0_{K^-\pi^+}$</th>
<th>$D^0_{K^+\pi^-\pi^+}$</th>
<th>$D^0_{\text{combined}}$</th>
<th>$D^+_{K^-\pi^+\pi^+}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absolute time scale</td>
<td>0.11%</td>
<td>0.11%</td>
<td>0.11%</td>
<td>0.11%</td>
</tr>
<tr>
<td>Split sample</td>
<td>0</td>
<td>0.37%</td>
<td>0.13%</td>
<td>0</td>
</tr>
<tr>
<td>Fit variant</td>
<td>0.19%</td>
<td>0.14%</td>
<td>0.17%</td>
<td>0.15%</td>
</tr>
<tr>
<td>Absorption</td>
<td>0.11%</td>
<td>0.20%</td>
<td>0.20%</td>
<td>0.38%</td>
</tr>
<tr>
<td>Acceptance</td>
<td>0.21%</td>
<td>0.21%</td>
<td>0.21%</td>
<td>0.52%</td>
</tr>
<tr>
<td>Total systematic error</td>
<td>0.32%</td>
<td>0.50%</td>
<td>0.38%</td>
<td>0.67%</td>
</tr>
</tbody>
</table>

Table 2
Measured lifetimes ($\times 10^{-12}$ s)

<table>
<thead>
<tr>
<th>Experiment</th>
<th>$D^0$</th>
<th>$D^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>E687 [8]</td>
<td>$0.413 \pm 0.004 \pm 0.003$</td>
<td>$1.048 \pm 0.015 \pm 0.011$</td>
</tr>
<tr>
<td>CLEO II [12]</td>
<td>$0.4085 \pm 0.0041 \pm 0.0034$</td>
<td>$1.0336 \pm 0.0221 \pm 0.0099$</td>
</tr>
<tr>
<td>E791 [13]</td>
<td>$0.413 \pm 0.003 \pm 0.004$</td>
<td>$1.0334 \pm 0.0127$</td>
</tr>
<tr>
<td>This measurement</td>
<td>$0.4096 \pm 0.0011 \pm 0.0015$</td>
<td>$1.0394 \pm 0.0043 \pm 0.0070$</td>
</tr>
</tbody>
</table>

Particle absorption effects (values from Particle Data Group [3]) have been simulated for the decay daughters. For the charm hadron absorption our simulation assumes 1/2 of the cross section for neutrons. We estimate the uncertainty of this contribution by varying the charm cross-section by 50% and the daughter particle interaction cross-sections by 25% in the Monte Carlo simulation. We verify that these estimates, reported in Table 1, are consistent with a determination of this contribution obtained comparing the lifetimes of decays with the $D$ produced in the upstream half of each target with those produced in the downstream half of the same target (see Ref. [11] for more details on the target setup). Each partition represents a different mixture of hadronic absorptions of decay daughters and $D$ mesons.

The acceptance could be another source of uncertainty. We analyzed this effect determining the $D^0$ lifetime without the correction function $f(t')$ and removing the fiducial volume cut from the set of analysis cuts (this makes the correction function almost flat). We obtained results in good agreement with the reported values. This check is not possible for the $D^+$ because of the longer lifetime, a geometric acceptance correction is always needed. A study was performed in FOCUS (see Ref. [10] for more details) comparing the acceptance part of the Monte Carlo correction with the high statistics $K_S^0 \rightarrow \pi^+\pi^-$ decays. The result of this study showed an excellent agreement between the acceptance observed in the data and the acceptance simulated by the Monte Carlo; however we assess a 2% uncertainty due to the finite statistics. This 2% uncertainty in the $f(t')$ correction function gives a 0.21% and 0.52% uncertainty in the lifetime of $D^0$ and $D^+$, respectively.

The finite Monte Carlo statistics give a negligible contribution to the systematic uncertainty.

Table 1 shows the contributions of each of these sources to the total systematic uncertainty. For the combined $D^0$ lifetime the systematic error (also shown in Table 1) is obtained combining the individual sources of systematic uncertainty from $D^0 \rightarrow K^-\pi^+$ and $D^0 \rightarrow K^-\pi^+\pi^-\pi^+$. We assume the absolute time scale and the absorption correlated. To obtain the final systematic error the uncertainties from the different sources are then added in quadrature.

The final lifetime values are $409.62 \pm 1.15$ (statistical) $\pm 1.55$ (systematic) fs for $D^0$ (weighted average) and $1039.42 \pm 4.28$ (statistical) $\pm 6.97$ (systematic) fs for $D^+$. A final check was performed for the $D^0$ lifetime. We compute the lifetime using a combined likelihood, that is forming a global likelihood for $K^-\pi^+$ and $K^-\pi^+\pi^-\pi^+$. The fit parameters are the two distinct backgrounds and one lifetime. The
lifetime from the combined likelihood, 409.62 ± 1.15 (statistical), is identical to the reported value.

This measurement of the $D^0$ lifetime value is in very good agreement with the result we obtained in our lifetime difference paper [4].

In conclusion we have measured the lifetimes of the $D^0$ and $D^+$ mesons. Our results are reported in Table 2 along with a comparison with the most recent published measurements.

Our results will significantly decrease the errors on the current world average values for the $D^0$ and $D^+$ lifetimes.

From our measurements of the $D^0$ and $D^+$ lifetimes we can update the determination of the ratio $\frac{\tau(D^+)}{\tau(D^0)}$: 2.538 ± 0.023. This result and the inclusive semileptonic branching ratios [3], $D^+ \rightarrow eX = (17.2±1.9)\%$ and $D^0 \rightarrow eX = (6.75±0.29)\%$, show that the $D^0$ and $D^+$ semileptonic decay widths are nearly equal:

$$\frac{\Gamma(D^0 \rightarrow eX)}{\Gamma(D^+ \rightarrow eX)} = \frac{B(D^0 \rightarrow eX) \times \tau(D^+)}{B(D^+ \rightarrow eX) \times \tau(D^0)} = 1.00 ± 0.12.$$  (5)

This implies that differences in the total decay widths between $D^0$ and $D^+$ must be due to differences in the hadronic decay sector.

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References

Experimental study of $\rho \to \pi^0\pi^0\gamma$ and $\omega \to \pi^0\pi^0\gamma$ decays


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Abstract

The $e^+e^- \to \pi^0\pi^0\gamma$ process was studied in the SND experiment at VEPP-2M $e^+e^-$ collider in the energy region $0.60$–$0.97$ GeV. From the analysis of the energy dependence of measured cross section the branching ratios $B(\omega \to \pi^0\pi^0\gamma) = (7.6^{+1.4}_{-0.8} \pm 0.6) \times 10^{-5}$ and $B(\rho \to \pi^0\pi^0\gamma) = (4.1^{+1.0}_{-0.9} \pm 0.3) \times 10^{-5}$ were obtained.

PACS: 13.65.+i; 14.40.Cs

1. Introduction

In 1998 and 2000 the experiments with Spherical Neutral Detector (SND) [1] at VEPP-2M $e^+e^-$ collider were carried out in the energy range $E = 360$–$970$ MeV where cross section of $e^+e^-$ annihilation into hadrons is determined by the $\rho$ and $\omega$ meson decays. The integrated luminosity of 9 pb$^{-1}$ collected in the experiment corresponds to $3.5 \times 10^6$ and $7 \times 10^6$ produced $\rho$ and $\omega$ mesons, respectively. One of the goals of the experiment was the investigation of the rare process

$$e^+e^- \to \rho, \omega \to \pi^0\pi^0\gamma.$$  \hspace{1cm} (1)

Our preliminary study [2] of the process (1) was based on 1/3 of collected statistics. Its results were the first measurement of

$$B(\rho \to \pi^0\pi^0\gamma) = (4.8^{+1.4}_{-1.8} \pm 0.2) \times 10^{-5}$$

and the measurement of

$$B(\omega \to \pi^0\pi^0\gamma) = (7.8 \pm 2.7 \pm 2.0) \times 10^{-5}$$

confirming the only previous measurement of this decay by GAMS:

$$B(\omega \to \pi^0\pi^0\gamma) = (7.2 \pm 2.5) \times 10^{-5}$$  \hspace{1cm} [3].

The theoretical study of the $\rho, \omega \to \pi^0\pi^0\gamma$ decays was begun by P. Singer in Ref. [4] where the transitions via $\omega\pi^0$ (Fig. 1(a)) and $\rho^0\pi^0$ intermediate states were suggested. The vector meson dominance (VMD)
calculation with these intermediate states leads to branching ratios $\sim 1 \times 10^{-5}$ and $\sim 3 \times 10^{-5}$ for $\rho \rightarrow \pi^{0}\pi^{0}\gamma$ and $\omega \rightarrow \pi^{0}\pi^{0}\gamma$, respectively [5]. For $\rho \rightarrow \pi^{0}\pi^{0}\gamma$ decay another mechanism through the pions loops (Fig. 1(b)) is also possible [5]. The branching ratios expected for this mechanism in chiral loops (Fig. 1(b)) is also possible [5]. The significantly larger branching ratios expected for this mechanism in chiral loops (Fig. 1(b)) is also possible [5]. The prediction of the chiral models [6–9] are in agreement with our previous experimental result. The significantly larger value of $B(\pi^{0}\pi^{0}\gamma)$ was obtained in Ref. [10] using $\sigma$ pole model. Their result contradicts to existing experimental data.

In $\omega \rightarrow \pi^{0}\pi^{0}\gamma$ decay the contribution of pion loops is $G$-parity suppressed while the contribution of kaon loop is small due to large kaon mass. Therefore, it is assumed that the $\omega \rightarrow \pi^{0}\pi^{0}\gamma$ decay proceeds through $\rho^{0}\pi^{0}$ intermediate state. The first measurement

$$B(\omega \rightarrow \pi^{0}\pi^{0}\gamma) = (7.2 \pm 2.5) \times 10^{-5} \quad [3]$$

significantly exceeded the existing prediction of VMD model: $3 \times 10^{-5}$ [5]. An attempt to explain this discrepancy was done in Ref. [11] where $\rho-\omega$ mixing was taken into account and coupling constants were extracted from experimental values of $\Gamma(\omega \rightarrow 3\pi)$ and $\Gamma(\rho^{0} \rightarrow \pi^{0}\gamma)$. As a result the estimated value of $B(\omega \rightarrow \pi^{0}\pi^{0}\gamma)$ increased up to $(4.6 \pm 1.1) \times 10^{-5}$. Similar results $(4.5–4.7) \times 10^{-5}$ were then obtained in Refs. [6,9]. In Ref. [12] the large experimental value of $B(\omega \rightarrow \pi^{0}\pi^{0}\gamma)$ was explained by additional contribution of the $\omega \rightarrow \sigma\gamma$ transition and used to extract the value of $g_{\omega\sigma\gamma}$ coupling constant.

In the present work we present the experimental results on $B(\omega \rightarrow \pi^{0}\pi^{0}\gamma)$ and $B(\rho \rightarrow \pi^{0}\pi^{0}\gamma)$ based on full SND data sample.

2. Event selection

For analysis five-photon events with the energy deposition in the calorimeter
and the total momentum measured by the calorimeter

$$P_{\text{tot}} < 0.15 \cdot E/c$$

were selected. Here $E$ is $e^+e^-$ center of mass energy.

Due to high beam background rate in 5% of events fake photons appear. This makes possible for lower photon multiplicity QED processes $e^+e^- \rightarrow 2\gamma$, $3\gamma$, $4\gamma$, and $\rho, \omega \rightarrow \pi^0\gamma, \eta\gamma \rightarrow 3\gamma$ decays to imitate five-photon events producing main background contribution for the process under study. Detector response to the beam background was studied using special events recorded with a random generator trigger. The information on the fired detector channels in these events was used for simulation of the process under study and the background processes. Considerable suppression (by a factor of 8) of the background from events with fake photons was achieved by imposing the following cuts:

$$E_{\text{min}} > 30 \text{ MeV}, \quad 30^\circ < \theta_{\text{min}} < 150^\circ,$$

where $E_{\text{min}}$ and $\theta_{\text{min}}$ are the energy and polar angle of the softest photon in an event. These cuts reduce the detection efficiency for the process under study by 25%. Another background source is the $e^+e^- \rightarrow \eta\gamma \rightarrow 3\pi^0\gamma \rightarrow 7\gamma$ reaction producing five-photon events mainly due to the merging of near photons. To suppress this background, the parameter $\chi_\gamma$ describing transverse energy deposition profile of the detected photon [13] was used. The cut

$$\chi_\gamma < 5$$

suppresses the $e^+e^- \rightarrow \eta\gamma\gamma$ background by a factor of 2 with a 5% loss of actual 5-photon events.

Further selection was based on the kinematic fitting of the events. Compatibility of the event kinematics with $e^+e^- \rightarrow 5\gamma$ and $e^+e^- \rightarrow 3\gamma$ hypotheses was checked. For the $3\gamma$ hypothesis two out of five photons were considered spurious: all $3\gamma$ subsets were tested and the best one with minimum $\chi^2$ value was selected. As a result of kinematic fitting the $\chi^2$ values, $\chi_{3\gamma}$ and $\chi_{5\gamma}$, were calculated for both hypotheses. The cut

$$\chi_{3\gamma} > 20$$

practically eliminates $e^+e^- \rightarrow 2\gamma, 3\gamma$ background with the loss only 2.5% of the events of the process under study. Fig. 2 depicts the $\chi_{3\gamma}$ distribution of the experimental and simulated events. The following cut was imposed on this parameter:

$$\chi_{3\gamma} < 20.$$  \hspace{1cm} (7)

Finally, the events with two $\pi^0$ mesons were selected. To do this the kinematic fit in $e^+e^- \rightarrow \pi^0\pi^0\gamma$ hypothesis was performed and the following cut was imposed:

$$\chi_{\pi\pi\gamma} - \chi_{5\gamma} < 10.$$  \hspace{1cm} (8)

Here $\chi_{\pi\pi\gamma}$ is the $\chi^2$ value of the kinematic fit for the $e^+e^- \rightarrow \pi^0\pi^0\gamma$ hypothesis. The $\chi_{\pi\pi\gamma} - \chi_{5\gamma}$ distributions for the experimental and simulated events are shown in Fig. 3.

The difference between distributions of the background events and events of the process under study (Figs. 2 and 3) was used to estimate the accuracy of the background calculation. The experimental distribution was fitted by the sum of simulated distributions for the process (1) and background processes. The shaded histogram shows the contribution of background processes.

$$E_{\text{tot}} > 0.7 \cdot E$$

Fig. 2. The $\chi_{3\gamma}$ distribution. Points with error bars represent experimental data. The histogram is a simulation of the process (1) and background processes. The shaded histogram shows the contribution of background processes.
was estimated to be 3%. The detection efficiency for the process (1) was determined by simulation. The differential cross section of the $e^+e^- \rightarrow \pi^0\pi^0\gamma$ process calculated in VMD model [14] was used for simulation. The systematic error of the detection efficiency, including the model error due to possible contribution from $\sigma\gamma$ intermediate state was estimated to be 5%.

### 3. Fitting of the cross section

The fitting procedure maximizes the logarithmic likelihood function

$$L = \sum_i \ln P_i(N_i^{\exp}, N_i^{\text{th}}),$$

where $P_i$ is a Poisson probability to detect observed number of events $N_i^{\exp}$ in the $i$th energy bin with a theoretical expectation of $N_i^{\text{th}}$. The theoretical expectations were calculated as

$$N_i^{\text{th}} = \varepsilon_i L_i \sigma(E_i)(1 + \delta(E_i)) + N_i^{\text{bkg}},$$

where $N_i^{\text{bkg}}$ is a calculated number of background events, $\varepsilon_i$ is a detection efficiency, $L_i$ is the integrated luminosity, $\sigma(E)$ is $e^+e^- \rightarrow \pi^0\pi^0\gamma$ cross section depending on a set of approximation parameters, $\delta$ is a radiative correction. The radiative correction, which is a functional of the cross-section energy dependence $\sigma(E)$ [15], was determined within the fitting procedure. The values of radiative correction evaluated for each experimental energy point are listed in Table 2. The model error of the $(1 + \delta)$ value does not exceed 3%. The values of the experimental cross section calculated as

$$\sigma_i^{\exp} = \frac{N_i^{\exp} - N_i^{\text{bkg}}}{\varepsilon_i L_i (1 + \delta(E_i))}$$

are shown in Fig. 4 and listed in Table 2. The systematic error of the cross section is determined by the errors of the detector efficiency, integrated luminosity, and radiative correction. It was estimated to be 7%.

To calculate the cross section $\sigma(E)$ the amplitude of the $e^+e^- \rightarrow \pi^0\pi^0\gamma$ process was parametrized as

$$A_{\pi\pi\gamma} = A_{\rho\pi\pi}(BW_\rho + \alpha_1 BW_\rho' + \alpha_2 BW_\rho')$$
$$+ \beta A_{\rho\rho\gamma} BW_\rho + \gamma A_{\omega\rho} BW_\omega.$$

(9)
The first term in Eq. (9) is the amplitude of the $e^+e^- \to \rho, \rho', \rho'' \to \omega \pi^0$, where $\rho'$ and $\rho''$ are excitations of the $\rho(770)$. Second and third terms are $e^+e^- \to \rho(770) \to \pi^0 \pi^0 \gamma$ and $e^+e^- \to \omega \to \pi^0 \pi^0 \gamma$ amplitudes. Each amplitude is written in a factorized form. The functions $A_{\rho \pi \pi \gamma}, A_{\rho \gamma \gamma}, A_{\omega \gamma}$ depending on the momenta of final particles describe the dynamics of vector mesons decays. The functions

<table>
<thead>
<tr>
<th>$E$ (MeV)</th>
<th>$\sigma_E$ (MeV)</th>
<th>$L$ (nb$^{-1}$)</th>
<th>$N_{\text{exp}}$</th>
<th>$N_{\text{bkg}}$</th>
<th>$\epsilon$</th>
<th>$1+\delta$</th>
<th>$\sigma_{\text{exp}}$ (nb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>600.1</td>
<td>0.29</td>
<td>88.3</td>
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<td>0.1</td>
<td>0.273</td>
<td>0.912</td>
<td>$-0.005^{+0.052}_{-0.002}$</td>
</tr>
<tr>
<td>630.1</td>
<td>0.30</td>
<td>116.1</td>
<td>0</td>
<td>0.1</td>
<td>0.269</td>
<td>0.906</td>
<td>$-0.004^{+0.041}_{-0.002}$</td>
</tr>
<tr>
<td>660.2</td>
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<td>271.6</td>
<td>2</td>
<td>0.3</td>
<td>0.273</td>
<td>0.900</td>
<td>$0.025^{+0.040}_{-0.019}$</td>
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<tr>
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<td>0.263</td>
<td>0.895</td>
<td>$0.046^{+0.067}_{-0.033}$</td>
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<tr>
<td>720.3</td>
<td>0.26</td>
<td>588.5</td>
<td>1</td>
<td>0.8</td>
<td>0.251</td>
<td>0.892</td>
<td>$0.003^{+0.018}_{-0.007}$</td>
</tr>
<tr>
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<td>0.32</td>
<td>219.0</td>
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<td>0.1</td>
<td>0.259</td>
<td>0.897</td>
<td>$0.057^{+0.057}_{-0.032}$</td>
</tr>
<tr>
<td>760.2</td>
<td>0.31</td>
<td>238.9</td>
<td>2</td>
<td>0.3</td>
<td>0.251</td>
<td>0.896</td>
<td>$0.032^{+0.049}_{-0.024}$</td>
</tr>
<tr>
<td>764.2</td>
<td>0.32</td>
<td>250.4</td>
<td>5</td>
<td>0.2</td>
<td>0.254</td>
<td>0.892</td>
<td>$0.085^{+0.060}_{-0.038}$</td>
</tr>
<tr>
<td>770.2</td>
<td>0.31</td>
<td>284.4</td>
<td>8</td>
<td>0.3</td>
<td>0.253</td>
<td>0.877</td>
<td>$0.125^{+0.063}_{-0.044}$</td>
</tr>
<tr>
<td>774.2</td>
<td>0.34</td>
<td>217.1</td>
<td>7</td>
<td>0.2</td>
<td>0.252</td>
<td>0.855</td>
<td>$0.145^{+0.081}_{-0.055}$</td>
</tr>
<tr>
<td>778.1</td>
<td>0.34</td>
<td>247.9</td>
<td>6</td>
<td>0.5</td>
<td>0.261</td>
<td>0.820</td>
<td>$0.104^{+0.068}_{-0.045}$</td>
</tr>
<tr>
<td>780.2</td>
<td>0.35</td>
<td>319.5</td>
<td>16</td>
<td>1.2</td>
<td>0.263</td>
<td>0.807</td>
<td>$0.219^{+0.075}_{-0.059}$</td>
</tr>
<tr>
<td>781.1</td>
<td>0.33</td>
<td>339.6</td>
<td>20</td>
<td>1.2</td>
<td>0.267</td>
<td>0.807</td>
<td>$0.257^{+0.061}_{-0.056}$</td>
</tr>
<tr>
<td>782.1</td>
<td>0.31</td>
<td>656.3</td>
<td>34</td>
<td>1.0</td>
<td>0.257</td>
<td>0.815</td>
<td>$0.246^{+0.050}_{-0.042}$</td>
</tr>
<tr>
<td>783.2</td>
<td>0.30</td>
<td>473.4</td>
<td>30</td>
<td>2.0</td>
<td>0.253</td>
<td>0.833</td>
<td>$0.280^{+0.066}_{-0.055}$</td>
</tr>
<tr>
<td>784.2</td>
<td>0.32</td>
<td>346.2</td>
<td>24</td>
<td>0.7</td>
<td>0.261</td>
<td>0.857</td>
<td>$0.301^{+0.077}_{-0.063}$</td>
</tr>
<tr>
<td>785.3</td>
<td>0.24</td>
<td>212.3</td>
<td>12</td>
<td>0.4</td>
<td>0.257</td>
<td>0.890</td>
<td>$0.238^{+0.094}_{-0.070}$</td>
</tr>
<tr>
<td>786.1</td>
<td>0.33</td>
<td>267.7</td>
<td>11</td>
<td>0.4</td>
<td>0.255</td>
<td>0.914</td>
<td>$0.170^{+0.071}_{-0.052}$</td>
</tr>
<tr>
<td>790.1</td>
<td>0.34</td>
<td>191.4</td>
<td>4</td>
<td>0.3</td>
<td>0.258</td>
<td>1.006</td>
<td>$0.074^{+0.064}_{-0.039}$</td>
</tr>
<tr>
<td>794.2</td>
<td>0.34</td>
<td>206.7</td>
<td>1</td>
<td>0.2</td>
<td>0.256</td>
<td>1.044</td>
<td>$0.014^{+0.022}_{-0.015}$</td>
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<tr>
<td>800.2</td>
<td>0.32</td>
<td>276.8</td>
<td>10</td>
<td>0.3</td>
<td>0.255</td>
<td>1.053</td>
<td>$0.130^{+0.085}_{-0.024}$</td>
</tr>
<tr>
<td>810.2</td>
<td>0.34</td>
<td>279.5</td>
<td>3</td>
<td>0.4</td>
<td>0.240</td>
<td>1.043</td>
<td>$0.037^{+0.030}_{-0.024}$</td>
</tr>
<tr>
<td>820.1</td>
<td>0.36</td>
<td>315.2</td>
<td>2</td>
<td>0.3</td>
<td>0.244</td>
<td>1.035</td>
<td>$0.021^{+0.031}_{-0.016}$</td>
</tr>
<tr>
<td>840.2</td>
<td>0.35</td>
<td>677.5</td>
<td>8</td>
<td>0.8</td>
<td>0.247</td>
<td>1.025</td>
<td>$0.043^{+0.023}_{-0.016}$</td>
</tr>
<tr>
<td>880.0</td>
<td>0.41</td>
<td>376.0</td>
<td>7</td>
<td>0.5</td>
<td>0.222</td>
<td>1.001</td>
<td>$0.079^{+0.045}_{-0.031}$</td>
</tr>
<tr>
<td>919.9</td>
<td>0.44</td>
<td>478.6</td>
<td>8</td>
<td>0.3</td>
<td>0.256</td>
<td>0.916</td>
<td>$0.065^{+0.035}_{-0.025}$</td>
</tr>
<tr>
<td>939.9</td>
<td>0.43</td>
<td>469.0</td>
<td>22</td>
<td>0.7</td>
<td>0.248</td>
<td>0.856</td>
<td>$0.214^{+0.058}_{-0.047}$</td>
</tr>
<tr>
<td>949.7</td>
<td>0.32</td>
<td>261.7</td>
<td>20</td>
<td>0.3</td>
<td>0.261</td>
<td>0.855</td>
<td>$0.339^{+0.095}_{-0.076}$</td>
</tr>
<tr>
<td>957.7</td>
<td>0.32</td>
<td>233.9</td>
<td>13</td>
<td>0.2</td>
<td>0.263</td>
<td>0.858</td>
<td>$0.242^{+0.089}_{-0.067}$</td>
</tr>
<tr>
<td>969.7</td>
<td>0.34</td>
<td>251.5</td>
<td>29</td>
<td>0.5</td>
<td>0.250</td>
<td>0.865</td>
<td>$0.524^{+0.119}_{-0.099}$</td>
</tr>
</tbody>
</table>
$BW_i$ describe the Breit–Wigner resonance shapes:

$$BW_i = \frac{m_i^2}{m_i^2 - E^2 - iE\Gamma_i(E)}, \quad i = \rho, \rho', \rho'', \omega.$$

Here $m_i$ and $\Gamma_i(E)$ are resonance mass and energy dependent width. The cross section is calculated from Eq. (9) by integration over the phase space of final particles: $\sigma(E) = \int \frac{d^3p}{3\pi^2} \frac{d^3q_\pi}{3\pi^2} |A_{\rho\pi\pi}|^2 d\Pi$. At the energy above $\omega\pi$ threshold the Breit–Wigner functions of $\rho$ mesons are modified $BW_{\rho_1} = BW_{\rho_0} C_{\rho\omega\pi}$, where $C_{\rho\omega\pi}$ are Blatt–Weisskopf factors, restricting fast growth of the $\Gamma_{\rho\omega\pi}$ partial widths [16]:

$$C_{\rho\omega\pi} = \sqrt{\frac{1}{1 + (Rq_{\omega}(m_{\rho}))^2}}, \quad C_{\rho_1\omega\pi} = \sqrt{\frac{1 + (Rq_{\omega}(m_{\rho}))^2}{1 + (Rq_{\omega}(E))^2}}, \quad \rho_0 = \rho', \rho''. \quad (10)$$

Here $q_{\omega}$ is the $\omega$ meson momentum in $\rho_0 \to \omega\pi$ decay. The range parameter $R$ is supposed to be the same for $\rho, \rho', \rho''$ mesons. The main decay modes of $\rho$ mesons were taken into account for calculation of the energy dependence of the resonance widths. For instance, in the case of $\rho(770)$ we use the following expression:

$$\Gamma_{\rho}(E) = \Gamma_{\rho}(m_{\rho}) \left(\frac{m_{\rho}}{E}\right)^2 \left(\frac{g_{\pi}(E)}{g_{\pi}(m_{\rho})}\right)^3 C_{\rho\pi\pi}^2 + \frac{g_{\rho\omega\pi}^2}{12\pi q_{\omega}(E)} C_{\rho\omega\pi}^2. \quad (11)$$

Here $g_{\omega\pi}$ is a pion momentum in the $\rho \to 2\pi$ decay. The Blatt–Weisskopf factor $C_{\rho\pi\pi}$ is expressed by the formula similar to Eq. (10).

The statistical errors are not shown because they are significantly smaller than model biases. For models with two excited $\rho$ states the $\rho'$ mass and width were fixed to 1400 and 500 MeV. These values are close to $\rho'$ parameters from $\pi^\pm\pi^0$ spectral function data [20, 21].

Below $\omega\pi^0$ threshold the amplitude of the $e^+e^- \to \omega\pi^0$ process drops rapidly and the product $|BW_{\rho}|^2 \propto \int |A_{\rho\omega\pi}|^2 d\Pi$ in contrast with the corresponding product for $\rho \to \sigma\gamma$ transition does not demonstrate resonance behavior. This allows to separate contributions of the two $\rho$ decay mechanisms by measurement of energy dependence of the $e^+e^- \to \pi^0\pi^0\gamma$ cross section. The $\rho \to \sigma\gamma$ decay amplitude was described by the $\chi PT$ and $L\sigma M$ models from Ref. [7]. The three sets of $\sigma$ parameters [7,22,23] used in the $L\sigma M$ model are listed in Table 5. The $\chi PT$ model corresponds to $m_\sigma \to \infty$. The $\beta$ parameter represents the difference between observed value of the $\rho \to \sigma\gamma$ decay amplitude and theoretical prediction.

For $\omega \to \pi^0\pi^0\gamma$ decay the variation of the final state phase in the CMS energy interval of the $\omega$ meson is small, so we cannot separate different decay mechanisms studying the cross section energy dependence. Thus the amplitude of the $\omega \to \pi^0\pi^0\gamma$ decay was written according to VMD model [14]. The $\rho\omega$ mixing was taken into account following Ref. [7]. Possible contributions of other mechanisms would result in a deviation of the complex parameter $\gamma$ from 1.

Full description of the energy dependence of the cross section below 1 GeV requires extra four parame-
Table 3
Parameters of the models describing $e^+e^-$ → $\omega\pi\rightarrow\pi^0\pi^0\gamma$ cross section for $E > 1$ GeV

<table>
<thead>
<tr>
<th>$g_{\rho\gamma\pi}$</th>
<th>$m_{\rho'}$</th>
<th>$\Gamma_{\rho'}$</th>
<th>$a_1$</th>
<th>$m_{\rho''}$</th>
<th>$\Gamma_{\rho''}$</th>
<th>$a_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14.3–15.8</td>
<td>–</td>
<td>–</td>
<td>1630–1710</td>
<td>630–1000</td>
<td>–(0.19–0.24)</td>
</tr>
<tr>
<td>2</td>
<td>14.1–15.7</td>
<td>1400</td>
<td>500</td>
<td>–(0.04–0.06)</td>
<td>1580–1620</td>
<td>420–580</td>
</tr>
<tr>
<td>3</td>
<td>15.4–16.6</td>
<td>1400</td>
<td>500</td>
<td>–(0.39–0.42)</td>
<td>1560–1640</td>
<td>380–780</td>
</tr>
</tbody>
</table>

Table 4
The branching ratios of $\rho$ and $\omega$ decays ($B \times 10^5$) and $P(x^2)$ values obtained as a result of cross section fitting with different values of $\phi_\beta$, $\phi_\gamma$. The $\omega\pi$ amplitude was described by the model 3 from Table 3. Model 1 from Table 5 was used for description of $\rho\pi\gamma$ amplitudes. But we prefer two other sets of parameters. The final results with statistical and systematic errors are given in the bottom line of the table.

<table>
<thead>
<tr>
<th>Model</th>
<th>$B(\omega \rightarrow \pi^0\pi^0\gamma)$</th>
<th>$B(\rho \rightarrow \pi^0\pi^0\gamma)$</th>
<th>$B(\rho \rightarrow \sigma\gamma)$</th>
<th>$P(x^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.3±1.4</td>
<td>4.1±0.9</td>
<td>1.9±0.9</td>
<td>35%</td>
</tr>
<tr>
<td>2</td>
<td>12.3±2.3</td>
<td>3.8±0.8</td>
<td>4.4±1.0</td>
<td>30%</td>
</tr>
<tr>
<td>3</td>
<td>25.5±2.3</td>
<td>5.1±1.0</td>
<td>1.9±1.0</td>
<td>6%</td>
</tr>
<tr>
<td>4</td>
<td>15.8±2.3</td>
<td>4.7±0.8</td>
<td>5.6±1.0</td>
<td>8%</td>
</tr>
</tbody>
</table>

Table 5
The probabilities of the $\rho$ and $\omega$ decays into $\pi^0\pi^0\gamma$ ($B \times 10^5$) obtained with different parameters of $\rho$ meson. The spreads in the parameter values correspond to the models listed in Table 3. Theoretical values of branching ratios are taken from Ref. [7]. The final results with statistical and systematic errors are given in the bottom line of the table.

<table>
<thead>
<tr>
<th>Model</th>
<th>$m_\rho$ (MeV)</th>
<th>$\Gamma_\rho$ (MeV)</th>
<th>$B(\omega \rightarrow \pi^0\pi^0\gamma)$</th>
<th>$B(\rho \rightarrow \pi^0\pi^0\gamma)$</th>
<th>$B(\rho \rightarrow \sigma\gamma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L\sigma M$</td>
<td>478</td>
<td>324</td>
<td>6.2–6.5</td>
<td>4.0–4.1</td>
<td>3.8</td>
</tr>
<tr>
<td>$L\sigma M$</td>
<td>555</td>
<td>540</td>
<td>6.3–6.6</td>
<td>4.2–4.3</td>
<td>2.8</td>
</tr>
<tr>
<td>$L\sigma M$</td>
<td>478</td>
<td>263</td>
<td>6.2–6.4</td>
<td>4.2–4.3</td>
<td>4.7</td>
</tr>
<tr>
<td>$\chi PT$</td>
<td>–</td>
<td>–</td>
<td>6.5–6.9</td>
<td>3.9–4.0</td>
<td>2.9</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$6.6^{+1.4}_{-1.3} \pm 0.6$</td>
<td>$4.1^{+1.0}_{-0.9} \pm 0.3$</td>
<td>$1.9^{+0.9}_{-0.8} \pm 0.4$</td>
</tr>
</tbody>
</table>

where $\sigma_V = 12\pi B(V \rightarrow e^+e^-)/M_V^2$ is a total vector meson production cross section in $e^+e^-$ collisions.

Characteristic feature of the process under study is a large interference between the contributions of $\rho$ and $\omega$ decays. For instance the cross section of $e^+e^- \rightarrow \omega \rightarrow \pi^0\pi^0\gamma$ process at $E = m_\omega$ evaluated using the table value of

$B(\omega \rightarrow \pi^0\pi^0\gamma) = (7.2 \pm 2.5) \times 10^{-5}$

is equal to 0.12 nb. The interference with $\rho$ meson increases this value up to approximately 0.25 nb (Fig. 4).

The experimental data on the energy dependence of the cross section are insufficient for determination of unambiguous solution for interference phases $\phi_\beta$, $\phi_\gamma$. There are four solutions listed in Table 4. The third and
fourth ones correspond to a large destructive contribution into ω decay from mechanisms other than ρ°π°. The $B(ω → π^0π^0γ)$ values obtained in this case disagree with existing experimental value

$$B(ω → π^0π^0γ) = (7.2 ± 2.5) \times 10^{-5}.$$  

The solution with $φ_β \approx π$, $φ_γ \approx 0$ can be ruled out for two reasons: $B(ω → π^0π^0γ)$ exceeds the table value by 1.7 standard deviations and consistency of the calculated spectrum of the recoil photon with the experimental one is poor. The analysis of the photon spectrum is described in the next section.

For the only survivor solution with both phases close to zero, the model dependence of the fit parameters was studied. Three models of excited ρ states (Table 3) and four sets of $σ$ parameters (Table 5) were tested. The $φ_γ$ was found ranging within $(20–80)^° ± 80^°$. These values are in agreement with theoretically expected zero value [7]. Therefore, the final fitting was performed with $φ_β = 0$. The phase $φ_γ$ was considered as a floating parameter to take into account its possible shift due to the contribution of mechanisms other than $ω → ρπ^0$. The fitted $φ_γ = -(2–20)^° ± 20^°$ is consistent with zero.

The probabilities of the ρ and ω decays into $π^0π^0γ$ obtained with different parameters of $σ$ meson are listed in Table 5. The spreads in parameter values correspond to different models describing $e^+e^- → ωπ^0$ cross section above 1 GeV. All models reproduce the experimental data well. Therefore parameter midrange was taken as a final result. Its spread was regarded as the model error. The branching ratios obtained this way with statistical and systematic errors are listed in the last row of Table 5. The systematic error includes the model error, uncertainties in the detection efficiency and integrated luminosity. The variation of the background level within its systematic error practically does not change the $B(ω → π^0π^0γ)$ central value and results in following additional uncertainties of the ρ meson branching ratios: 7% for $B(ρ → π^0π^0γ)$ and 12% for $B(ρ → σγ)$. Since these uncertainties affect statistical significance of the results they were added to statistical errors. The energy dependence of the cross section in the model with $m_σ = 478$ MeV and $Γ_σ = 324$ MeV is shown in Fig. 4 together with the curve corresponding to $B(ρ → σγ) = 0$. The $P(χ^2)$ value for the latter model is equal to 0.5%.

4. The energy and angular spectra

From Table 2 it is seen that selected events are mainly concentrated in two energy regions: 180 events $ω$ peak and 92 events in the range 920–970 MeV above the $e^+e^- → ωπ^0$ reaction threshold. The angular and energy distributions in the latter region agree with $ωπ^0$ mechanism. Recoil photon spectrum for events from 760–800 MeV energy range is shown in Fig. 6. Although this energy region is dominated by $ω$ peak the contributions of both $ω$ and $ρ$ decays must be taken into account to obtain the theoretical spectrum. The spectrum calculated in model 3 from Table 5 (Fig. 6) is in a good agreement with the experimental one. In this model it is supposed that $ω → π^0π^0γ$ decay proceeds through $ρ^0π^0$ intermediate state. Another way is to describe the $ω$ decay by a sum of contributions of $ω → ρ^0π^0$ and $ω → σγ$ mechanisms. To do this we fix $B(ω → ρ^0π^0 → π^0π^0γ)$ at $2.5 × 10^{-5}$ and fit the $ω → σγ$ decay contribution to a value yielding observed $ω → π^0π^0γ$ branching ratio. In case of constructive interference this leads to

$$B(ω → σγ → π^0π^0γ) = 1.3 × 10^{-5}$$

and the photon spectrum close to that expected for $ρ^0π^0$ mechanism. As was shown in the previous section, assumption of destructive interference results in $B(ω → π^0π^0γ)$ inconsistent with the PDG table value.

![Fig. 6. The recoil photon spectrum for the experimental events of the reaction (1) in the energy range 760 < E < 800 MeV (points with error bars) and the results of simulation in the model 3 from Table 5 (solid line) and in the model 2 from Table 4 (dashed line).](image-url)
The second theoretical spectrum in Fig. 6 corresponds to the model 2 from Table 4 with destructive interference of $\omega\pi$ and $\sigma\gamma$ amplitudes in $\rho$ decay. For this model the consistency between theoretical and experimental spectra calculated using Kolmogorov test [24] is about 1%, which was one of the reasons to discard this model.

Additional information about mechanism of $\omega$ decay can be obtained from the analysis of angular distributions. One of such distribution is shown in Fig. 7. The same figure displays the theoretical distributions obtained under assumptions that $\omega$ decay proceeds through either pure $\rho^0\pi^0$ intermediate state or a mixture of $\rho^0\pi^0$ and $\sigma\gamma$. One can see that our limited statistics does not allow to distinguish these two models.

5. Summary

The branching ratios measured in this work,

$$B(\omega \rightarrow \pi^0\pi^0\gamma) = (6.6^{+1.4}_{-1.3} \pm 0.6) \times 10^{-5},$$

$$B(\rho \rightarrow \pi^0\pi^0\gamma) = (4.1^{+1.0}_{-0.9} \pm 0.3) \times 10^{-5}$$

are in a good agreement with our preliminary results [2] and GAMS measurement

$$B(\rho \rightarrow \pi^0\pi^0\gamma) = (7.2 \pm 2.5) \times 10^{-5}$$

[3], but have higher accuracy.

The probability of $\rho \rightarrow \pi^0\pi^0\gamma$ decay significantly exceeds VMD model prediction $(1.3 \pm 1.5) \times 10^{-5}$. This excess can be explained by the contribution of the decay via scalar state $\rho \rightarrow \sigma\gamma$. Two mechanisms, $\rho \rightarrow \sigma\gamma$ and $\rho \rightarrow \omega\pi$, can be separated using difference in energy dependence of their amplitudes. Our result on $\rho \rightarrow \sigma\gamma$ decay

$$B(\rho \rightarrow \sigma\gamma \rightarrow \pi^0\pi^0\gamma) \equiv (1.9^{+0.9}_{-0.8} \pm 0.4) \times 10^{-5}$$

(14) differs from zero by 2.4 standard deviations and is consistent with the predictions of chiral models [7,9]. The magnitude of $B(\rho \rightarrow \sigma\gamma)$ is sensitive to $\sigma$ parameters. As can be seen from Table 5, the models with $\Gamma_\rho \sim 300$ MeV give the most consistent description of the experimental data.

The value of the branching ratio

$$B(\omega \rightarrow \pi^0\pi^0\gamma) = (6.7 \pm 1.2) \times 10^{-5},$$

obtained by averaging of our measurement with the GAMS result exceeds theoretical predictions, $(4.6 \pm 1.1) \times 10^{-5}$ [7,11] and $(4.7 \pm 0.9) \times 10^{-5}$ [9], by 1.3 standard deviations. It is necessary to make some remarks about these predictions. The result of Ref. [11] is based on table value of

$$\Gamma(\rho \rightarrow \pi^0\gamma) = 102 \pm 26 \text{ keV} \ [25].$$

It must be corrected taking into account newer measurement

$$\Gamma(\rho \rightarrow \pi^0\gamma) = 76 \pm 22 \text{ keV} \ [26],$$

which is close to the value for charged $\rho$,

$$\Gamma(\rho^\pm \rightarrow \pi^\pm\gamma) = 68 \pm 8 \text{ keV} \ [25].$$

This decreases the predicted $B(\omega \rightarrow \pi^0\pi^0\gamma)$ and worsens agreement with the experiment. In the Refs. [7,9] the values of $g_{\rho\omega\pi}$ equal to 15 and 15.9 GeV$^{-1}$ were used to calculate $B(\omega \rightarrow \pi^0\pi^0\gamma) \propto g^2_{\rho\omega\pi} g_{\rho\pi\gamma} \propto g^4_{\rho\omega\pi} / g^2_{\rho\pi\gamma}$. On the other hand, the use of these $g_{\rho\omega\pi}$ values for VMD calculation of $\Gamma(\omega \rightarrow 3\pi)$ and $\Gamma(\omega \rightarrow \pi^0\gamma)$ leads to too large values conflicting with experimental data. For example, the $g_{\rho\omega\pi}$ obtained from $\omega \rightarrow 3\pi$ width assuming intermediate $\rho\pi$ state is equal to $(14.3 \pm 0.2)$ GeV$^{-1}$ [14]. Therefore, our opinion is that $4.6 \times 10^{-5}$ is the maximum branching ratio acceptable within VMD model and the additional theoretical study is required to explain the large value of $B(\omega \rightarrow \pi^0\pi^0\gamma)$. 

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**Fig. 7.** Distribution of cosine angle between photon and $\pi^0$ meson in the $\pi^0\pi^0$ rest frame. Point with error bars are experimental data. Solid line is a simulation with $\omega \rightarrow \rho^0\pi^0\pi^0$ state. Dashed line is a simulation with $\omega \rightarrow \rho^0\pi^0\pi^0$ and $\sigma\gamma$ states.
Acknowledgements

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References

Search for the decay $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ in the momentum region $P_\pi < 195$ MeV/c

E787 Collaboration

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Abstract

We have observed the decay $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ in the kinematic region with pion momentum below the $K^+ \rightarrow \pi^+ \pi^0$ peak. One event was observed, consistent with the background estimate of $0.73 \pm 0.18$. This implies an upper limit on $B(K^+ \rightarrow \pi^+ \nu \bar{\nu}) < 4.2 \times 10^{-9}$ (90% C.L.), consistent with the recently measured branching ratio of $(1.57^{+1.75}_{-0.82}) \times 10^{-10}$, obtained using the standard model spectrum and the kinematic region above the $K^+ \rightarrow \pi^+ \pi^0$ peak. The same data were used to search for $K^+ \rightarrow \pi^+ X^0$, where $X^0$ is a weakly interacting neutral particle or system of particles with $150 < M_{X^0} < 250$ MeV/c$^2$.

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In a recent Letter we reported the branching ratio for the rare decay $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ to be $(1.57^{+1.75}_{-0.82}) \times 10^{-10}$ based on the observation of two events in the phase space region $p_\pi > 211$ MeV/c [1]. This decay is sensitive to the coupling of top to down quarks, $V_{td}$, in the Cabibbo–Kobayashi–Maskawa mixing matrix. The standard model (SM) predicted branching ratio, $B(K^+ \rightarrow \pi^+ \nu \bar{\nu})$, is $(0.75 \pm 0.29) \times 10^{-10}$ [2]. Loop diagrams involving new heavy particles in extensions of the SM can interfere with SM diagrams and alter the decay rate, and also the kinematic spectrum [3]. Exotic scenarios such as $K^+ \rightarrow \pi^+ X^0$ where $X^0$ is a hypothetical stable weakly interacting particle or system of particles have also been suggested [4,5]. It is, therefore, important to obtain higher statistics for this decay and to extend the measurement to other regions of phase space. The results in [1] are from analysis of data with the $\pi^+$ momentum above the $K^+ \rightarrow \pi^+ \pi^0$ ($K_{\pi 2}$) peak (Region 1). The $\pi^+$ from $K_{\pi 2}$ decay has a kinetic energy ($E$), momentum ($P$), and range ($R$) in plastic scintillator of 108 MeV, 205 MeV/c, and 30 cm, respectively. In this Letter we report the analysis of data below the $K_{\pi 2}$ peak (Region 2) obtained from Experiment E787 [6–12] at the Alternating Gradient Synchrotron (AGS) of Brookhaven National Laboratory. The previous limit using Region 2, $1.7 \times 10^{-8}$ (90% C.L.), was obtained from an earlier version of the E787 detector [13].

The signature for $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ in the E787 experiment is a single $K^+$ stopping in a target (TG), decaying to a single $\pi^+$ with no other accompanying photons or charged particles. In Region 1, the major backgrounds were found to be the two body decays $K_{\pi 2}$ and $K^+ \rightarrow \mu^+ \nu_\mu (K_{\mu 2})$, scattered beam pions, and $K^+$ charge exchange (CEX) reactions resulting in decays $K^+_L \rightarrow \pi^+ l^- \bar{\nu}_l$, where $l = e$ or $\mu$. Region 2 has larger potential acceptance than Region 1 because the phase space is more than twice as large and the loss of pions due to nuclear interactions in the detector is smaller at the lower pion energies. However, there are additional sources of background for Region 2. These include $K_{\pi 2}$ in which the $\pi^+$ loses energy by scattering in the material of the detector (primarily in the TG), $K^+ \rightarrow \pi^+ \pi^0 \gamma (K_{\pi 2 \gamma})$, $K^+ \rightarrow \mu^+ \nu \gamma (K_{\mu 2 \gamma})$, $K^+ \rightarrow \mu^+ \nu \pi^0 (K_{\mu 3})$, and $K^+ \rightarrow \pi^+ \pi^- e^+ e^- (K_{e4})$ decays in which both the $\pi^-$ and the $e^+$ are invisible because of absorption.

The data were obtained with a flux of $6 \times 10^6$ kaons per 1.6 s spill at 730 MeV/c (with 24% pion contamination) entering the apparatus. The kaons were identified by a Cerenkov detector; two multi-wire-proportional-chambers were used to determine that there was only one entering particle. After slowing in a BeO degrader the kaons traversed a 10-cm-thick lead-glass detector read out by 16 fine-mesh photomultiplier tubes (PMT) and a scintillating target hodoscope (TH) placed before the TG. The lead-glass detector was designed to be insensitive to kaons and detect electromagnetic showers originating from kaon decays in the TG. The TH was used to verify that there was only one kaon as well as determine the position, time, and energy loss of the kaon before it entered and stopped in the TG. The TG consisted of 413 5.0-mm-square, 3.1-m-long plastic scintillating fibers, each connected to a PMT. The fibers were packed axially to form a cylinder of ~12 cm diameter. Gaps in the outer edges of the TG were filled with smaller fibers which were connected to PMTs in groups. The PMTs were read out by ADCs, TDCs, and 500 MHz transient digitizers based on GaAs charge-coupled devices (CCDs) [7]. Photons were detected in a hermetic calorimeter mainly consisting of a 14-radiation
length thick barrel detector made of lead/scintillator sandwich and 13.5-radiation length thick endcaps of undoped CsI crystals [8]. The rest of the detector consisted of a central drift chamber (UTC) [9], and a cylindrical stack (RS) of 21 layers of plastic scintillator with two layers of embedded tracking chambers, all within a 1-T solenoidal magnetic field. The TG, UTC, and RS allowed the measurement of the \( P, R, \) and \( E \) of the charged decay products. The \( \pi \rightarrow \mu \rightarrow e \) decay sequence from pions that came to rest in the RS was observed using another set of 500 MHz transient digitizers (TD) [10].

The data reduction and offline analysis for Region 2 was similar to the analysis of Region 1 [11,12,14], although the final cuts to enhance signal and suppress background to less than one event were different. Here we will emphasize the key instrumentation and analysis tools used to suppress the background in Region 2. The TG, CCDs, and the photon veto system were the important elements for Region 2 analysis. A multilevel trigger selected events by requiring an identified \( K^+ \) to stop in the TG, followed, after a delay of at least 1.5 ns, by a single charged particle track that traversed TG and RS with a hit-pattern consistent with the expectation for \( K^+ \rightarrow \pi^+ \nu \bar{\nu} \). Events with photons were suppressed by vetos on the barrel and endcap detectors. In the offline analysis, the single charged particle was required to be identified as a \( \pi^+ \) with \( P, R, \) and \( E \) consistent for a \( \pi^+ \), and the TD pulse information consistent with the decay sequence \( \pi \rightarrow \mu \rightarrow e \) in the last RS counter on the pion trajectory. The signal region was defined by the intervals \( 140 < P < 195 \text{ MeV}/c, \ 12 < R < 27 \text{ cm}, \) and \( 60 < E < 95 \text{ MeV} \).

The background was found to be dominated by \( K_{\pi 2} \) events in which the pion had a nuclear interaction near the kaon decay vertex, most probably on a carbon nucleus in the TG plastic scintillator. This scatter left the pion with reduced kinetic energy, putting it in Region 2. We suppressed this background by removing events in which the pion track had a scattering signature in the TG. These signatures included kinks, tracks that did not point back to the vertex fiber in which the kaon decayed, or energy deposits inconsistent with the ionization energy loss for a pion of the measured momentum. The remaining \( K_{\pi 2} \) background consisted of events in which the pion scattered in one of the fibers traversed by the kaon. The extra energy deposits from the pion scatters were obscured by the earlier large energy deposits of the kaon. For these events, we examined the pulse shapes recorded in the CCDs in each kaon fiber using a \( \chi^2 \) fit and eliminated events in which an overlapping second pulse, in time with the pion, was found to have energy larger than 1 MeV. To obtain sufficient separation of the \( \Kp\p 2 \) and \( \pi^+ \) induced pulses in the CCDs we required a minimum delay of 6 ns between the kaon and the pion. Finally, additional \( K_{\pi 2} \) rejection was obtained by removing events with photon interactions in detectors surrounding the kaon beam-line; these cuts caused substantial \((\sim 42\%)\) loss of efficiency because of accidental hits due to the high flux of particles.

We formed multiple independent constraints on each source of background. These constraints were grouped in two independent sets of cuts, designed to have little correlation. One set of cuts was relaxed (or inverted) to enhance the background so that the other set could be evaluated to determine its power of rejection, as summarized below. The sum of the background due to \( K_{\mu 2\gamma} \) and \( K_{\mu 3} \) was obtained by separately measuring the rejection factors of the TD particle identification and kinematic \( (R \) and \( P \)) particle identification. The background from beam pions was evaluated by separately measuring the rejections of Cerenkov, TH beam particle identification, and the delay time between pion and kaon. The dominant background from \( K_{\pi 2} \) decay was measured by evaluating the rejection of the photon veto system on events tagged by scattering signatures in the TG and target CCDs. Similarly, the rejection of the target CCD cut was determined by using events that failed the photon veto criteria. It should be noted that the Region 1 analysis measured photon veto rejection using the unscattered events in the momentum peak \( (205 \text{ MeV}/c) \) [12]. This method could not be used for Region 2 because the scattering in the TG spoiled the back-to-back correlation between the detected \( \pi^+ \) track in the RS and the undetected \( \pi^0 \), leading to different photon veto rejection factors for scattered and unscattered \( K_{\pi 2} \) background events.

We employed Monte Carlo (MC) simulation to evaluate the backgrounds from \( K_{e 4} \) and \( K_{\pi 2\gamma} \) because these could not be distinguished, on the basis of the \( \pi^+ \) track alone, from the much larger \( K_{\pi 2} \) background with a \( \pi^+ \) scatter. In the case of \( K_{e 4} \), we first identi-
The errors include statistics of the data and Monte Carlo as well as systematic uncertainties. The errors in Table 1 include corrections for small correlations among background samples.

<table>
<thead>
<tr>
<th>Decay</th>
<th>Method</th>
<th>Number (± Error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K^+ \to \pi^+\pi^0 )</td>
<td>d</td>
<td>0.630 ± 0.170</td>
</tr>
<tr>
<td>( K^+ \to \pi^+\pi^0\gamma )</td>
<td>dm</td>
<td>0.027 ± 0.004</td>
</tr>
<tr>
<td>( K_{\mu2\gamma} + K_{\mu3} )</td>
<td>d</td>
<td>0.007 ± 0.007</td>
</tr>
<tr>
<td>Beam</td>
<td>d</td>
<td>0.033 ± 0.033</td>
</tr>
<tr>
<td>( K^+ \to \pi^+\pi^-e^+\nu_e )</td>
<td>dm</td>
<td>0.026 ± 0.032</td>
</tr>
<tr>
<td>CEX</td>
<td>dm</td>
<td>0.011 ± 0.011</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>0.734 ± 0.177</td>
</tr>
</tbody>
</table>

Systematic uncertainties on the largest background, \( K_{\pi2} \), were estimated by measuring the rejection of photon veto cuts on many different event ensembles, tagged in different ways for a TG scattering signature. The event ensembles were designed to have little contamination by other background sources such as \( K_{\pi2\gamma} \) and \( K_{\pi4} \). As a final check, each cut was relaxed to admit background events in a predictable way. Examination of these background events, which are close to the signal region, provided no indication of background sources other than those in Table 1. For example, the kaon decay time region between 2 to 6 ns, with acceptance of 0.254 ± 0.004 (less than the naive expectation due to other lifetime-dependent cuts) relative to the signal region, was examined. This region has a total estimated background of 0.45 ± 0.14, dominated by \( K_{\pi2} \) decays due to the reduced background rejection from the CCDs. One event, consistent with the background estimate, was observed in this background region.

After the background study, the signal region was examined, yielding one candidate event with \( P = 180.7 \text{ MeV}/c, R = 22.1 \text{ cm}, \) and \( E = 86.3 \text{ MeV} \) with a kaon decay time of 17.7 ns, consistent with the background estimate of 0.73 ± 0.18. Fig. 1 shows the kinematics of the remaining events before and after the cut on measured momentum, \( P \).

Using the total number of \( K^+ \) incident on TG for these data, \( 1.12 \times 10^{12} \), the acceptance reported in Table 2, and the observation of one event in Region 2 we calculate the upper limit of \( B(K^+ \to \pi^+\nu\bar{\nu}) < 4.2 \times 10^{-9} \) (90% C.L.) [20]. This is consistent with the branching ratio reported from Region 1 and the SM decay spectrum [1]; combining the measurements from Region 1 and Region 2 does not alter the branching ratio measurement significantly because it is dominated by the sensitivity of Region 1. However, for non-standard scalar and tensor interactions, Region 2 has larger acceptance than Region 1. We have combined the sensitivity of both regions to obtain the 90% C.L. upper limits, \( 4.7 \times 10^{-9} \) and \( 2.5 \times 10^{-9} \), for scalar and tensor interactions, respectively.

This measurement is also sensitive to \( K^+ \to \pi^+X^0 \), where \( X^0 \) is a hypothetical stable weakly interacting particle, or system of particles. Fig. 2 shows 90% C.L. upper limits on \( B(K^+ \to \pi^+X^0) \) together with the previous limit from [13]. The dotted line in Fig. 2 is the single event sensitivity defined as the inverse of the acceptance for \( K^+ \to \pi^+X^0 \).
Fig. 1. Range (cm in plastic scintillator) and kinetic energy (MeV) of events remaining after all cuts except the momentum cut (top), and including the momentum cut (bottom). The dark points represent the data. The simulated distribution of expected events from $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ is indicated by the light dots. The group of events around 108 MeV is due to the $K\pi$ background. The events at higher energy are due to $K\mu_2$ and $K\mu_2\gamma$ background. All events except for the one in the signal box are eliminated by the $140 < P < 195$ MeV/c cut on momentum.

Table 2
Acceptance factors used in the measurement of $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ in Region 2. The “$K^+$ stop efficiency” is the fraction of kaons entering the TG that stopped. “Other kinematic constraints” include particle identification cuts

<table>
<thead>
<tr>
<th>Acceptance factors</th>
<th>Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K^+$ stop efficiency</td>
<td>0.670</td>
</tr>
<tr>
<td>$K^+$ decay after 6 ns</td>
<td>0.591</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^+ \nu \bar{\nu}$ phase space</td>
<td>0.345</td>
</tr>
<tr>
<td>Geometry</td>
<td>0.317</td>
</tr>
<tr>
<td>$\pi^+$ nucl. int. and decay in flight</td>
<td>0.708</td>
</tr>
<tr>
<td>Reconstruction efficiency</td>
<td>0.957</td>
</tr>
<tr>
<td>Other kinematic cuts</td>
<td>0.686</td>
</tr>
<tr>
<td>$\pi^-\mu^-\nu$ decay chain</td>
<td>0.545</td>
</tr>
<tr>
<td>Beam and target analysis</td>
<td>0.479</td>
</tr>
<tr>
<td>CCD acceptance</td>
<td>0.401</td>
</tr>
<tr>
<td>Accidental loss</td>
<td>0.363</td>
</tr>
<tr>
<td>Total acceptance</td>
<td>$7.65 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Fig. 2. The 90% C.L. upper limit for $B(K^+ \rightarrow \pi^+ X^0)$ as a function of $M_{X^0}$, the mass of the recoiling system. The solid (dashed) line is from this analysis (from [13]). The limit for $M_{X^0} < 140$ MeV/c$^2$ is derived from the result for Region 1. The observation of one event in Region 2 reported in this Letter causes a bump in the limit at 194 MeV/c$^2$. Similarly the 2 events, consistent with the observation of $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ above background, in Region 1 reported in [1] increase the limit at 105 and 86 MeV/c$^2$. We have also included the single event sensitivity as a function of $M_{X^0}$ (the dotted line), defined in the text, obtained by E787.

In conclusion, the use of GaAs charged-coupled devices to record pulse shapes as well as highly efficient photon detection has allowed us to suppress background in Region 2. This has resulted in new limits on the spectrum of the pion in the decay $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ as well as improvement in the sensitivity to $K^+ \rightarrow \pi^+ X^0$ by a factor between 4 and 40 over the accessible mass range. The detailed enumeration of backgrounds in Region 2 will be important for new experiments that intend to precisely measure $K^+ \rightarrow \pi^+ X^0$ with large statistics [21,22].

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References

[21] An upgraded version of the experiment, E949, is now pursuing a precise measurement of K+ → π+νν.
Observing CP violating MSSM Higgs bosons at hadron colliders?

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Abstract

We report on the possibility of observing Higgs sector CP violation of the minimal supersymmetric standard model at a hadron machine. The CP phase dependent cross-sections for the $VH_i$ associated production processes are given for the Fermilab Tevatron and CERN LHC. Substantial production cross-sections for channels of all three Higgs bosons simultaneously are shown to be possible, giving a direct indication of the CP violation. The observability of the Higgs signals are discussed.

The search for the Higgs bosons, scalar particles from the electroweak symmetry breaking multiplet(s), in the Standard Model (SM) and beyond is a major goal of present and future colliders. One or more light Higgs boson within relatively easy reach of the upgraded Tevatron or CERN LHC is particularly favorable to the most popular extension of the SM, namely the minimal supersymmetric standard model (MSSM). Hence, it is very important that we study all possible scenarios under the framework in careful details. Here in this Letter, we make an effort in the direction focusing on the scenario with radiatively induced Higgs sector CP violation [1,2].

In the recent years, the so-called CP violating MSSM has became a subject of many phenomenological studies. A major part of the latter focus on Higgs physics, especially with application to the LEP machine at CERN (see, for example, Ref. [3] and references therein). Implications for a $e^+e^-$ machine is quite well studied. In particular, Ref. [4] has pushed the analysis to the prospective Next Linear Collider. Nevertheless, after the closing of the LEP machine and before another lepton machine is commissioned, we have no choice but to focus on the not as clean environment of the hadron colliders. Hence, it is the time for detailed careful studies of the topic at hadron machines. Some steps in the direction have been taken. More notable ones include works on aspects of the production phase in Refs. [5,6] restricting to the gluon fusion mechanism [7], as well as analysis of the subsequent decays of the Higgs bosons produced [8,9]. A complete analysis of the collider signature from
production to decays with the inclusion of signal-background studies is of course the final goal. However, the topic is a complicated one, and may have to be taken one step at a time. The present Letter reports the first step by the present authors in the direction. We aim here at presenting explicit production cross-sections for all the three neutral Higgs bosons through the Higgstrahlung processes from quark–quark collisions, i.e., \( q\bar{q} \rightarrow Z^0 \rightarrow Z^0 H_i \) and \( q\bar{q} \rightarrow W^{\pm*} \rightarrow W^{\pm*} H_i \). The gluon fusion process is generally expected to have the largest cross-section but suffer from very large background \[11\]. Hence, other production channels may also prove to be useful in probing the CP violation in MSSM Higgs physics. It has been emphasized, in Ref. \[4\] for example, that due to the absence of scalar–pseudoscalar mixings, and hence Higgs sector CP violation, in the MSSM. We want to emphasize that this would be a qualitative result, pretty much independent of the details of the exact cross-sections themselves and the explicit determination of which region of parameter space the model lives in. Seeing all the three Higgs channels basically says that the MSSM without the Higgs sector CP violation is not right. One can always go on to models with a richer Higgs spectrum. The CP violating scenario studied here would, however, preferred by many as the viable alternative. We present here production cross-sections of the processes at both the Tevatron and LHC, as our first probe towards the possibility of signals for the CP violation at the two machines. The observability of such Higgs signals is a deeply involved, but obviously very important, question. While a detailed quantitative study is beyond our present report, we will try our best to address the issues involved qualitatively, drawing lessons and comparison from related results in the literature. We would particularly like to draw attention to some plausibly important aspects beyond what has been studied in the literature.

It should be noted that CP violating phases in MSSM are stringently constrained by their contribution to electric dipole moments (EDMs). The topic has been studied extensively \[12\]. It suffices here to emphasize that the EDM constraints can be by-passed, for instance, by effectively decoupling the sparticles of the first family and/or cancellations among the various contributions. Such constraints are not explicitly imposed in the present study. The rationale being that the Higgs sector CP violation involves flavor dependent parameters only of the third family and the complex phase combination \( \Phi_{\text{CP}} = \arg(A_{\mu}) \). This certainly leaves much room for getting around the EDM constraints, by tuning the other parameters and phases for instance.

The tree-level Higgs potential of the MSSM conserves CP. This ensures that the three neutral Higgs mass eigenstates can be divided into the CP-even \( h^0 \) and \( H^0 \) and CP-odd \( A^0 \). The 1-loop effective potential, however, may violate CP. When this happens, three Higgs mass eigenstates with no definite CP parity would be resulted. The Higgs bosons are denoted by \( H_1^0, H_2^0, \) and \( H_3^0 \) (in ascending order of mass). It has been shown that the CP violation may be generated by complex phases residing in the \( \mu \) term and the soft SUSY breaking parameters \( A_t \) (and \( A_b \)). These phases generate contributions to the off-diagonal block \( M_{\text{SP}}^2 \) in neutral Higgs mass-squared matrix \( M_{ij}^2 \) mixing the scalar and pseudoscalar fields. These may be given approximately by \[13\]

\[
M_{\text{SP}}^2 \approx \mathcal{O} \left( \frac{m_t^2 |\mu| |A_t|}{v^2 32\pi^2 M_{\text{SUSY}}^2} \right) \sin \Phi_{\text{CP}} \times \left[ 6, \frac{|A_t|^2}{M_{\text{SUSY}}^2}, \frac{|\mu|^2}{\tan \beta M_{\text{SUSY}}^2}, \frac{\sin 2 \Phi_{\text{CP}} |A_t| |\mu|}{\sin \Phi_{\text{CP}} M_{\text{SUSY}}^2} \right],
\]

(1)

where \( \Phi_{\text{CP}} = \arg(A_t/\mu) \). Here, we have only displayed the contributions from the top squarks \((\tilde{t}_1, \tilde{t}_2)\) which are dominant for small \( \tan \beta \) \[14\]. Sizeable scalar–pseudoscalar mixing is possible for large \(|\mu|\) and \(|A_t|\) (> \( M_{\text{SUSY}} \)).

In our numerical Higgs mass computation, we follow Ref. \[3\] and use the public code available at \[15\]. This involves one-loop effective potential with large two-loop nonlogarithmic corrections induced by one-loop threshold effects on the top and bottom quark Yukawa couplings included. We are interested in regions of parameter space where all three Higgs boson of masses are relatively close to each other, say
all smaller than 200 GeV. Otherwise, the model would be close to the decoupling limit where the radiative CP effect on the Higgs sector is known to be unimportant [1,3]. We are also restricting to relatively small $\tan \beta$ value, with demonstrated substantial scalar-pseudoscalar mixings. For simplicity, we take nonzero a common phase for $A_t$ and $A_b$ as the sole source of $\Phi_{\text{CPR}}$.

At the patron level, to the leading order (LO), the cross-section for a $V H_i$ associated production process is given by

$$\hat{\sigma}_{LO}(q \bar{q} \rightarrow V H_i) = \frac{G_F^2 M_V^4}{288\pi^2 Q^2} (v_q^2 + a_q^2) \overline{\lambda(m_V^2, m_{H_i}^2, Q^2) + 12m_V^2/Q^2} / (1 - m_V^2/Q^2)^2 \times \sqrt{\lambda(m_V^2, m_{H_i}^2, Q^2)} C_i^2,$$  \hspace{1cm} (2)

where $\lambda(x, y, z) = (1 - x/z - y/z)^2 - 4xy/z^2$ and $v_q = -a_q = \sqrt{2}$ for $V = W^\pm$ and $v_q = 2f_q^2 - 4e_q \sin^2 \theta_W$, $a_q = 2f_q^2$ for $V = Z^0$ ($q' = q$); while $C_i$ is the $VVH_i$ coupling renormalized to $e M_V^2 / \cos \theta_W$.

A crucial point here is that the three Higgs bosons mix through an “orthogonal” matrix, leading to a sum rule for the $C_i$ couplings [16]. Namely,

$$C_1^2 + C_2^2 + C_3^2 = 1.$$  \hspace{1cm} (3)

The sum rule is well appreciated among researchers on the subject. It guarantees that at least one of the three production cross-sections is not suppressed. This feature is much exploited in Higgs discovery studies. The sum rule also suggests that all the three cross-sections can be simultaneously substantial for some particular set of the relevant SUSY parameters. The latter feature plays a central role in our present analysis. Explicit presentations of the variations of the $C_i$’s, or $VVH_i$ couplings, for the CP violating case of interest here have been given in many of the earlier works.\(^2\) Hence, we skip explicit numerical presentation here.

We convolute the admissible patron sub-process cross-section given above with the CTEQ4L patron distribution functions. QCD corrections are known to give an enhancement of about 40% for the Tevatron (Run II) and 30% for the LHC [18]. Other SUSY corrections are generally small [19]. In the latter reference, it is shown to be less than 1.5%, being smaller for large squark/gluino masses. In the explicit plots given in the figures, we scaled the LO cross-sections calculated with the corresponding enhancement $K = \sigma_{\text{NLO}} / \sigma_{\text{LO}}$ factor, to give a better indication of the next-leading order (NLO) results. The $K$ factor is taken simply to be 1.4 and 1.3 for the Tevatron and LHC, respectively. This should be good enough for the present purpose. Readers interested in further details on the tiny variations in the exact $K$ factor value along with the changes of the model parameters are referred to Ref. [19].

We illustrate our results with two representative set of chosen input parameters. Parameter Set A is chosen in accordance with the benchmark scenario (CPX) introduced in Ref. [17] aimed at maximizing the CP violating effects. The CPX scenario is as follows:

$$M_Q = \tilde{M}_t = \tilde{M}_b = 1 \text{ TeV}, \quad \mu = 4 \text{ TeV},$$
$$|A_t| = |A_b| = 2 \text{ TeV},$$
$$|m_{\tilde{g}}| = 1 \text{ TeV} \quad \text{and} \quad |m_{\tilde{\mu}}| = |m_{\tilde{\nu}}| = 0.3 \text{ TeV},$$  \hspace{1cm} (4)

added to which we fixed $\tan \beta = 6$ for Set A inputs. Parameter Set B is similar, with little variations, namely

$$M_Q = \tilde{M}_t = \tilde{M}_b = 1 \text{ TeV}, \quad \mu = 2 \text{ TeV},$$
$$|A_t| = |A_b| = 2 \text{ TeV},$$
$$|m_{\tilde{g}}| = 1 \text{ TeV} \quad \text{and} \quad |m_{\tilde{\mu}}| = |m_{\tilde{\nu}}| = 0.3 \text{ TeV},$$
$$\tan \beta = 15.$$  \hspace{1cm} (5)

The charged Higgs mass is taken as the control parameter on the scale of the Higgs masses. We show only results for two choices of the charged Higgs mass, at 150 GeV and 200 GeV. The 150 GeV gives $H_1$ mass very close to the known bound from LEP. The 200 GeV case gives a relatively high mass scale value getting close to, while still staying away from, the decoupling limit. Hence, the two choices roughly envelope the range of interest. The production cross-sections are plotted as a function of $\Phi_{\text{CPR}}$, which comes here from a common phase of $A_t$ and $A_b$. Explicit plots of the masses are also given for easy cross-reference. Figs. 1 and 2 are results for the Tevatron,
Fig. 1. Figure (a) and (d) show the variation of three Higgs masses with the CP violating phase $\Phi_{CP}$. Figures (b), (e) and (c), (f) show the variation of $ZH_i$ and $WH_i$ ($i = 1, 2, 3$) production cs at Tevatron Run II energy with the phase $\Phi_{CP}$. The left panel and right panel correspond to the charged Higgs mass = 150 GeV and 200 GeV, respectively. Other MSSM parameter is fixed to Set A.

Based on Set A and Set B inputs, respectively, while Figs. 3 and 4 give the corresponding results for the LHC. The figures do illustrate the existence of the exciting possibility we go after, namely, having substantial production cross-sections simultaneously for all the three $VH_i$ channels. This should not come as a surprise. Naively, the sum rule [cf. Eq. (3)] suggests there might be a "democratic" limit where the three channels share the overall coupling equally. In that situation, each $H_i$ would have a production cross-section at the same order of magnitude as that of the SM Higgs of the same mass, only suppressed by a small factor. Our results do indicate explicitly that the scenario could be more or less achieved for some optimal $\Phi_{CP}$ value.

Let us also briefly comment on the basic features of the plots. The general features of the dependence of the masses and gauge couplings [represented by the $C_i$'s in Eq. (2) above] of the three (physical) Higgs bosons upon the CP phase $\Phi_{CP}$ through the stop mixing parameter $|X_t| = |A_t - \mu \cot \beta|$ have been studied by various groups (see Ref. [3] and references therein). In each of the top panels of Figs. 1–4, our explicitly illustrated Higgs masses agree well with previous studies. A particularly note-worthy aspect is that the (one loop corrected) $H_1$ mass increases...
Fig. 2. Figure (a) and (d) show the variation of three Higgs masses with the CP violating phase $\Phi_{\text{CP}}$. Figures (b), (e) and (c), (f) show the variation of $ZH_i$ and $WH_i$ ($i = 1, 2, 3$) production cross-sections at Tevatron Run II energy with the phase $\Phi_{\text{CP}}$. The left panel and right panel correspond to the charged Higgs mass of 150 and 200 GeV, respectively. Other MSSM parameter is fixed to Set B.

with $|X_t|$ till reaching its maximum at $|X_t|/M_{\text{squark}} \simeq 2.45$, and drops with further increase in $|X_t|$. Here within each panel, the latter is tuned with $\Phi_{\text{CP}} \simeq \arg(A_t)$. In the plot of Fig. 1(a), for example, $H_1$ mass is maximum at $\Phi_{\text{CP}} \simeq 80^\circ$. The large effect of the CP phase enhancing stop mixing here promotes scalar-pseudoscalar mixings, hence suppresses $C_1$. This, together with the $H_1$ mass enhancement, gives a minimum for the $VH_1$ production cross-sections at $\Phi_{\text{CP}} \simeq 88^\circ$, as shown in the plots right below [Fig. 1(b) and (c)].

3 More explicitly, from the relation $C_i = \cos \beta \, O_{1i} + \sin \beta \, O_{2i}$, we have $C_1 = \sin \beta \, O_{21}$ for large $\tan \beta$. $C_1$ is then dominated by $O_{21}$ which flips sign. That is why we have a dip in the plot.

the character of the pseudoscalar around this point. The situation is almost completely reversed for $H_2$, which simply corresponds to the pseudoscalar at the vanishing $\Phi_{\text{CP}}$ limit. In this case, $H_3$ is not much affected by variation in $\Phi_{\text{CP}}$ except when it assumes the character of the pseudoscalar at the other CP conserving limit of $\Phi_{\text{CP}} = 180^\circ$. With larger mass splitting between $H_2, H_3$ and the lightest Higgs $H_1$ as given by the second case in Fig. 1 [plots (d), (e), and (f)], we are getting closer to the decoupling limit. The $H_1$ then behaves like the SM Higgs. It not much affected by $\Phi_{\text{CP}}$ variation, which now mainly describes mixing between the two heavier states. With the same set of inputs, the features in Fig. 3 are more or less the same as in Fig. 1. Figs. 2 and 4
Fig. 3. Figure (a) and (d) show the variation of three Higgs masses with the CP violating phase $\Phi_{CP}$. Figures (b), (e) and (c), (f) show the variation of $Z H_i$ and $W H_i$ ($i = 1, 2, 3$) production $\sigma$ at LHC with the phase $\Phi_{CP}$. The left panel and right panel correspond to the charged Higgs mass of 150 and 200 GeV, respectively. Other MSSM parameter is fixed to Set A.

are results from a different input set of parameters (Set B). The set of input parameters is not very different from the previous case though, as we are strongly confined by our special interest in large $\Phi_{CP}$ induced mass mixings. The point where $V H_1$ cross-sections are strongly suppressed while $V H_2$ cross-sections enhanced in Fig. 2 (cf. Fig. 1) is now shifted to $\Phi_{CP} \approx 110^\circ$. In all the cases, roughly at the central values of $\Phi_{CP}$ between two of the dips of the three $V H_1$ cross-section curves, one gets the solutions for substantial cross-sections for all three Higgs channels simultaneously.

One of our major result is that in the most favorable situation, there is a chance that all three Higgs boson could be simultaneously produced with around or above 0.01 pb cross-sections at the Tevatron. The mass region of interest to us is in fact happens to be just well covered by the machine. This suggests the exciting possibility of seeing Higgs sector CP violation, assuming MSSM. At the LHC, the cross-sections could all go simultaneously to around or above 0.1 pb. The more or less optimal case correspond to results illustrated in the sets of left panels in the figures, with masses for all three $H_i$ below 150 GeV. The sets of right panels in the figures, however, illustrate roughly the cases of limiting capacity. Here, $H_2$ and $H_3$ are a bit heavier. They are around 180–190 GeV. As we will discuss below, this might be really pushing the limit on
Fig. 4. Figure (a) and (d) show the variation of three Higgs masses with the CP violating phase $\Phi_{CP}$. Figures (b), (e) and (c), (f) show the variation of $ZH_i$ and $WH_i$ ($i = 1, 2, 3$) production cs at LHC with the phase $\Phi_{CP}$. The left panel and right panel correspond to the charged Higgs mass of 150 and 200 GeV, respectively. Other MSSM parameter is fixed to Set B.

signal observability. However, it is our opinion that a careful and detailed analysis is required to set the definite mass reach. The latter may get somewhat close to the situation illustrated in these panels. This is especially true in the case of LHC.

One certainly should not be too optimistic about the scenario. We put the question mark in the title of this Letter because there are good reasons to be cautious. Assuming that the MSSM really falls into such a parameter space region where the three CP violating Higgs boson can all be produced with substantial cross-sections at the Tevatron or LHC, identifying the signals may be daunting task. In the spirit of our present analysis, one can claim that the CP violating Higgs bosons are unambiguously observed only after we have successfully identified all the three $H_i$’s produced through their decays. Otherwise, one may have to rely on details on cross-section measurements and further inputs from other possible SUSY signals (if at all available) to pin down the Higgs signals as coming from CP violating MSSM. We emphasize again that identifying three Higgs mass eigenstates produced through the $V H_i$ channels in itself establishes Higgs sector CP violation, i.e., assuming MSSM. With anything less than that, it is going to be an extremely involved task to confirm that the Higgs boson(s) observed is/are more than generic MSSM Higgs boson(s).
Observing “intermediate mass” neutral Higgs bosons are notoriously difficult (see, for example, Ref. [7]). The “gold-plated” 4-lepton decay mode $H \rightarrow Z^0Z^0 \rightarrow l^+l^-\bar{l}'\bar{l}'$ [20] has good branching ratio only for heavier Higgs bosons, while the $2\gamma$ mode $H \rightarrow \gamma\gamma$ has steeply increasing background [20]. Associated productions, the presently considered $VH$ [20,21], $t\bar{t}H$ [22], or even $t\bar{t}H$ [23] have been advocated as viable alternatives to the gluon fusion process to be the focus of Higgs hunting exactly for that reason. A recent paper [24] gives detailed discussions on the various associated production processes focusing on the effects of large stop mixing. The latter, while closer to our present study, is still limited to the CP-conserving MSSM. This is also the case for most of the previous studies [25]. Such studies, while of some use in giving a rough idea on the observability of the Higgs signals under discussion, certainly cannot take the place of the necessary specific studies, as along the line recently started by Refs. [8,9]. However, results from the latter references are not enough to help us to reach any definite conclusion.

Ref. [9] in particular gives interesting results on partial branching fractions of the various decays for the three $H_i$’s separately. It is a big step in the direction. What is still missing though are signal versus background analyses. The reference gives no definite mass reach numbers for the $H_i$’s. One would like to have the definite reach of the hadron machines in terms of the mass control parameter (i.e., charged Higgs mass) as a function of $\Phi_{CP}$ reflecting the range of simultaneous observability of all three Higgs signals. This ambitious task is beyond the present short Letter. Mass reach numbers for MSSM Higgs bosons are not widely available, even for the general case without Higgs sector CP violation. We can only present below discussions based on available information in the literature. The discussion aims at giving some idea on what might be expected. Hence, it may have some speculative element. We will quote some mass reach number for the SM, or SM-like lightest MSSM, Higgs. These numbers are of course not directly applicable to our scenario. However, the SM case is much better studied. We have pointed out above that the coupling sum rule [cf. Eq. (3)] close to the “democratic” limit suggests a scenario in which the three $H_i$’s kind of share equally the role of the SM Higgs. Each $H_i$ then behaves like “a third of” the SM Higgs at the same mass. If one take the latter statement seriously while assuming all the other SUSY particles are heavier, we have a situation where the SM Higgs numbers do provide a useful guideline.

For Higgs mass in the upper intermediate range, 135–180 GeV, the planned focus is rather on production through gluon fusion with subsequent decay $H \rightarrow W^+W^-$ [28]. The reference claims that the mass reach for a SM like Higgs, with an integrated luminosity of 30 fb$^{-1}$, would be up to 190 GeV. A reason behind is the strong background for the $b\bar{b}$ signal. In our scenario, at least for masses of around or above 150 GeV, $H_i \rightarrow WW^{(*)}$ could be sizeable for all three Higgs bosons [9]. On the other hand, $b\bar{b}$ coupling(s) would be suppressed for the heavier Higgs states. Obviously, we cannot rely on the $H \rightarrow b\bar{b}$ channel to see all the three $H_i$’s then. Nevertheless, unlike the SM case, using gluon fusion does not help confirming the CP violating setting we are interested in here. In this regard, a previous study on trilepton Higgs signal [31] is particularly relevant. We will very likely have to rely on the $WH \rightarrow WW\gamma \rightarrow 3l$ decay to identify at least one or two of the $VH_i$ channels. With an integrated luminosity of 100 fb$^{-1}$, Ref. [31] claims a limiting 3$\sigma$ reach for the SM Higgs in mass range 140–175 GeV, with suggestions on further gains to be achieved. We conclude that a combined study of the $VH$ production processes with decay channels and signal–background

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4 We should add that the $t\bar{t}H$ with $H \rightarrow b\bar{b}$ has also been advocated as a discovery mode at Tevatron [29].
analyses specifically for the CP violating MSSM scenario will have to take the trilepton signal as one of the major focus.

The situation looks much better at the LHC. Studies for SM-like Higgs shown that, for the $VH$ associated production under consideration, the $H \to b\bar{b}$ channel should have reasonable efficiency in identifying the signal [30], at least in the lower intermediate mass range. $H \to \gamma\gamma$ would also be useful [21,25]. Again, in the upper intermediate mass range, even for a SM-like Higgs one will have to go back to $H \to WW^{(*)}$ (or $H \to Z^0Z^0 \to l^+l^-l^+l^-$). In addition, we emphasize again that the $H b\bar{b}$ couplings are very unlikely to be simultaneously unsuppressed. Hence, the $WH \to WWW^{(*)} \to 3l$ type trilepton signals are definitely useful for probing the CP violating model. Ref. [31] claims a 5$\sigma$ discovery reach for the mass range 140–180 (125–200) GeV with 30 (100) fb$^{-1}$ for a SM Higgs. One may naively expect the signal for each $H_i$ to be weaken by a third or so in the optimal case of “democratically shared” couplings (equal $C_i$’s). That sounds quite encouraging.

It should be noted that, in general, possible decays of $H_2$ and $H_3$ through $H_1$ itself may compete with the $H_i \to WW^{(*)}$ channels and complicates analyses of the latter. However, in the region of parameter space of interest here, such decays are unlikely to be important and hence not taken into consideration here. Finally, we should mention that decays into superparticles are likely to dominate if their masses put them within kinematic threshold of the Higgs decays. This is very unlikely for the scenario we are interested in here, as one can easily see from the illustrative parameter input Set A and B given above, hence not discussed.

Perhaps we should also mention a related production mechanism, the weak boson fusion channels (see Ref. [32] and references therein). Similar to the associated productions, the processes exploit the $VVH$ couplings. It is advocated as a possible Higgs discovery channel at LHC, and a useful tool to determine the CP nature of the Higgs boson involved. From the present perspective, it will be interesting to explore the possibility of simultaneous observation of all three $VV \to H_i$ channels as a probe for the Higgs sector CP violation.

In summary, we illustrate in this Letter explicit results on the production cross-sections of the three $VH_i$ channels at the Tevatron and the LHC. Our results indicate that a simultaneous observation of the three channels may be a possibility, though could be quite marginal at the Tevatron. In the best scenario, MSSM with radiative Higgs sector CP violation could give rise to cross-sections of the order or above 0.01 pb and 0.1 pb, for the Tevatron and the LHC, respectively. Detail signal–background analyses exploring various decay modes are called for. Nevertheless, we hope that the above discussions have convinced the readers that there are good reasons to be optimistic. In particular, we point out that the $WH_i \to WWW^{(*)} \to 3l$ decays are going to be useful. Assuming MSSM, the simultaneous observations of the three $VH_i$ channels will confirm the radiative Higgs sector CP violation scenario.

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CPT violation and the nature of neutrinos

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Abstract

In order to accommodate the neutrino oscillation signals from the solar, atmospheric, and LSND data, a sterile fourth neutrino is generally invoked, though the fits to the data are becoming more and more constrained. However, it has recently been shown that the data can be explained with only three neutrinos, if one invokes CPT violation to allow different masses and mixing angles for neutrinos and antineutrinos. We explore the nature of neutrinos in such CPT-violating scenarios. Majorana neutrino masses are allowed, but in general, there are no longer Majorana neutrinos in the conventional sense. However, CPT-violating models still have interesting consequences for neutrinoless double beta decay. Compared to the usual case, while the larger mass scale (from LSND) may appear, a greater degree of suppression can also occur. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In recent years, stronger and stronger experimental evidence for neutrino oscillations has been accumulating. As is well known, this evidence would extend the Standard Model by requiring neutrino masses and mixings. While knowing the values of the mass and mixing parameters may be an important clue to physics beyond the Standard Model, more information is needed. For example, it is presently unknown whether neutrino processes violate lepton number or not, or whether the neutrinos are their own antiparticles or not.

Neutrino masses and mixings can be straightforwardly included in the Standard Model, as can the choice of Majorana or Dirac neutrinos. But in fact, the present neutrino oscillation evidence may imply more profound extensions to the Standard Model. With three neutrinos, there are only two independent mass-squared differences. However, since the mass-squared differences required by the solar [1], atmospheric [2], and LSND [3] neutrinos are \( \delta m^2 \approx 10^{-5} \text{ eV}^2, 10^{-3} \text{ eV}^2, \) and \( 1 \text{ eV}^2 \), respectively, three different mass-squared scales are needed to explain all the data. The usual way out is to postulate a fourth neutrino, thus allowing three independent mass-squared...
differences. This fourth neutrino must be sterile under the weak interactions since the invisible width of the $Z^0$ only allows three light active neutrinos.

The LSND results are based on an appearance experiment, namely $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$. For years, the solar and atmospheric neutrino results only gave clear evidence for the disappearance of $\nu_e$ or $\nu_\mu$, $\bar{\nu}_\mu$, respectively. Thus one or the other could have been accommodated by oscillations to sterile neutrinos. However, the difficulties that seem to require the introduction of a fourth neutrino have recently become more acute as the combined solar neutrino results from the Sudbury Neutrino Observatory and Super-Kamiokande indicate that $\nu_e \rightarrow \nu_\mu$, $\nu_\tau$, and the atmospheric neutrino results from Super-Kamiokande indicate that $\nu_\mu$, $\bar{\nu}_\mu \rightarrow \nu_\tau$, $\bar{\nu}_\tau$. Thus with three signals of neutrino oscillations among active flavors, there is not only a problem of not enough independent mass-squared differences, but also a problem of where to incorporate the required mixing with the sterile neutrino. While four-neutrino models may still work, it is only with difficulty (see, e.g., Ref. [4]).

Recently, an intriguing but speculative suggestion to accommodate these results has been made [5–7]. If CPT is violated in the neutrino sector, then the mixing parameters which govern solar neutrino oscillations of $\nu_e \rightarrow \nu_\mu$, $\nu_\tau$, and the atmospheric neutrino results from Super-Kamiokande indicate that $\nu_\mu$, $\bar{\nu}_\mu \rightarrow \nu_\tau$, $\bar{\nu}_\tau$. Thus with three signals of neutrino oscillations among active flavors, there is not only a problem of not enough independent mass-squared differences, but also a problem of where to incorporate the required mixing with the sterile neutrino. While four-neutrino models may still work, it is only with difficulty (see, e.g., Ref. [4]).

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We show that there are interesting practical consequences for neutrinoless double beta decay, a process that violates lepton number by two units. One way to explain the present null1 results [17] in the usual CPT-conserving scenario would be to say that the neutrino masses are all below say, 0.1 eV (larger masses can be accommodated if the mixing angles and Majorana phases cause cancellations). Given present constraints on the oscillation parameters, there can be a nonzero minimum allowed effective mass. In the CPT-violating model the overall scale may be set by the large LSND mass scale of $\sim 1$ eV, i.e., predicting a larger signal. However, a larger degree of suppression can occur, so that the effective mass can also be arbitrarily small.

2. CPT violation and the neutrino masses

In the Standard Model, extended to include neutrino masses, the interactions are lepton-number ($L$) conserving. Any nonconservation of $L$ would come from Majorana mass terms, which turn a neutrino into an antineutrino. When CPT is conserved and Majorana mass terms are present, the neutrino mass eigenstates $\nu_i$ are Majorana particles. That is, each $\nu_i$ is its own

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1 Though evidence for neutrinoless double beta decay has very recently been claimed in Ref. [15], it has been disputed in Ref. [16].
violated, we consider the simplest case of a single neutrino \( \nu \) (i.e., \( \nu_e \)), coupled to the electron by the Standard Model weak coupling, and its CPT conjugate \( \bar{\nu} \), coupled to the positron. When allowing CPT violation, we do not assume that \( \nu \) and \( \bar{\nu} \) have the same mass.

Rather, we suppose that for a given spin direction, the \( \nu, \bar{\nu} \) mass matrix \( M_\nu \) has the form

\[
M_\nu = \begin{bmatrix}
\mu + \Delta & y^* \\
y & \mu - \Delta
\end{bmatrix},
\]

where the first and second rows correspond to \( \nu \) and \( \bar{\nu} \), respectively.

Under CPT, \( \nu \) and \( \bar{\nu} \) transform as

\[
\text{CPT}(\nu) = e^{i\xi} |\bar{\nu}\rangle, \quad \text{CPT}(\bar{\nu}) = e^{i\xi} |\nu\rangle,
\]

where we have left the phase \( \xi \) free. Since CPT is an antiunitary operator,

\[
\text{CPT}(\text{CPT}(\nu)) = \text{CPT}(e^{i\xi} |\bar{\nu}\rangle) = e^{-i\xi} \text{CPT}(|\bar{\nu}\rangle) = e^{i(\xi - \theta)} |\nu\rangle = |\nu\rangle.
\]

From Eqs. (3), (4),

\[
\text{CPT}(\nu_+) = e^{i(\xi - \phi)} [\sin \theta_{\nu\bar{\nu}} |\nu\rangle + e^{i\phi} \cos \theta_{\nu\bar{\nu}} |\bar{\nu}\rangle]
\]

and

\[
\text{CPT}(\nu_-) = -e^{i(\xi - \phi)} [-\cos \theta_{\nu\bar{\nu}} |\nu\rangle + e^{i\phi} \sin \theta_{\nu\bar{\nu}} |\bar{\nu}\rangle].
\]

Comparing with Eqs. (3), (4), one sees that the mass eigenstates \( \nu_{\pm} \) are Majorana states (that is, CPT self-conjugate apart from a phase) if and only if \( \theta_{\nu\bar{\nu}} = \pi/4 \). But, from Eq. (5), this value of \( \theta_{\nu\bar{\nu}} \) corresponds to \( \Delta = 0 \); i.e., to an absence of CPT violation. If

Fig. 1. Diagram for neutrinoless double beta decay and the leptonic amplitude considered below. This diagram corresponds to the CPT-conserving case; the modifications for the CPT-violating case are considered below.

antiparticle in the sense that

\[
\text{CPT}(|\nu_i\rangle) = e^{i\xi} |\bar{\nu}_i\rangle,
\]

where \( \xi_i \) is a phase. Through the process pictured in Fig. 1, exchange of the Majorana \( \nu_i \) leads to neutrinoless double beta decay (\( \beta\beta_0 \)). This is the \( L \)-nonconserving reaction in which one nucleus decays to another with the emission of two electrons. Conversely, even if \( \beta\beta_0 \) should arise predominantly from some mechanism other than light neutrino exchange, the observation of this decay would imply that the electron neutrino \( \nu_e \) has a nonzero Majorana mass.

When CPT is conserved, this would in turn imply that the mass eigenstates \( \nu_i \) are Majorana particles. In a CPT-conserving world, when there are no Majorana mass terms, \( L \) is conserved, \( \beta\beta_0 \) is forbidden, and the mass eigenstates \( \nu_i \) are Dirac particles. That is, each \( \nu_i \) differs from its CPT conjugate by the value of the conserved quantum number \( L \): \( \nu_i \) has \( L = +1 \) while \( \bar{\nu}_i \) has \( L = -1 \).

To see what becomes of this picture when CPT is violated, we consider the simplest case of a single neutrino \( \nu \) (i.e., \( \nu_e \)), coupled to the electron by the Standard Model weak coupling, and its CPT conjugate \( \bar{\nu} \), coupled to the positron. When allowing CPT violation, we do not assume that \( \nu \) and \( \bar{\nu} \) have the same mass.

Rather, we suppose that for a given spin direction, the \( \nu, \bar{\nu} \) mass matrix \( M_\nu \) has the form

\[
M_\nu = \begin{bmatrix}
\mu + \Delta & y^* \\
y & \mu - \Delta
\end{bmatrix},
\]

where the first and second rows correspond to \( \nu \) and \( \bar{\nu} \), respectively. We neglect the possibility of neutrino decay, so that \( M_\nu \) must be Hermitian, which implies that the mass parameters \( \mu \) and \( \Delta \) (Dirac masses) are real. Any nonvanishing \( \Delta \) is a violation of the CPT constraint that a particle and its antiparticle must have the same mass. Any nonvanishing \( y \) (Majorana mass), which mixes \( \nu \) and \( \bar{\nu} \), is a violation of \( L \) conservation. (Note that limits on CPT violation entering via a heavy Majorana mass have been considered in Ref. [13]).

The eigenstates of \( M_\nu \) are

\[
|\nu_+\rangle = \cos \theta_{\nu\bar{\nu}} |\nu\rangle + e^{i\phi} \sin \theta_{\nu\bar{\nu}} |\bar{\nu}\rangle
\]

with mass \( m_+ = \mu + \sqrt{|y|^2 + \Delta^2} \); and

\[
|\nu_-\rangle = -\sin \theta_{\nu\bar{\nu}} |\nu\rangle + e^{i\phi} \cos \theta_{\nu\bar{\nu}} |\bar{\nu}\rangle
\]

with mass \( m_- = \mu - \sqrt{|y|^2 + \Delta^2} \). The neutrino–antineutrino mixing angle is given by

\[
\tan 2\theta_{\nu\bar{\nu}} = \frac{|y|}{\Delta},
\]

and \( \phi = \arg(y) \).

Under CPT, \( \nu \) and \( \bar{\nu} \) transform as

\[
\text{CPT}(\nu) = e^{i\xi} |\bar{\nu}\rangle, \quad \text{CPT}(\bar{\nu}) = e^{i\xi} |\nu\rangle,
\]

where we have left the phase \( \xi \) free. Since CPT is an antiunitary operator,

\[
\text{CPT}(\text{CPT}(\nu)) = \text{CPT}(e^{i\xi} |\bar{\nu}\rangle) = e^{-i\xi} \text{CPT}(|\bar{\nu}\rangle) = e^{i(\xi - \theta)} |\nu\rangle = |\nu\rangle.
\]

From Eqs. (3), (4),

\[
\text{CPT}(\nu_+) = e^{i(\xi - \phi)} [\sin \theta_{\nu\bar{\nu}} |\nu\rangle + e^{i\phi} \cos \theta_{\nu\bar{\nu}} |\bar{\nu}\rangle]
\]

and

\[
\text{CPT}(\nu_-) = -e^{i(\xi - \phi)} [-\cos \theta_{\nu\bar{\nu}} |\nu\rangle + e^{i\phi} \sin \theta_{\nu\bar{\nu}} |\bar{\nu}\rangle].
\]

Comparing with Eqs. (3), (4), one sees that the mass eigenstates \( \nu_{\pm} \) are Majorana states (that is, CPT self-conjugate apart from a phase) if and only if \( \theta_{\nu\bar{\nu}} = \pi/4 \). But, from Eq. (5), this value of \( \theta_{\nu\bar{\nu}} \) corresponds to \( \Delta = 0 \); i.e., to an absence of CPT violation. If
\( \Delta \neq 0 \), so that CPT is not conserved, the neutrino mass eigenstates can no longer be Majorana particles. Nevertheless, if \( \gamma \) is nonzero, then there is \( \nu \rightleftharpoons \bar{\nu} \) mixing, \( L \) is not conserved, and \( \beta \beta_{0v} \) can occur. But if it does occur, that would imply only that \( L \) is violated and that there is a “Majorana” (i.e., \( \nu \rightleftharpoons \bar{\nu} \) mixing) mass term, and not that the neutrino mass eigenstates are CPT self-conjugate.

It is interesting to compare this situation with the neutral kaon system when a possible CPT-violating term is introduced as a difference between the \( K^0 \) and \( \bar{K}^0 \) masses (e.g., see Ref. [18]). To compare to neutrinos, we neglect kaon decay. Then the kaon mass matrix is identical to the \( M_\nu \) of Eq. (2), but with the first and second rows now corresponding to \( K^0 \) and \( \bar{K}^0 \), respectively. The neutral kaon mass eigenstates (these correspond to \( K_S \) and \( K_L \) in the usual case) are described by Eqs. (3) and (4), with \( \nu \) replaced by \( K^0 \) and \( \bar{\nu} \) replaced by \( \bar{K}^0 \). The CPT conjugates of these states are described by Eqs. (8) and (9), with the same replacements. Once again, the mass eigenstates are CPT self-conjugate if and only if the CPT-violating parameter \( \Delta \) vanishes. Otherwise, each of them differs from its CPT conjugate. Nevertheless, so long as the “Majorana” \( K^0-\bar{K}^0 \) mixing term \( \gamma \) is present, then strangeness (playing the role of \( L \)) is not conserved. A kaon born as a \( K^0 \) can evolve into a \( K^0-\bar{K}^0 \) mixture. Observing, via the scattering of the kaon, that such evolution had occurred, would imply that strangeness is not conserved, and in particular, that \( K^0 \) and \( \bar{K}^0 \) mix, but not that the mass eigenstates are self-conjugate.

### 3. Neutrinoless double beta decay

Neutrinoless double beta decay [17] is the process

\[
(A, Z) \rightarrow (A, Z + 2) + 2e^{-}
\]

in which a nucleus decays into another nucleus with the emission of two electron; see Fig. 1. The emission of two \( W \) bosons by the initial nucleus is described by a nuclear matrix element, not considered here. The full amplitude for this process is proportional to just the leptonic amplitude, which is

\[
A = \sum_{i,h} \langle e^{-} e^{-} | H_W | e^{-} \nu_i W^- \rangle \langle e^{-} \nu_i W^- | H_W | W^- W^- \rangle,
\]

where \( i \) labels the neutrino mass eigenstates and \( h \) their helicities. Each mass eigenstate can be represented as a superposition of neutrinos and antineutrinos, so that

\[
|\nu_i\rangle = \sum_{\alpha=e,\mu,\tau} U_{\alpha i} |\nu_\alpha\rangle + \sum_{\alpha=e,\mu,\tau} \bar{U}_{\alpha i} |\bar{\nu}_\alpha\rangle,
\]

which defines the matrices \( U \) and \( \bar{U} \) for neutrinos and antineutrinos, respectively. Absorbing a \( \nu_l \) yields an \( l^- \), and similarly for a \( \bar{\nu}_l \) and an \( l^+ \). Because of the helicity mismatch between the neutrino and antineutrino interaction, the leptonic amplitude is proportional to

\[
A \propto \sum_i m_i U_{ei} \bar{U}_{ei},
\]

where the sum is over mass eigenstates. In particular, for the one-flavor case discussed above,

\[
A \propto \sin \theta_{\nu \bar{\nu}} \cos \theta_{\nu \bar{\nu}} (m_+ - m_-)
\]

\[
= \sin 2\theta_{\nu \bar{\nu}} \sqrt{|y|^2 + \Delta^2} = |y|.
\]

As expected, the Majorana mass term \( y \) contributes to the amplitude. In fact, with just a single neutrino flavor, it is the only contribution, i.e., the value of \( \Delta \) is irrelevant. In the more relevant case where more than one neutrino family is involved, there are in general contributions depending on the CPT-violating mass term \( \Delta \), and hence the LSND scale of 1 eV. We show this with an explicit example below. Even at this stage, it is easy to understand that if the neutrino and antineutrino mass matrices are different, the diagonalizing matrices must also be different. The difference will be reflected in the matrices \( U \) and \( \bar{U} \) in Eq. (13). The degree of difference depends on \( \Delta \), the CPT-violating mass term, and of course vanishes when \( \Delta = 0 \).

Note that the usual CPT-conserving amplitude is obtained with \( \bar{U} = U \) in Eq. (13). At this point two remarks are in order. First, one should remember that in a CPT-violating world \( \bar{U} \neq U \) and second, Eq. (13) is obtained without depending on the usual Feynman rules by constructing the amplitudes from quantum mechanics and therefore is quite safe regardless whether our neutrinos conserve CPT or not. It must be emphasized that the same expression can also be obtained by using the method suggested in [6], i.e., by calculating the matrix elements as if they belonged to
a CPT-conserving neutrino in an artificial background of matter. In this case (as it happens in field theory at finite temperature) the Feynman rules for the physical vertices are the same as those in the CPT-conserving theory. The propagators, however, are different. For momenta much higher than the neutrino masses, the propagator in matter can be written as an expansion in powers of the CPT-violating mass term, whose first term is the standard propagator.

Double beta decay will have no manifest CPT violation, in the sense that the CPT-conjugated process involving two positrons will have exactly the same rate. This observation follows from the fact that the corresponding amplitude for two positrons is simply the complex conjugate of that for two electrons. However, CPT violation does have an experimentally observable consequence. Namely, since in the CPT-violating case the double beta decay amplitude involves two independent matrices \(U\) and \(\bar{U}\), the decay rate can reach values that are outside those allowed in the usual CPT-conserving case.

In order to make this statement even more transparent, we first recall the result for the CPT-conserving case. The usual expression (assuming only two flavors and degenerate masses) is

\[
m_{\beta\beta} = m_0 |\cos^2 \theta + \sin^2 \theta \ e^{2i\alpha}|, \tag{15}\]

where \(\theta\) is the angle involved in the solar neutrino solution and \(\alpha\) is a CP-violating phase characteristic of Majorana neutrinos which does not appear in oscillation-related phenomena. Since SNO sees 1/3 of the expected \(\nu_e\) flux (i.e., less than 1/2), the mixing angle must be less than maximal (the best-fit value is about 30°) [1]. Thus there is a minimum value for \(m_{\beta\beta}\), obtained when \(\alpha = \pi/2\). The range for \(m_{\beta\beta}\) is thus

\[
m_0 \cos(2\theta_\odot) < m_{\beta\beta} < m_0. \tag{16}\]

Presently, \(\cos(2\theta_\odot) \simeq 0.4\), and the overall mass scale can be very small. In the more general three-flavor CPT-conserving case, the expression for \(m_{\beta\beta}\) is more complicated, but it retains the feature of a minimum value (except at a singular point) [19].

When CPT is violated, the neutrino and antineutrino mixing matrices can be different, and so can the neutrino and antineutrino masses. If MiniBooNE confirms LSND, then we know that a mass of about 1 eV exists (this is the case of most interest to CPT violation). This mass scale is larger than required in the absence of the LSND oscillation signal, raising hope that it might give a large neutrinoless double beta decay signal. In general, it is much more complicated to treat the full problem, where there are now six mass eigenstates, each containing components of both neutrinos and antineutrinos of all three flavors. This can be visualized by merging the two spectra in Fig. 1 in Ref. [5], and allowing neutrino–antineutrino admixtures, permitted by the Majorana masses. That combined spectrum is consistent with the solar, atmospheric, and LSND data.

However, we know that for this spectrum the state with mass corresponding to the LSND mass scale is the dominant term in the sum for \(m_{\beta\beta}\) given by Eq. (13), which is then reduced to

\[
m_{\beta\beta} \approx m_{\text{LSND}} U_{e,\text{LSND}} \bar{U}_{e,\text{LSND}}. \tag{17}\]

In this picture, \(|U_{e,\text{LSND}}|^2\) represents the electron neutrino content of the highest mass eigenstate, and similarly for \(|\bar{U}_{e,\text{LSND}}|^2\) and the electron antineutrino content. In order to explain all the data, the electron antineutrino content should be very large (\(\simeq 99\%\)) [6], and the remainder can be filled by other neutrino and antineutrino flavors. These proportions in particular can satisfy the bounds on reactor \(\bar{\nu}_e\) disappearance. In the case where this remainder is mostly the electron neutrino component, then \(m_{\beta\beta}\) reaches its maximal value, given by \(\simeq 0.1 \ m_{\text{LSND}}\). On the other hand, there is no lower limit of the electron neutrino content, so finally,

\[
0 < m_{\beta\beta} < 0.1 \ m_{\text{LSND}}. \tag{18}\]

Though beyond the scope of this work, it would be very interesting to consider in more detail the allowed numerical range of \(m_{\beta\beta}\) in realistic three-flavor models with CPT violation.

### 4. Conclusions

Present and forthcoming experiments devoted to studying neutrino oscillations should give us much more information about the neutrino mass-squared differences and mixing angles. However, in order to really understand the neutrino sector we will also need to know the absolute mass scale and the nature (Dirac or Majorana) of the neutrino mass terms. Neutrinoless
double beta decay experiments are crucial in this respect, as well as for studying lepton flavor violation.

We have explored the nature of neutrinos when CPT is violated. Contrary to the widespread belief that CPT-violating neutrinos can have only a Dirac character and therefore no neutrinoless beta decay can be expected, we have shown that CPT violation can also be seen in neutrinoless double beta decay experiments. As an important general point, though Majorana neutrino masses are allowed, in general, there are no longer Majorana neutrino states in the conventional sense. If the CPT-violating neutrino mixing model is chosen to explain the LSND result, then at least one mass is of order the large LSND scale of 1 eV. This can increase the neutrinoless double beta decay rate relative to the usual case in which all neutrino masses can be small. On the other hand, due to the freedom in the mixing between the neutrino and antineutrino in each mass eigenstate, a greater degree of suppression in the effective mass that appears in neutrinoless beta decay is also possible.

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Corrections for two photon decays of $\chi_{c0}$ and $\chi_{c2}$ and color octet contributions

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Abstract

Using the fact that the $c$-quark inside a charmonium moves with a small velocity $v$ in the charmonium rest-frame, one can employ an expansion in $v$ to study decays of charmonia and results at the leading order for $\chi_{c0,2} \to \gamma\gamma$ exist in the literature. We study corrections at the next-to-leading order in the framework of nonrelativistic QCD (NRQCD) factorization. The study presented here is different than previous approaches where $\chi_{c0,2}$ is taken as a bound-state of a $c\bar{c}$ pair and a nonrelativistic wave-function is used for the pair. We find that the corrections consist not only of relativistic corrections, but also of corrections from Fock state components of $\chi_{c0,2}$ in which the $c\bar{c}$ pair is in a color-octet state. For $\chi_{c2}$ there is also a contribution from a Fock state component in which the pair is in a $F$-wave state. We determine the factorization formula for decay widths in the form of NRQCD matrix elements representing nonperturbative effects related to $\chi_{c0,2}$, and calculate the perturbative coefficients at tree-level. Because the NRQCD matrix elements are unknown, a detailed prediction for the decay $\chi_{c0,2} \to \gamma\gamma$ cannot be made, but the effect of these corrections can be determined at certain level. Estimations show that the effect is significant and cannot be neglected. © 2002 Elsevier Science B.V. All rights reserved.


1. Introduction

Because of their simple final states, decays of $P$-wave charmonium into two photons can be good channels to determine the value of $\alpha_s$ and for study of properties of charmonium. In recent years various experiment groups have published their results on $\Gamma_{\gamma\gamma}(\chi_{c0})$ and $\Gamma_{\gamma\gamma}(\chi_{c2})$ [1–5]. Some major and newest results are listed in Table 1, where errors from different sources are combined. From these data it is easy to find that although there is a tendency of consistency in latest results of CLEO and E835, values of different groups have a large discrepancy from each other. On the theoretical side, because charm quark can be taken as heavy quark and it moves with a small velocity $v$ inside a charmonium in its rest-frame, one may describe the $c$- or $\bar{c}$-quark inside charmonia with a nonrelativistic wave-functions by taking charmonia as a bound-state of a $c\bar{c}$ pair. In most previous calculations for decays of charmonium such a nonrelativistic wave-functions is employed for $\chi_{c0,2} \to \gamma\gamma$ [6–10]. By expanding the small
velocity $v$, the decay width at the leading order of $v$ can be expressed with the derivatives of the wave functions at origin, where the leading order is at $v^2$. The corrections from the next-to-leading order of $v$ are also studied with the nonrelativistic wave-functions [7–10], in which the corrections are only relativistic corrections. This also implies that at the next-to-leading order the charmonia $\chi_{c0,2}$ are still taken as a bound-state of a $c\bar{c}$ pair and the pair has the same quantum number of $\chi_{c0,2}$.

From point of view of a quantum field theory like QCD, a hadron state is a bound state of quarks and gluons and has many components composed with different number of quarks and of gluons. For charmonia it can be expanded in $v$ and for $\chi_{c0,2}$ it looks like:

$$|\chi_{c0,2}\rangle = |(c\bar{c})_1\rangle + O(v) |(c\bar{c})_8\rangle + \text{higher orders},$$  

(1)

where the index 1 or 8 denotes color singlet or color octet, respectively. From this expansion the state $|(c\bar{c})_1\rangle$ is a dominant component, while other components have a suppressed probability to be observed in $\chi_{c0,2}$, the suppression parameter is $v$. Hence, in a systematic study of higher-order corrections one should also include contributions from these suppressed components, i.e., a systematic expansion in $v$ is needed. Recently, a novel approach based on NRQCD for productions and decays of quarkonia is proposed [12]. In this approach a decay width can be systematically expanded in the velocity $v$. For example, the decay width of $\chi_{c0}$ can be written as:

$$\Gamma(\chi_{c0} \rightarrow \gamma\gamma) = \sum_n c_n \langle \chi_{c0} | \hat{O}_n | \chi_{c0} \rangle,$$

(2)

where $\hat{O}_n$'s are operators defined in NRQCD and $c_n$ can be calculated with perturbative theory. There is a rule of power counting of $v$ for the operators $\hat{O}_n$ [11]. We will follow this approach to study the next-to-leading order corrections for the decay $\chi_{c0,2} \rightarrow \gamma\gamma$. We will determine the form of relevant operators in NRQCD and calculate the corresponding coefficients. It should be noted that the role of the suppressed component $|(c\bar{c})_8\rangle$ has been studied before in inclusive processes. It has been shown [13] that only by including the contribution from $|(c\bar{c})_8\rangle$ the inclusive decay of $\chi_{cJ}$ can be consistently predicted with QCD. It has been also shown that such components with a color octet $c\bar{c}$ pair of $\psi$ are very important for explaining the $\psi$ surplus at the Tevatron [14]. In this work we will show the effect of such components in exclusive processes. The effect of $|(c\bar{c})_8\rangle$ has been studied in $\chi_{cJ} \rightarrow \pi\pi$ [15], in which a light-cone wave-function for the component was introduced and the effect was not parameterized with matrix elements of NRQCD.

At the next-to-leading order of $v$, the corrections, as we will see, consist of relativistic corrections and also corrections from the state $|(c\bar{c})_8\rangle$. Beside these, for $\chi_{c2}$ there is also a correction from the state $|(c\bar{c})_1\rangle$ where the $c\bar{c}$ pair is in a state with the orbit angular momentum $l = 3$. These corrections take a factorized form as products of perturbative coefficients and matrix elements, in which the matrix elements are defined in NRQCD and they parameterize nonperturbative effects. We will determine the form of these matrix element and calculate these coefficients at leading order of $\alpha_s$. Our work is organized as the following: in Section 2 we calculate these coefficients and give our main results. In Section 3 we discuss the effect of these corrections and Section 4 is our summary.

### Table 1

List of major experimental results

<table>
<thead>
<tr>
<th>Experiment</th>
<th>$\Gamma_{\gamma\gamma}(\chi_{c0})$ (keV)</th>
<th>$\Gamma_{\gamma\gamma}(\chi_{c2})$ (keV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CLEO (1994)</td>
<td>$\pm 0.25$</td>
<td>$1.08 \pm 0.28$</td>
</tr>
<tr>
<td>CLEO (2001)</td>
<td>$3.76 \pm 1.07$</td>
<td>$0.53 \pm 0.16$</td>
</tr>
<tr>
<td>ES35 (2000)</td>
<td>$1.61 \pm 0.75$</td>
<td>$0.270 \pm 0.042$</td>
</tr>
<tr>
<td>L3 (1999)</td>
<td>$1.02 \pm 0.25$</td>
<td>$1.76 \pm 0.37$</td>
</tr>
</tbody>
</table>
2. The correction at the next-to-leading order

We consider $\chi_{c0.2}$ to two photon decay in its rest frame:

$$\chi_{c0.2}(P) \rightarrow \gamma(k_1) + \gamma(k_2),$$

where $P$, $k_1$, $k_2$ denote momentum of $\chi_{c0.2}$ and two photons. We decompose the decay width as

$$\Gamma = \Gamma_1 + \Gamma_8,$$

where $\Gamma_{1,8}$ denotes the contribution from a color-singlet and color-octet $c\bar{c}$ pair, respectively. The color-singlet contribution can be written as:

$$\Gamma_1 = 16Q^4 e^4 \int d\Gamma \int \frac{d^4p_1 d^4p'_1}{(2\pi)^4} \int d^4x e^{i(2k_1+k_2)x} \int d^4y e^{-ip(x-y)} \langle 0|\bar{\epsilon}_i(x)c_j(y)|\chi_{c0.2}\rangle$$

$$\times \int d^4y' e^{-ip'(y'+x)} \langle \chi_{c0.2}|\bar{\epsilon}_i'(y')c_j(-x)|0\rangle \times H_{ij,i'j'},$$

$$d\Gamma = \frac{d^4k_1 d^4k_2}{(2\pi)^4}(2\pi)^2\delta_+(k_1^2)\delta_+(k_2^2),$$

$$H_{ij,i'j'} = \left[\gamma^\nu \frac{1}{p_1 - \not{k}_2 - m_c} \gamma^\rho + \frac{1}{p_1 - \not{k}_1 - m_c} \gamma^\rho \right]_{ij} \times \left[\gamma_4 \frac{1}{p'_1 - \not{k}_2 - m_c} \gamma_4 \right]_{j'i'},$$

where $c(x)$ stands for the Dirac field of $c$-quark and $i$, $j$, $i'$ and $j'$ stand for color- and spin-indices. This contribution can be represented by the diagrams in Fig. 1. The matrix element with Dirac fields of $c$-quark represents the nonperturbative effect related to the initial hadron. The quantity $H$ is calculated with perturbative theory. An expansion in $v$ can be performed by expanding the Dirac fields with fields of NRQCD. This can be done by using Foldy–Wouthuysen transformation [16]:

$$c(x) = e^{-im_v t} \left[ 1 - i\gamma_j D_j \frac{1}{2m_c} + \frac{1}{8m_c^2}(-i\gamma_j D_j)^2 + \frac{-1}{4m_c^3} \gamma_j [D_0, D_j] + \frac{3}{16m_c^4}(-i\gamma_j D_j)^3 \right] \psi(x)$$

$$+ e^{im_v t} \left[ 1 - i\gamma_j D_j \frac{1}{2m_c} + \frac{1}{8m_c^2}(-i\gamma_j D_j)^2 + \frac{1}{4m_c^3} \gamma_j [D_0, D_j] + \frac{3}{16m_c^4}(-i\gamma_j D_j)^3 \right] \chi(x) + O(v^4),$$

where $\psi(x)$ and $\chi(x)$ are NRQCD fields and $\psi(\chi^\dagger)$ annihilates a $c(\bar{c})$-quark, respectively, $m_c$ is the pole mass of $c$-quark. Using this transformation the matrix elements are then of the NRQCD fields $\gamma$, $\psi$ and their derivatives. Operators $D$ and $\partial_i$ are of $v^1$ order and $D_0$ and $\partial_0$ are of $v^2$ order in NRQCD according the power counting rule [12]. The space–time dependence of matrix element are controlled with different scales in the different directions. In the time direction it is $m_c v^2$, while in the spatial direction it is $m_c v$. The expansion in $v$ is then completed by doing Taylor expansion of fields $\gamma$ and $\psi$ around origin. Using the power counting rule in [12] we collect all local matrix elements up to order of $v^2$, which are allowed by rotation-, charge conjugation- and parity symmetries for $\chi_{c0.2}$, and calculate the corresponding contributions. Doing the calculation in this way is similar to the matching

![Fig. 1. Diagrams of contribution for decay width from $|\langle c\bar{c}\rangle_3\rangle$ state.](image-url)
method by taking a free $c\bar{c}$ pair, as proposed in [12], the difference is that we only collect the contributions from operators with correct quantum numbers in the problem considered here. The calculation is rather tedious but straightforward. We obtain:

$$\Gamma_1(\chi_{c0} \rightarrow \gamma \gamma) = \frac{6\pi Q_c^2\alpha^2}{m_c^2} \langle \chi_{c0}|\mathcal{O}_{\text{EM}}(\mathcal{P}_0)|\chi_{c0}\rangle - \frac{14\pi Q_c^4\alpha^2}{m_c^2} \langle \chi_{c0}|\mathcal{G}_{\text{EM}}(\mathcal{P}_0)|\chi_{c0}\rangle + \mathcal{O}(v^6).$$  \hspace{1cm} (9)

$$\mathcal{O}_{\text{EM}}(\mathcal{P}_0) = \frac{1}{3} \bar{\psi} \left( \frac{i}{2} D \cdot \sigma \right) \chi \langle 0|0\rangle \chi^\dagger \left( \frac{i}{2} D \cdot \sigma \right) \psi,$$  \hspace{1cm} (10)

$$\mathcal{G}_{\text{EM}}(\mathcal{P}_0) = \frac{1}{2} \left\{ \frac{1}{3} \bar{\psi} \left( \frac{i}{2} D \right) \left( \frac{i}{2} D \cdot \sigma \right) \chi \langle 0|0\rangle \chi^\dagger \left( \frac{i}{2} D \cdot \sigma \right) \psi + \text{h.c.} \right\}.$$  \hspace{1cm} (11)

where $\sigma_i$ ($i = 1, 2, 3$) is the Pauli matrix. With the above results we find the correction to the color-singlet contribution is only the relativistic correction. Similarly, the decay width of $\chi_{c2}$ is:

$$\Gamma_1(\chi_{c2} \rightarrow \gamma \gamma) = \frac{8\pi Q_c^2\alpha^2}{5m_c^4} \langle \chi_{c2}|\mathcal{O}_{\text{EM}}(\mathcal{P}_2)|\chi_{c2}\rangle - \frac{172\pi Q_c^4\alpha^2}{105m_c^6} \langle \chi_{c2}|\mathcal{G}_{\text{EM}}(\mathcal{P}_2)|\chi_{c2}\rangle + \mathcal{O}(v^6).$$  \hspace{1cm} (12)

$$\mathcal{O}_{\text{EM}}(\mathcal{P}_2) = \bar{\psi} \left( \frac{i}{2} \frac{D}{\sigma} \right) \chi \langle 0|0\rangle \chi^\dagger \left( \frac{i}{2} \frac{D}{\sigma} \right) \psi,$$  \hspace{1cm} (13)

$$\mathcal{G}_{\text{EM}}(\mathcal{P}_2) = \frac{1}{2} \left\{ \bar{\psi} \left( \frac{i}{2} \frac{D}{\sigma} \right) \chi \langle 0|0\rangle \chi^\dagger \left( \frac{i}{2} \frac{D}{\sigma} \right) \psi + \text{h.c.} \right\},$$  \hspace{1cm} (14)

$$\mathcal{G}_{\text{EM}}(\mathcal{P}_2) = \frac{1}{2} \left\{ \bar{\psi} \left( \frac{i}{2} \frac{D}{\sigma} \right) \chi \langle 0|0\rangle \chi^\dagger \left( \frac{i}{2} \frac{D}{\sigma} \right) \psi + \text{h.c.} \right\}. $$  \hspace{1cm} (15)

where the notation $(ij)$ means $T^{(ij)} = (T^{ij} + T^{ji})/2 - \delta_{ij} T^{kk}/3$, i.e., the tensor is symmetric and trace-less. Unlike the case with $\chi_{c0}$ we find not only a relativistic correction but also a contribution from a mixing with a $F$-wave $c\bar{c}$ pair. This is allowed by symmetries, the $c\bar{c}$ pair is in a spin triplet state, a spin triplet state with an orbital momentum $l = 3$ can have a total angular momentum $J = 2, 3, 4$. The state with $J = 2$ is just the $\chi_{c2}$ state.

Now we consider color octet contributions from $|\langle (c\bar{c})_8 g \rangle|$. At the order we consider the contributions can be represented by diagrams in Fig. 2. The contributions before the expansion in $v$ can be written as:

$$\Delta \Gamma_{8a} = Q_c^4 e^4 \int d\Gamma \int d^{4}y \int d^{4}x \int \left[ \frac{d^4k}{(2\pi)^4} \delta(k^2+i(k_1+k_2)x) \langle 0|\tilde{c}_i(\chi)c_j(y)|\chi_{c0,2}\rangle \right.$$

$$\times \left. \int d^{4}y' d^{4}x' \int \left[ \frac{d^4q}{(2\pi)^4} \delta(q^2+iq'x') \langle \chi_{c0,2}|\tilde{c}_i(0)g_\mu G^\mu(\chi)\bar{c}_j(y')|0\rangle \times M_{ij, j', i'} \right] \right]$$  \hspace{1cm} (16)

$$\Delta \Gamma_{8b} = Q_c^4 e^4 \int d\Gamma \int d^{4}y \int d^{4}x \int \left[ \frac{d^4k}{(2\pi)^4} \delta(k^2+i(k_1+k_2)x) \langle 0|\tilde{c}_i(\chi)c_j(y)|\chi_{c0,2}\rangle \right.$$

$$\times \left. \int d^{4}y' d^{4}x' \int \left[ \frac{d^4q}{(2\pi)^4} \delta(q^2+iq'x') \langle \chi_{c0,2}|\tilde{c}_i(0)g_\mu G^\mu(\chi)\bar{c}_j(y')|0\rangle \times M_{ij, j', i'} \right] \right]$$  \hspace{1cm} (17)

$$\Delta \Gamma_{8c} = Q_c^4 e^4 \int d\Gamma \int d^{4}y \int d^{4}x \int \left[ \frac{d^4k}{(2\pi)^4} \delta(k^2+i(k_1+k_2)x) \langle 0|\tilde{c}_i(\chi)c_j(y)|\chi_{c0,2}\rangle \right.$$

$$\times \left. \int d^{4}y' d^{4}x' \int \left[ \frac{d^4q}{(2\pi)^4} \delta(q^2+iq'x') \langle \chi_{c0,2}|\tilde{c}_i(0)g_\mu G^\mu(\chi)\bar{c}_j(y')|0\rangle \times M_{ij, j', i'} \right] \right]$$  \hspace{1cm} (18)
Fig. 2. Diagrams contribute to decay width from $|\langle cc \rangle \rangle$ state, the diagrams for conjugated contributions are not shown.

with

$$M^a_{ij,i'j'} = \left[ \begin{array}{cc} y^\alpha \frac{1}{k - \bar{k}_2 - m_c + i\epsilon} & y^\beta \frac{1}{k - \bar{k}_1 - m_c + i\epsilon} \\ y^\beta \frac{1}{k' + \bar{q} + \bar{k}_1 - m_c + i\epsilon} & y^\alpha \frac{1}{k' + \bar{q} - m_c + i\epsilon} \end{array} \right]_{ij} \times \left[ \begin{array}{cc} y^\mu \frac{1}{k' + \bar{q} - m_c + i\epsilon} & y^\alpha \frac{1}{k' + \bar{q} - \bar{k}_2 - m_c + i\epsilon} \\ y^\mu \frac{1}{k' + \bar{q} - \bar{k}_2 - m_c + i\epsilon} & y^\alpha \frac{1}{k' + \bar{q} - m_c + i\epsilon} \end{array} \right]_{i'j'} \right]$$

$$M^b_{ij,i'j'} = \left[ \begin{array}{cc} y^\alpha \frac{1}{k - \bar{k}_2 - m_c + i\epsilon} & y^\beta \frac{1}{k - \bar{k}_1 - m_c + i\epsilon} \\ y^\beta \frac{1}{k' + \bar{q} - m_c + i\epsilon} & y^\alpha \frac{1}{k' + \bar{q} - \bar{k}_2 - m_c + i\epsilon} \end{array} \right]_{ij} \times \left[ \begin{array}{cc} y^\mu \frac{1}{k' + \bar{q} - m_c + i\epsilon} & y^\alpha \frac{1}{k' + \bar{q} - \bar{k}_2 - m_c + i\epsilon} \\ y^\mu \frac{1}{k' + \bar{q} - \bar{k}_2 - m_c + i\epsilon} & y^\alpha \frac{1}{k' + \bar{q} - m_c + i\epsilon} \end{array} \right]_{i'j'}$$

$$M^c_{ij,i'j'} = \left[ \begin{array}{cc} y^\alpha \frac{1}{k - \bar{k}_2 - m_c + i\epsilon} & y^\beta \frac{1}{k - \bar{k}_1 - m_c + i\epsilon} \\ y^\beta \frac{1}{k' + \bar{q} - m_c + i\epsilon} & y^\alpha \frac{1}{k' + \bar{q} - \bar{k}_2 - m_c + i\epsilon} \end{array} \right]_{ij} \times \left[ \begin{array}{cc} y^\mu \frac{1}{k' + \bar{q} - m_c + i\epsilon} & y^\alpha \frac{1}{k' + \bar{q} - \bar{k}_2 - m_c + i\epsilon} \\ y^\mu \frac{1}{k' + \bar{q} - \bar{k}_2 - m_c + i\epsilon} & y^\alpha \frac{1}{k' + \bar{q} - m_c + i\epsilon} \end{array} \right]_{i'j'}$$

Similarly we need to perform the expansion in $v$ for matrix element like $\langle \langle \chi c_0 | \bar{c}_i(x) g_s G^\mu (\gamma) c_j(0) |0 \rangle \rangle$. Again with the power counting rule, one can find that the leading term for the matrix element is related to the matrix element of the operator $\psi^\dagger g_s E \cdot \sigma \chi$. This matrix element is at order of $v^3$ and will lead to a contribution to $\Gamma_8$ at order of $v^4$. To calculate the contribution we take the gauge: $G^0 = 0$, in this gauge the electric chromo field is

$$E = \frac{\partial}{\partial t} G.$$

Performing the expansion for matrix elements in Eq. (19), results can be obtained in a straightforward way. We obtain the contribution at $v^4$:

$$\Gamma_8(\chi c_0 \rightarrow \gamma\gamma) = -\frac{4\pi \alpha^2 Q^4}{m_c^5} \langle \chi c_0 | F_{EM}(\gamma P_0) | \chi c_0 \rangle + \mathcal{O}(v^6),$$

$$F_{EM}(\gamma P_0) = \frac{1}{6} \left[ \psi \cdot \sigma \cdot g_s E \chi |0\rangle |0\rangle^\dagger \sigma \cdot D \psi + h.c. \right].$$
As discussed before, the first relation is not useful here unless we know the correction at \( O(\alpha_s^2) \) appearing in Eq. (10) can be written as:

\[
\langle \chi_{c2} | \mathcal{O}_{\text{EM}}(\mathbf{P}_2) | \chi_{c2} \rangle = \frac{1}{2} \left[ \psi \gamma^\mu g_\mu \bar{\chi} \chi \right] \left( 0 \right) \left( 0 \right) \psi \gamma^\nu \bar{\chi} \chi + \text{h.c.} \].
\]

(Eq. (26)

Eqs. (9), (12), (23) and (25) are our main results. From these results we can see that the corrections consist not only of relativistic correction which is represented by the matrix elements \( \langle \chi_{c0} | \mathcal{O}_{\text{EM}}(\mathbf{P}_0) | \chi_{c0} \rangle \) and \( \langle \chi_{c2} | \mathcal{O}_{\text{EM}}(\mathbf{P}_2) | \chi_{c2} \rangle \), but also of contributions from the color octet component. For \( \chi_{c2} \) there is also a contribution from the component in which the \( c\bar{c} \) pair is in color singlet state with the orbit angular momentum \( l = 3 \).

3. Numerical impact of the correction

To make theoretical predictions for decay widths one needs to know all matrix elements in the last section. Unfortunately, there is no information available for the matrix elements appearing in the correction. There are estimations for the matrix element at the leading order of \( v \), but for consistency we need an estimation for them at the accuracy of \( O(v^4) \). All this prevents us from detailed predictions. However, numerical impact of the correction can be studied by noting that there are several relations among these operators. Because of spin symmetry of NRQCD we can have:

\[
\langle \chi_{c2} | \mathcal{O}_{\text{EM}}(\mathbf{P}_2) | \chi_{c2} \rangle = \langle \chi_{c0} | \mathcal{O}_{\text{EM}}(\mathbf{P}_0) | \chi_{c0} \rangle \left( 1 + O(v^2) \right),
\]

\[
\langle \chi_{c2} | \mathcal{O}_{\text{EM}}(\mathbf{P}_2) | \chi_{c2} \rangle = \langle \chi_{c0} | \mathcal{O}_{\text{EM}}(\mathbf{P}_0) | \chi_{c0} \rangle \left( 1 + O(v^2) \right),
\]

\[
\langle \chi_{c2} | \mathcal{O}_{\text{EM}}(\mathbf{P}_2) | \chi_{c2} \rangle = \langle \chi_{c0} | \mathcal{O}_{\text{EM}}(\mathbf{P}_0) | \chi_{c0} \rangle \left( 1 + O(v^2) \right).
\]

(27)

As discussed before, the first relation is not useful here unless we know the correction at \( O(v^2) \). Beside these relations one can also use equation of motion to obtain another type of relations. Generally there are subtleties for using equation of motion for operators. However, it is shown that one can still use equation of motion for operators which are sandwiched between physical states [18]. For operators relevant to \( J/\Psi \) one has used this to derive this type of relations [17].

The equation of motion of NRQCD reads:

\[
\left( i D_t + \frac{D^2}{2m} \right) \psi = 0,
\]

\[
\left( i D_t - \frac{D^2}{2m} \right) \chi = 0.
\]

(28)

We can use these equations to replace the operator \( D^2 \) with \( D_t \). For doing this we note that the matrix element appearing in Eq. (10) can be written as:

\[
\langle 0 | \chi \left( \frac{i}{2} \frac{\mathbf{D} \cdot \sigma}{2} \left( \frac{i}{2} \frac{\mathbf{D} \cdot \sigma}{2} \right) \right)^2 \psi | \chi_{c0} \rangle = \frac{i}{2} \langle 0 | \chi \mathbf{D} \cdot \sigma \mathbf{D}^2 \psi + \left( \mathbf{D}^2 \chi \right) \mathbf{D} \cdot \sigma \psi | \chi_{c0} \rangle \left( 1 + O(v^2) \right),
\]

(29)

where the corrections terms are due to interchange of the gauge covariant derivative \( \mathbf{D} \), and they are at order of \( O(v^2) \) and can be neglected here. Using the equation of motion we obtain:

\[
\langle 0 | \chi \mathbf{D} \cdot \sigma \mathbf{D}^2 \psi + \left( \mathbf{D}^2 \chi \right) \mathbf{D} \cdot \sigma \psi | \chi_{c0} \rangle = -2m_c \eta \left[ \langle 0 | \chi \mathbf{D} \cdot \sigma \mathbf{D} \psi + \left( \mathbf{D} \chi \right) \mathbf{D} \cdot \sigma \psi | \chi_{c0} \rangle \right]
\]

\[
= -2m_c \left( \langle 0 | \chi \mathbf{D} \cdot \sigma \psi | \chi_{c0} \rangle = -2m_c \langle \chi | \mathbf{E} \cdot \sigma \psi | \chi_{c0} \rangle, \right.
\]

(30)

where \( E_{\chi_{c0}} \) is the binding energy. Doing the same for \( \chi_{c2} \) we obtain the relations:

\[
\langle \chi_{c0} | \mathcal{O}(\mathbf{P}_0) | \chi_{c0} \rangle = m_c E_{\chi_{c0}} \langle \chi_{c0} | \mathcal{O}(\mathbf{P}_0) | \chi_{c0} \rangle + m_c \langle \chi_{c0} | \mathcal{F}(\mathbf{P}_0) | \chi_{c0} \rangle,
\]

\[
\langle \chi_{c2} | \mathcal{O}(\mathbf{P}_2) | \chi_{c2} \rangle = m_c E_{\chi_{c2}} \langle \chi_{c2} | \mathcal{O}(\mathbf{P}_2) | \chi_{c2} \rangle + m_c \langle \chi_{c2} | \mathcal{F}(\mathbf{P}_2) | \chi_{c2} \rangle,
\]

(31)
These parameters are order of \( v^2 \) also the effects from color octet contributions and from mixing with the brackets in Eq. (34), respectively. From these numbers, we see that the effect of binding energies is very significant, and the binding energy is defined as:

\[
E_{\chi,0} = m_{\chi,0} - 2m_c, \quad E_{\chi,2} = m_{\chi,2} - 2m_c. \tag{32}
\]

It should be noted that the difference \( E_{\chi,2} - E_{\chi,0} \) is at order higher than \( v^2 \), hence the above relations are in consistent with those in Eq. (27). To study the numerical impact we define the following parameters:

\[
a_s = \frac{(\chi_0|\mathcal{F}(\mathcal{P}_0)|\chi_0)}{m_c(\chi_0|\mathcal{O}(\mathcal{P}_0)|\chi_0)} = \frac{(\chi_2|\mathcal{F}(\mathcal{P}_2)|\chi_2)}{m_c(\chi_2|\mathcal{O}(\mathcal{P}_2)|\chi_2)} \cdot (1 + \mathcal{O}(v^2)),
\]

\[
a_F = \frac{(\chi_2|\mathcal{G}(\mathcal{P}_2)|\chi_2)}{m_c^2(\chi_2|\mathcal{O}(\mathcal{P}_2)|\chi_2)}. \tag{33}
\]

These parameters are order of \( v^2 \). With these parameters and with the relations in Eq. (31) the decay widths can be written:

\[
\Gamma'(\chi_0 \rightarrow \gamma \gamma) = \frac{6\pi Q^2 \alpha^2}{m^2_e} (\chi_0|\mathcal{O}(\mathcal{P}_0)|\chi_0) \cdot \left(1 + \frac{0.1787}{\alpha_s(m_c)} - \frac{2.333}{m_c} - 3a_s \right) + \mathcal{O}(v^6),
\]

\[
\Gamma'(\chi_2 \rightarrow \gamma \gamma) = \frac{8\pi Q^2 \alpha^2}{5m^2_e} (\chi_2|\mathcal{O}(\mathcal{P}_2)|\chi_2)
\]

\[
\cdot \left[1 - 5.333 \frac{\alpha_s(m_c)}{\pi} - 1.024 \frac{E_{\chi,2}}{m_c} - 2.738a_s - 1.690a_F \right] + \mathcal{O}(v^6), \tag{34}
\]

where we have added the one-loop correction to the contribution at leading order of \( v \) [6]. If we take \( m_c = 1.5 \) GeV and \( \alpha_s(m_c) \approx 0.3 \), we obtain by using experimental values of hadron masses:

\[
\Gamma'(\chi_0 \rightarrow \gamma \gamma) \approx \frac{6\pi Q^2 \alpha^2}{m^2_e} (\chi_0|\mathcal{O}(\mathcal{P}_0)|\chi_0) \cdot [1 + 0.017 - 0.645 - 0.3] + \mathcal{O}(v^6),
\]

\[
\Gamma'(\chi_2 \rightarrow \gamma \gamma) \approx \frac{8\pi Q^2 \alpha^2}{5m^2_e} (\chi_2|\mathcal{O}(\mathcal{P}_2)|\chi_2) \cdot [1 - 0.510 - 0.380 - 0.227 - 0.169] + \mathcal{O}(v^6). \tag{35}
\]

where we have taken \( a_s = a_F = 0.1 \). In the above equations, the numbers in brackets correspond to the terms in brackets in Eq. (34), respectively. From these numbers, we see that the effect of binding energies is very significant, also the effects from color octet contributions and from mixing with the \( F \)-wave state are not small, although their effects cannot be determined precisely. However, our results show that each correction at the next-to-leading order is significant. Unless among those corrections some cancellation happens, they cannot be neglected.

4. Summary and conclusion

We have presented a systematic study of high order corrections in two photon decays of \( \chi_{c,0,2} \). Using NRQCD velocity power counting rule, implicit expansion of decay width in relative velocity \( v \) is achieved. We calculated not only first order relativistic corrections but also corrections from suppressed component \( |(c\bar{c})_k g \rangle \) in which \( c\bar{c} \) pair is in a color octet state. The effect of color octet has been studied extensively in inclusive processes. In this work we have studied the effect in exclusive processes. Unlike most previous theoretical calculations in which nonrelativistic wave function is employed, our calculation is based on a field theoretic framework of NRQCD. The results are expressed by products of perturbatively calculable short distance coefficients and nonperturbative matrix elements which concern with initial bound states. In the case of \( \chi_{c,2} \), a correction from orbital angular momentum \( l = 3 \) is also obtained in our calculation. In previous calculations only relativistic corrections can be estimated.

Unfortunately, the matrix elements at the next-to-leading order are not known at all, although some relations among them can be obtained by symmetries and by using equation of motion. This prevents us from numerical
predictions for the decays. However, numerical impact of these corrections can be estimated at certain level and with this estimation the effect of the corrections is significant and can be not neglected. It would be interesting to study these matrix elements by nonperturbative methods, especially by lattice QCD. With a nonperturbative method one may determine how important the color octet component is.

References

Radiative M1-decays of heavy–light mesons in the relativistic quark model

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Abstract

Radiative magnetic dipole decays of heavy–light vector mesons into pseudoscalar mesons $V \rightarrow P \gamma$ are considered within the relativistic quark model. The light quark is treated completely relativistically, while for the heavy quark the $1/m_Q$ expansion is used. It is found that relativistic effects result in a significant reduction of decay rates. Comparison with previous predictions and recent experimental data is presented. © 2002 Elsevier Science B.V. All rights reserved.


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In this Letter we consider radiative magnetic dipole (M1) transitions of the ground state vector ($V$) heavy–light mesons to the pseudoscalar ($P$) ones, $V \rightarrow P \gamma$ (in quark model notations, $1^3S_1 \rightarrow 1^1S_0 + \gamma$). For this purpose, we use the relativistic quark model based on the quasipotential approach in quantum field theory. Recently, this model has been successfully applied for the description of different properties of the heavy–light mesons, such as their mass spectra [1] and rare radiative decays [2]. Our analysis showed that the light quark in the heavy–light mesons should be treated completely relativistically, while for the heavy quark it is useful to apply the expansion in powers of the inverse heavy quark mass $1/m_Q$, which considerably simplifies calculations. The first- and sometimes second-order corrections in $1/m_Q$ are also important for the heavy–light meson decay description. Analogously, for the radiative decay calculations considered here, the $1/m_Q$ expansion is carried out up to the second order, and the light quark is treated completely relativistically (i.e., without the unjustified expansion in inverse powers of the light quark mass). It follows from the obtained results that the relativistic effects give substantial contributions to the calculated decay rates.

The radiative $V \rightarrow P \gamma$ decay rate is given by [3]

$$\Gamma = \frac{\omega^3}{3\pi} |M|^2,$$

where

$$\omega = \frac{M_V^2 - M_P^2}{2M_V},$$

(1)

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$M_V$ and $M_P$ are the vector and pseudoscalar meson masses. The matrix element of the magnetic moment $\mathcal{M}$ is defined by

$$\mathcal{M} = -\frac{i}{2} \left[ \frac{\partial}{\partial \Delta} \times \langle P|J(0)|V \rangle \right]_{\Delta=0}, \quad \Delta = P - Q,$$

where $\langle P|J_\mu(0)|V \rangle$ is the matrix element of the electromagnetic current between initial vector ($V$) and final pseudoscalar ($P$) meson states with momenta $Q$ and $P$, respectively.

We use the relativistic quark model for the calculation of the matrix element of the magnetic moment $\mathcal{M}$ (2). In our model a meson is described by the wave function of the bound quark–antiquark state, which satisfies the quasipotential equation of the Schrödinger type in the center-of-mass frame [1]:

$$\left( \frac{b^2(M)}{2\mu_R} - \frac{p^2}{2\mu_R} \right) \psi_M(p) = \int \frac{d^3q}{(2\pi)^3} V(p, q; M) \psi_M(q),$$

where the relativistic reduced mass is

$$\mu_R = \frac{M^4 - (m_1^2 - m_2^2)^2}{4M^3},$$

and $b^2(M)$ denotes the on-mass-shell relative momentum squared

$$b^2(M) = \frac{[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]}{4M^2}.$$  

Here $m_{1,2}$ and $M$ are quark masses and a heavy–light meson mass, respectively.

The kernel $V(p, q; M)$ in Eq. (3) is the quasipotential operator of the quark–antiquark interaction. It is constructed with the help of the off-mass-shell scattering amplitude, projected onto the positive energy states. An important role in this construction is played by the Lorentz-structure of the confining quark–antiquark interaction in the meson. In constructing the quasipotential of the quark–antiquark interaction we have assumed that the effective interaction is the sum of the usual one-gluon exchange term and the mixture of vector and scalar linear confining potentials. The quasipotential is then defined by [4]

$$V(p, q; M) = \bar{u}_1(p)\bar{u}_2(-p)\mathcal{V}(p, q; M)\gamma_\mu u_1(q)\gamma_\mu u_2(-q),$$

with

$$\mathcal{V}(p, q; M) = 4\alpha_sD_{\mu\nu}(k)\gamma_\mu^\nu + V^{V}_{\text{conf}}(k)\Gamma_{1\mu}^\nu \Gamma_{2;\mu} + V^{S}_{\text{conf}}(k),$$

where $\alpha_s$ is the QCD coupling constant, $D_{\mu\nu}$ is the gluon propagator in the Coulomb gauge and $k = p - q$, $\gamma_\mu$ and $u(p)$ are the Dirac matrices and spinors. The effective long-range vector vertex is given by

$$\Gamma_\mu(k) = \gamma_\mu + \frac{ik}{2m}\sigma_{\mu\nu}k^\nu, \quad k^\nu = (0, k),$$

where $\kappa$ is the Pauli interaction constant characterizing the nonperturbative anomalous chromomagnetic moment of quarks. Vector and scalar confining potentials in the nonrelativistic limit reduce to

$$V^V_{\text{conf}}(r) = (1 - \varepsilon)(Ar + B), \quad V^S_{\text{conf}}(r) = \varepsilon(Ar + B),$$

reproducing

$$V_{\text{conf}}(r) = V^V_{\text{conf}}(r) + V^S_{\text{conf}}(r) = Ar + B,$$

where $\varepsilon$ is the mixing coefficient.

The quasipotential for the heavy quarkonia, expanded in $p^2/m^2$, can be found in Ref. [4] and for heavy–light mesons in [1]. All the parameters of our model, such as quark masses, parameters of the linear confining potential,
mixing coefficient $\varepsilon$ and anomalous chromomagnetic quark moment $\kappa$, were fixed from the analysis of heavy quarkonium spectra [4] and radiative decays [3]. The quark masses $m_b = 4.88$ GeV, $m_c = 1.55$ GeV, $m_s = 0.50$ GeV, $m_u,d = 0.33$ GeV and the parameters of the linear potential $A = 0.18$ GeV$^2$ and $B = -0.30$ GeV have the usual quark model values. In Ref. [5] we have considered the expansion of the matrix elements of weak heavy quark currents between pseudoscalar and vector meson ground states up to the second order in inverse powers of the heavy quark masses. It has been found that the general structure of the leading, first, and second order $1/m_Q$ corrections in our relativistic model is in accord with the predictions of HQET. The heavy quark symmetry and QCD impose rigid constraints on the parameters of the long-range potential in our model. The analysis of the first-order corrections [5] fixes the value of the Pauli interaction constant $\kappa = -1$. The same value of $\kappa$ was found previously from the fine splitting of heavy quarkonia $^3P_J$-states [4]. The value of the parameter characterizing the mixing of vector and scalar confining potentials, $\varepsilon = -1$, was found from the comparison of the second-order $1/m_Q^2$ corrections in our model [5] with the same order contributions in HQET. This value is very close to the one determined from considering radiative decays of heavy quarkonia [3], especially the M1-decays (e.g., the calculated decay rate of $J/\Psi \rightarrow \eta_c \gamma$ can be brought in accord with the experiment only with the above value of $\varepsilon$).

In the quasipotential approach, the matrix element of the electromagnetic current $J_{\mu}$ between the states of a vector $V$ meson and a pseudoscalar $P$ meson has the form [6]

$$
\langle P | J_{\mu} (0) | V \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \overline{\Psi}_P(p) \Gamma_{\mu}(p,q) \Psi_V(q),
$$

where $\Gamma_{\mu}(p,q)$ is the two-particle vertex function and $\Psi_V,P$ are the meson wave functions projected onto the positive energy states of quarks and boosted to the moving reference frame. The contributions to $\Gamma$ come from Figs. 1 and 2. The contribution $\Gamma^{(2)}$ is the consequence of the projection onto the positive-energy states. Note that the form of the relativistic corrections resulting from the vertex function $\Gamma^{(2)}$ explicitly depends on the Lorentz structure of the $q\bar{q}$-interaction. Thus the vertex function is given by

$$
\Gamma_{\mu}(p,q) = \Gamma^{(1)}_{\mu}(p,q) + \Gamma^{(2)}_{\mu}(p,q) + \cdots,
$$

where $\Gamma^{(1)}_{\mu}(p,q)$ and $\Gamma^{(2)}_{\mu}(p,q)$ are the lowest order and second order contributions, respectively.
where
\[ \Gamma^{(1)}_\mu(p, q) = e_1 \bar{u}_1(p_1) \gamma_\mu u_1(q_1)(2\pi)^3 \delta(p_2 - q_2) + (1 \leftrightarrow 2), \] (12)
and
\[ \Gamma^{(2)}_\mu(p, q) = e_1 \bar{u}_1(p_1) \bar{u}_2(p_2) \left\{ \mathcal{V}(p_2 - q_2) \frac{A_1^-(k_1')}{\epsilon_1(k_1') + \epsilon_1(q_1)} \gamma_\mu \right. \\
+ \left. \gamma_\mu \frac{A_1^-(k_1)}{\epsilon_1(k_1) + \epsilon_1(p_1)} \gamma_0 \mathcal{V}(p_2 - q_2) \right\} u_1(q_1) u_2(q_2) + (1 \leftrightarrow 2). \] (13)

Here \( e_{1,2} \) are the quark charges, \( k_1 = p_1 - \Delta; \quad k_1' = q_1 + \Delta; \quad \Delta = P - Q \);
\[ A^-(p) = \frac{\epsilon(p) - (m\gamma^0 + \gamma^0(p\rho))}{2\epsilon(p)}, \quad \epsilon(p) = \sqrt{p^2 + m^2}. \]

It is important to note that the wave functions entering the current matrix element (10) cannot be both in the rest frame. In the initial \( V \) meson rest frame, the final \( P \) meson is moving with the recoil momentum \( \Delta \). The wave function of the moving \( P \) meson \( \Psi_{P\Delta} \) is connected with the wave function in the rest frame \( \Psi_{P0} \) by the transformation [6]
\[ \Psi_{P\Delta}(p) = D_{1/2}^1(R_{k1}^W) D_{1/2}^2(R_{\Delta}^W) \Psi_{P0}(p), \] (14)
where \( R^W \) is the Wigner rotation, \( L_\Delta \) is the Lorentz boost from the rest frame to a moving one, and \( D^{1/2}(R) \) is the rotation matrix in the spinor representation.

We substitute the vertex functions \( \Gamma^{(1)} \) and \( \Gamma^{(2)} \) given by Eqs. (12) and (13) in the decay matrix element (10) and take into account the wave function transformation (14). To simplify calculations we note that the mass of the heavy–light mesons \( M_{V, P} \) is large (due to presence of the heavy quark \( M_{V, P} \sim m_Q \)) and carry out the expansion in inverse powers of this mass up to the second order. Then we calculate the matrix element of the magnetic moment operator (2) and get:

(a) For the vector potential
\[ \mathcal{M}_V = \int \frac{d^3p}{(2\pi)^3} \bar{\Psi}_{P\Delta}(p) \frac{e_1}{2\epsilon_1(p)} \]
\[ \times \left\{ \sigma_1 + \frac{(1 - \epsilon)(1 + \epsilon)}{2\epsilon_1(p)[\epsilon_1(p) + m_1]} \left[ \epsilon_1(p)[\epsilon_2(p) + m_2] \right] \right. \\
+ \left. \frac{1}{2M_V} \left[ p \times \left( \sigma_1 \left( \frac{\sigma_1}{\epsilon_1(p) + m_1} - \frac{\sigma_2}{\epsilon_2(p) + m_2} \right) \right) \right] \right\} \Psi_{P0}(p) + (1 \leftrightarrow 2). \] (15)

(b) For the scalar potential
\[ \mathcal{M}_S = \int \frac{d^3p}{(2\pi)^3} \bar{\Psi}_{P\Delta}(p) \frac{e_1}{2\epsilon_1(p)} \]
\[ \times \left\{ \left( 1 + \frac{\epsilon_1(p) + \epsilon_2(p) - M_V}{\epsilon_1(p)} \right) \left[ \epsilon_1 - \frac{\epsilon_2(p)}{M_V} \left( p \times \frac{\partial}{\partial p} \right) \right] - \frac{\epsilon[p \times [\sigma_1 \times p]]}{2\epsilon_1(p)[\epsilon_1(p) + m_1]} \right. \\
+ \left. \frac{1}{2M_V} \left[ p \times \left( \sigma_1 \left( \frac{\sigma_1}{\epsilon_1(p) + m_1} - \frac{\sigma_2}{\epsilon_2(p) + m_2} \right) \right) \right] \right\} \Psi_{P0}(p) + (1 \leftrightarrow 2). \] (16)
to evaluate spin matrix elements using the relation $\langle \sigma_1 \rangle = -\langle \sigma_2 \rangle$. Then assuming one quark to be light $q$ and the other one $Q$ to be heavy and further expanding Eqs. (15), (16) in the inverse powers of the heavy quark mass $m_Q$ up to the second order corrections to the leading contribution we get:

(a) For the purely vector potential ($\varepsilon = 0$)

$$
\mathcal{M}_V = \frac{e_q}{2m_q} \left\{ \left[ \frac{m_q}{\epsilon_q(p)} - \frac{m_q}{3} \frac{\mathbf{p}^2}{\epsilon_q(p)\epsilon_q(p) + m_q} \left( \frac{1}{\epsilon_q(p)} + \frac{1}{M_V} \right) \right] \\
+ \frac{(1 + \kappa)m_q}{3} \left[ \frac{\mathbf{p}^2}{\epsilon^2_q(p)} \left( \frac{2}{\epsilon_q(p) + m_q} - \frac{1}{m_Q} \right) \right] \\
- \frac{e_q}{2m_Q} \left[ 1 - \frac{2(p^2)}{3m_Q^2} + \frac{1 + \kappa}{3} \left( \frac{\mathbf{p}^2}{m_Q} \left( \frac{1}{\epsilon_q(p)} - \frac{2}{m_Q} \right) \right) - \left( \frac{\mathbf{p}^2}{6M_V} \left( \frac{1}{m_Q} + \frac{2}{\epsilon_q(p) + m_q} \right) \right) \right] \right\}. (17)
$$

(b) For the purely scalar potential ($\varepsilon = 1$)

$$
\mathcal{M}_S = \frac{e_q}{2m_q} \left\{ 2 \frac{m_q}{\epsilon_q(p)} - \frac{m_q(M_V - m_Q)}{\epsilon^2_q(p)} - \frac{m_q}{6M_V m_Q} \left( \frac{\mathbf{p}^2}{\epsilon_q(p)} \right) \right\} \\
+ \frac{m_q}{2} \left( \frac{\mathbf{p}^2}{\epsilon^2_q(p)} \left( \frac{1}{m_Q} - \frac{2}{3[\epsilon_q(p) + m_q]} - \frac{2[\epsilon_q(p)]}{3M_V[\epsilon_q(p) + m_q]} \right) \right) \\
- \frac{e_q}{2m_Q} \left[ 2 - \frac{M_V - \epsilon_q(p)}{m_Q} - \frac{(\mathbf{p}^2)}{6m_Q} \left( \frac{1}{m_Q} + \frac{1}{M_V} \right) - \frac{1}{3M_V} \left( \frac{\mathbf{p}^2}{\epsilon_q(p) + m_q} \right) \right]. (18)
$$

Here $\langle \cdots \rangle$ denotes the matrix element between radial meson wave functions. For these matrix element calculations we use the wave functions of heavy–light mesons obtained in Ref. [1]. It is important to note that in this reference while calculating the heavy–light meson mass spectra only the heavy quark was treated using the $1/m_Q$ expansion but the light quark was treated completely relativistically.

The values of decay rates of mesons with open flavour calculated on the basis of Eqs. (1), (17), (18) are displayed in Table 1. In the second column ($\Gamma^{NR}$) we give predictions for decay rates obtained in the nonrelativistic approximation ($p/m \to 0$) for both heavy and light quarks. In the third ($\Gamma^V$) and fourth ($\Gamma^S$) columns we show the results obtained for the purely vector and scalar confining potentials, respectively. And in the last column ($\Gamma$) we present predictions for the mixture of vector and scalar confining potentials (8) with the mixing parameter $\varepsilon = -1$. As seen from this table relativistic effects significantly influence the predictions. Their inclusion results in a significant reduction of decay rates ($\Gamma^{NR}/\Gamma = 2–4.5$). Both relativistic corrections to the heavy quark and the relativistic treatment of the light quark play an important role. The dominant decay modes of $D^*$ mesons are the strong decay $D^* \to D\pi$, which is considerably suppressed by the phase space, and the electromagnetic decay $D^* \to \gamma \gamma$. The corresponding branching ratios are known already for a long time and listed in PDG tables [7]. However, the total decay rates of $D^*$ mesons were not measured until recently. In Ref. [8] CLEO Collaboration reported the first measurement of the $D^*$ decay width $\Gamma(D^{*+}) = 96 \pm 4 \pm 22$ keV. Combining this value with the measured $BR(D^{*+} \to D^{*+}\gamma) = (1.6 \pm 0.4)\%$ [7], the following experimental value of the decay rate can be obtained: $\Gamma(D^{*+} \to D^{*+}\gamma) = (1.5 \pm 0.6)$ keV. Our model prediction is in agreement with this experimental value. However, the experimental errors are still large in order to discriminate the relativistic and nonrelativistic results. In
the case of $B$ mesons the pion emission is kinematically forbidden, so the dominant decay mode is electromagnetic decay $B^+ \to B \gamma$. None of $B^+$ widths has been measured yet. We also present our predictions for the radiative decay rate of $B_s^+$ meson, which consists of two heavy quarks ($b$ and $c$). Therefore the expressions (17) and (18) can be further expanded in inverse powers of both quark masses up to the second order.

In Table 2 we compare our predictions for radiative decay rates of vector heavy–light mesons with other theoretical results. We show the predictions obtained in quark models [9–11], in the framework of heavy quark effective theory (HQET) combined with vector meson dominance (VMD) hypothesis [12] and in QCD sum rules [13–15]. These predictions vary quite significantly from each other. Our predictions are in rough agreement with the quark model calculations of Ref. [10], with HQET+VMD results of Ref. [12] and with some of the predictions of the QCD sum rules. It is important to note that in our calculations we do not need to introduce the anomalous electromagnetic moment of the light quark as it is done in Ref. [11], where it was found that its value should be rather large ($\sim 0.5$) in order to get agreement with the experimental (CLEO) value for $D^{*+} \to D^+ \gamma$ decay rate. The large value of the anomalous electromagnetic moment is not justified phenomenologically (see, e.g., Refs. [3,16]). The other differences of our calculations from those of Ref. [11] are the Lorentz structure of the confining potential and a more comprehensive account of relativistic effects. In particular, the relativistic transformation of the meson wave function from the rest frame to a moving one given by Eq. (14) is missing in Ref. [11].

In Table 3 we present the comparison of our results with the predictions of different quark models [17–19] for the rates of the radiative M1-transitions ($1^3S_1 \to 1^1S_0 + \gamma$) in the heavy–heavy $B_c$ meson. There we also give the

---

**Table 1**

Radiative decay rates of mesons with an open flavour (in keV)

<table>
<thead>
<tr>
<th>Decay</th>
<th>$\Gamma^{NR}$</th>
<th>$\Gamma^{V}$</th>
<th>$\Gamma^{S}$</th>
<th>$\Gamma^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D^{\pm} \to D^{\pm} \gamma$</td>
<td>2.08</td>
<td>0.60</td>
<td>0.28</td>
<td>1.04</td>
</tr>
<tr>
<td>$D^{*0} \to D^{0} \gamma$</td>
<td>37.0</td>
<td>14.3</td>
<td>17.4</td>
<td>11.5</td>
</tr>
<tr>
<td>$D_s^+ \to D_s \gamma$</td>
<td>0.36</td>
<td>0.13</td>
<td>0.08</td>
<td>0.19</td>
</tr>
<tr>
<td>$B^{*\pm} \to B^{\pm} \gamma$</td>
<td>0.89</td>
<td>0.24</td>
<td>0.29</td>
<td>0.19</td>
</tr>
<tr>
<td>$B^{*0} \to B^{0} \gamma$</td>
<td>0.27</td>
<td>0.087</td>
<td>0.101</td>
<td>0.070</td>
</tr>
<tr>
<td>$B_s^+ \to B_s \gamma$</td>
<td>0.132</td>
<td>0.064</td>
<td>0.074</td>
<td>0.054</td>
</tr>
<tr>
<td>$B_{sJ}^+ \to B_{sJ} \gamma$</td>
<td>0.073</td>
<td>0.048</td>
<td>0.066</td>
<td>0.033</td>
</tr>
</tbody>
</table>

**Table 2**

Comparison of different theoretical predictions for radiative decays of heavy–light mesons (in keV)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$D^{\pm} \to D^{\pm} \gamma$</td>
<td>1.04</td>
<td>0.36</td>
<td>1.72</td>
<td>0.050</td>
</tr>
<tr>
<td>$D^{*0} \to D^{0} \gamma$</td>
<td>11.5</td>
<td>17.9</td>
<td>7.18</td>
<td>7.3</td>
</tr>
<tr>
<td>$D_s^+ \to D_s \gamma$</td>
<td>0.19</td>
<td>0.118</td>
<td>0.101</td>
<td>0.24 ± 0.24</td>
</tr>
<tr>
<td>$B^{*\pm} \to B^{\pm} \gamma$</td>
<td>0.19</td>
<td>0.261</td>
<td>0.272</td>
<td>0.084</td>
</tr>
<tr>
<td>$B^{*0} \to B^{0} \gamma$</td>
<td>0.070</td>
<td>0.092</td>
<td>0.064</td>
<td>0.037</td>
</tr>
<tr>
<td>$B_{sJ}^+ \to B_{sJ} \gamma$</td>
<td>0.054</td>
<td>0.051</td>
<td>0.035</td>
<td>0.045</td>
</tr>
</tbody>
</table>

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1 The QCD sum rule results [13] for the $D^*$ decays were obtained using the $1/m_c$ expansion, which could be inaccurate due to the large value of the $1/m_c$ corrections.

2 In Table 2 we show predictions of Ref. [11] for the value of anomalous electromagnetic quark moment equal to zero.
Table 3
Comparison of theoretical predictions for the radiative $B_c^+ \rightarrow B_c \gamma$ decay

<table>
<thead>
<tr>
<th>Photon energy (MeV)</th>
<th>Our</th>
<th>[17]</th>
<th>[18]</th>
<th>[19]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma(B_c^+ \rightarrow B_c \gamma)$ (eV)</td>
<td>61</td>
<td>72</td>
<td>64</td>
<td>55</td>
</tr>
<tr>
<td>33</td>
<td>135</td>
<td>60</td>
<td>60</td>
<td>59</td>
</tr>
</tbody>
</table>

predicted values of the photon energy, which is determined by the mass splitting of the vector and pseudoscalar ground states. In previous calculations [17–19] the nonrelativistic expression for the matrix element of the magnetic moment was used. We see that even in the heavy–heavy $B_c$ meson inclusion of the relativistic effects results in considerable reduction of the radiative M1-decay rate. As can be seen from Table 1, this reduction is evoked by significant contributions of relativistic effects for the $c$ quark, since it is not heavy enough, as well as by the special choice of the mixture of vector and scalar confining potentials in our model (8).

In summary we calculated radiative M1-decay rates of mesons with open flavour in the framework of the relativistic quark model. In our analysis the light quark was treated relativistically, while for the heavy quark the $1/m_Q$ expansion was carried out up to the second order. Relativistic consideration of the light quark, relativistic heavy quark corrections as well as Lorentz-structure of the confining potential considerably influence the predictions. We find that only the mixture of vector and scalar confining potentials (8), with the mixing coefficient fixed previously from quarkonium radiative decays [3] and weak decays of heavy–light mesons [5], is in agreement with recent CLEO data for the $D^{*+} \rightarrow D^+ \gamma$ decay rate. More precise measurement of this decay rate and the measurement of radiative M1-decays of other heavy–light mesons will be crucial for testing the relativistic quark dynamics.

Acknowledgements

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References

CP and T trajectory diagrams for a unified graphical representation of neutrino oscillations

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Abstract

Recently the CP trajectory diagram was introduced to demonstrate the difference between the intrinsic CP violating effects to those induced by matter for neutrino oscillation. In this Letter we introduce the T trajectory diagram. In these diagrams the probability for a given oscillation process is plotted versus the probability for the CP- or the T-conjugate processes, which forms an ellipse as the CP- or T-violating phase is varied. Since the CP- and the T-conjugate processes are related by CPT symmetry, even in the presence of matter, these two trajectory diagrams are closely related with each other and form a unified description of neutrino oscillations in matter. © 2002 Published by Elsevier Science B.V.

1. Introduction

Accumulating evidences for neutrino oscillation in the atmospheric [1], the solar [2], and the accelerator [3] neutrino experiments make it realistic to think about exploring the full structure of the lepton flavor mixing. One of the challenging goals in such attempt would be to measure the leptonic CP- or T-violating phase $\delta$ in the MNS matrix [4]. $^1$ It appears that the long baseline neutrino oscillation experiments are the most feasible way to actually detect these effects.

It has been known since sometime ago that the measurement of CP- and T-violating effects in the long baseline neutrino oscillation experiments can be either contaminated or enhanced by the matter effect inside the earth; see, e.g., Refs. [5] and [6]. Therefore, it is one of the most important issues to achieve a complete understanding of the interplay between the CP-phase and the matter effects in parameter regions relevant for such experiments.

Toward the goal, we have introduced in a previous paper [7] the “CP trajectory diagram in bi-probability space” as a useful tool for pictorial representation of the CP and the matter effects in neutrino oscillation. This diagram enables us to display three effects; (a) genuine CP violation due to the $\sin\delta$ term, (b) CP conserving $\cos\delta$ term, and (c) fake CP violation due to the earth’s matter, separately in a single diagram.

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$^1$ In this Letter we will assume that the light neutrino sector consists of only three active neutrinos.
2. CP and T trajectory diagrams in bi-probability space and their symmetries

Now we introduce the T trajectory diagram in bi-probability space spanned by $P(v) \equiv P(\nu_e \rightarrow \nu_\mu)$ and its T-conjugate $T[P(v)] \equiv P(\nu_\mu \rightarrow \nu_e)$. We fully explain in this section its notable characteristic properties, the relationship (or unity) with the CP trajectory diagram, and the symmetry relations obeyed by them. Suppose that we compute the oscillation probability $P(v)$ and $T[P(v)]$ with a given set of oscillation and experimental parameters. Then, we draw a dot on the two-dimensional plane spanned by $P(v)$ and $T[P(v)]$. When $\delta$ is varied we have a set of dots which forms a closed trajectory, closed because the probability must be a periodic function of $\delta$, a phase variable. Let us remind the reader that the T trajectory diagram has very similar structure with the CP trajectory diagram introduced in Ref. [7]; the abscissa is the same and the ordinate for the CP diagram is $CP[P(v)] \equiv P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu)$.

In Fig. 1 the CP and the T trajectory diagrams, denoted as CP± and T±, are plotted in the same figure. Here, the ordinate of the diagram is meant to be $CP[P(v)]$ for the CP and $T[P(v)]$ for the T diagrams, respectively. In the center there exist two vacuum diagrams, $\pm$, one for each of the signs of $\Delta m^2_{31}$. When the matter effect is turned on, the positive and the negative $\Delta m^2_{31}$ trajectories split.

In vacuum, the CP and the T trajectories are identical with each other. This is because the CPT theorem tells us that $T[P(v)] = P(\nu_\tau \rightarrow \nu_\mu) = P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu) = CP[P(v)]$. Depending upon the sign of $\Delta m^2_{31}$ we have two CP (or T) trajectories which are almost degenerate with each other in vacuum due to the approximate symmetry under the simultaneous transformation [7]

$$\delta = 3D(\pi - \delta) \mod(2\pi),$$

$$\Delta m^2_{31} \rightarrow -\Delta m^2_{31}. \tag{1}$$

It was noticed that the CP and the T trajectories are elliptical exactly in vacuum and approximately in matter [7]. Recently, it was shown in a remarkable paper by Kimura, Takamura, and Yokomakura [8] that it is exactly elliptic even in matter; the $\delta$-dependence of the oscillation probabilities in constant matter density can be written on general ground in the form

$$P(v) = A(a) \cos \delta + B(a) \sin \delta + C(a), \tag{2}$$

$$CP[P(v)] = A(-a) \cos \delta - B(-a) \sin \delta + C(-a),$$

$$T[P(v)] = A(a) \cos \delta - B(a) \sin \delta + C(a).$$
where $a$ denotes neutrino’s index of refraction in matter, $a = \sqrt{2} G_F N_e$, with electron number density $N_e$ and the Fermi constant $G_F$. The dependence on other variables are suppressed. Throughout this paper, we use the standard parameterization of the MNS matrix.

Matter effects split the two trajectories in quite different manners. Namely, the T trajectories split along the straight line $P(\nu) = T[P(\nu)]$ whereas the CP trajectories split in the orthogonal direction. The T trajectories must move along the diagonal line because the $\delta = 0$, $\pi$ points on T trajectories must stay on the diagonal because T violation vanishes at $\delta = 0$, $\pi$ for matter distributions which are symmetric about the midpoint between production and detection. In constant matter density, this stems from the Naumov–Harrison–Scott relation (i.e., the proportionality between the vacuum and the matter Jarlskog factors) [9, 10] and the absence of $\sin \delta$ dependence [11] in the other parts of the oscillation probability, or simply from the KTY formula in Eq. (2).

On the other hand, the CP trajectories must split along the direction orthogonal to the diagonal line for small matter effect. This is because the first order matter correction, which is proportional to $a = \sqrt{2} G_F N_e(x)$, has opposite sign for the neutrino and the anti-neutrino oscillation probabilities; $\Delta P(\nu) = -\Delta P(\bar{\nu})$ [7]. No matter how large the matter effect the line connecting the two CP trajectories with opposite signs of $\Delta m^2_{31}$ is orthogonal to the diagonal line. As we will explain below this reflects a symmetry relationship obeyed by the oscillation probabilities, which we denote as the “CP–CP relation”.

One may also notice, as indicated by the eye-guided lines in Fig. 1 that the CP and the T trajectories have identical lengths when projected onto either the abscissa or the ordinate. It should be the case for the projection onto the abscissa because the abscissa is common for both of the CP and the T diagrams. What is nontrivial is the equality in length when projected onto the ordinate. It again represents a symmetry which we want to call the “T–CP relation”. We note that neither the CP–CP nor the T–CP relations are exact, as one may observe from Fig. 1.

The precise statement of the CP–CP relation is (see Fig. 1)

$$P(\nu_e \rightarrow \nu_\mu; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) = P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu; -\Delta m^2_{31}, -\Delta m^2_{21}, \delta, a)$$

$$\approx P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu; -\Delta m^2_{31}, +\Delta m^2_{21}, \pi + \delta, a). \quad (3)$$

Whereas the precise statement of the T–CP relation is

$$P(\nu_\mu \rightarrow \nu_e; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) = P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e; -\Delta m^2_{31}, -\Delta m^2_{21}, 2\pi - \delta, a)$$

$$\approx P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e; -\Delta m^2_{31}, +\Delta m^2_{21}, \pi - \delta, a). \quad (4)$$

The equalities have the opposite sign for all the $\Delta m^2_{ij}$'s from the left-hand side to the right-hand side. But because of the hierarchy, $|\Delta m^2_{31}| \gg |\Delta m^2_{21}|$, changing the sign of $\Delta m^2_{21}$ produces only a small deviation whose precise origin and magnitude will be become clear in the derivation of these relations. These relations imply that when the CP+ trajectory winds counter-clockwise as $\delta$ increases the CP– and T– (T+) trajectories wind clockwise (counter-clockwise) as in Fig. 1.

To derive the CP–CP and T–CP relationships we need a number of identities. The first set of identities comes from taking the time reversal and the complex conjugate of the neutrino evolution equation assuming that the matter profile is symmetric about the midpoint between production and detection;

$$P(\nu_\alpha \rightarrow \nu_\beta; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) = P(\bar{\nu}_\beta \rightarrow \bar{\nu}_\alpha; \Delta m^2_{31}, \Delta m^2_{21}, \delta, -a). \quad (5)$$

These identities are just CPT invariance in the presence of matter.

The second set of identities comes from taking the complex conjugate of the neutrino evolution equation for an arbitrary matter distribution, they are

$$P(\nu_\alpha \rightarrow \nu_\beta; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) = P(\nu_\alpha \rightarrow \nu_\beta; -\Delta m^2_{31}, -\Delta m^2_{21}, 2\pi - \delta, -a)$$

$$= P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta; -\Delta m^2_{31}, -\Delta m^2_{21}, \delta, a)$$

$$= P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta; \Delta m^2_{31}, \Delta m^2_{21}, 2\pi - \delta, -a). \quad (6)$$

These identities relate the (anti-)neutrino oscillation probabilities in matter to those in anti-matter with

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2 While it may be easier for the readers to understand the following discussions by referring the KTY formula in Eq. (2), our subsequent discussion will be entirely independent from the constant density approximation. We however rely on the adiabatic approximation when we utilize perturbative formulas.
opposite signs for all the $\Delta m^2$'s. They also relate the neutrino oscillation probabilities in matter to the anti-neutrino oscillation probabilities in matter with opposite signs for all the $\Delta m^2$.

Combinations of these identities give the equalities in the CP–CP and the T–CP relations, the first lines in Eqs. (3) and (4). Now we will use the hierarchy that $|\Delta m^2_{31}| \gg |\Delta m^2_{21}|$ to get an approximate equality if we flip the sign of the $\Delta m^2_{21}$ with the appropriate change in the phase $\delta$. The probability for $\nu_\alpha \rightarrow \nu_\beta$ consists of three terms as in Eq. (2). The coefficients $A$ and $B$ vanish like $\Delta m^2_{21}$ as $\Delta m^2_{21} \rightarrow 0$. Thus, a change in the sign of $\Delta m^2_{31}$ in these two coefficients can be compensated by replacing $\delta$ with $\pi + \delta$, whereas the change in the coefficients $C$ is further suppressed by an extra factor of $\sin \theta_{13}$. This proves the approximate equalities in the CP–CP and the T–CP relations.

The CP–CP relation guarantees that the size and the shape of two sign-conjugate ($\Delta m^2_{31} = \pm |\Delta m^2_{31}|$) diagrams are identical, whereas there is no such relation in T diagrams. The CP phase relation between two sign-conjugate CP diagrams implies, among other things, that the two-fold ambiguity in $\delta$ which still remains after accurate determination of $\theta_{13}$ [7] are related by $\delta_2 = \delta_1 + \pi$ apart from the correction of order $\sim \Delta m^2_{21}/\Delta m^2_{31}$.

Finally, we note that from the first line of Eq. (6) that

$$P(\nu_\alpha \rightarrow \nu_\beta; \Delta m^2_{31}, \Delta m^2_{21}, \delta, a) 
\approx P(\nu_\alpha \rightarrow \nu_\beta; -\Delta m^2_{31}, \Delta m^2_{21}, \pi - \delta, -a)$$

which is a generalization of the approximate symmetry under the transformation (1) into the case in matter, from which the CP–CP and the T–CP relations also follow.

3. Further examples

While Fig. 1 is for the preferred parameters of a neutrino factory [12] it is interesting to see how the T–CP trajectory diagram changes as we change the energy and path length of the experiment. In Fig. 2 we have increased the path length to 6000 km and given the CP–T diagram for neutrino energies of 13 and 26 GeV. Again one can see how well the CP–CP and T–CP relationships hold.

The next examples use an energy which is approximately half the resonance energy corresponding to $\Delta m^2_{31}$ as suggested by Parke and Weiler [6] (Fig. 3). At a distance of 3500 km this energy maximizes T-violation effects which is reflected in the size of ellipses. Halving the distance between source and

Fig. 3. The T (CP) trajectory diagrams (ellipses) in the plane $P(\nu_e \rightarrow \nu_\mu)$ versus $P(\bar{\nu}_e \rightarrow \nu_\mu)$ for an average neutrino energy (spread 20%) and baseline of (a) 6.5 GeV and 3500 km and (b) 6.5 GeV and 1750 km. Labels and mixing parameters are as in Fig. 1.

The final examples are using the energies and baselines of NUMI/MINOS and JHF/SK see Fig. 4. For NUMI/MINOS there is reasonable separation between the two $\Delta m^2_{31}$ ellipse whereas for JHF/SK there is some overlap. From the viewpoint of simultaneous determination of $\delta$ and the sign of $\Delta m^2_{31}$ the longer baseline of NUMI/MINOS would be more advanta-
geous, while there are possibilities that it can be done at JHF/SK if $P$ and CP[$P$] are asymmetric [7,13].

An another feature which is worth noting in the JHF/SK case in Fig. 4 is that the problem of parameter degeneracy in T-violation measurements is milder than that in CP measurements. It is because the T (or CP) ellipse is flatter in the radial direction, i.e., along the movement of T trajectory due to matter effect. The underlying reason for the phenomenon is that the coefficient of the cos $\delta$ term when averaged over energy width of a beam. In this sense, T measurement, if feasible experimentally, is more advantageous than CP measurement for simultaneous determination of $\delta$ and the sign of $\Delta m^2_{31}$.

4. Parameter degeneracy in T-violation measurements

Armed by the T as well as the CP trajectory diagrams we are now ready to discuss the problem of parameter degeneracy with T-violation measurement. We do not aim at complete treatment of the problem in this Letter but briefly note the new features that arise in T-violation measurement as compared to the CP-violation measurement. They arise because of the more symmetric nature of the T-conjugate probability as apparent in Eq. (2).

We work in the same approximation as in Ref. [14] and write the oscillation probability in small $\delta_{13}$ approximation. We have four equations:

$$P(\nu)_\pm = X_\pm \theta^2 + Y_\pm \cos \left( \delta \mp \frac{\Delta_{31}}{2} \right) + P_0,$$

$$T[P(\nu)]_\pm = X_\pm \theta^2 + Y_\pm \cos \left( \delta \pm \frac{\Delta_{31}}{2} \right) + P_0,$$  

where $X_\pm$ and $Y_\pm$ are given in Ref. [14], $P_0$ indicates the term which is related with solar neutrino oscillations, and $\Delta_{31} \equiv \frac{|\Delta m^2_{31}|}{2E}L$. Note that $\pm$ here refers to the sign of $\Delta m^2_{31}$ and $\theta$ is an abbreviation of $\theta_{13} \simeq s_{13}$.

It is easy to show that Eq. (8) can be solved to obtain the same-sign $\Delta m^2_{31}$ degenerate solutions as

$$\delta_2 = \pi - \delta_1$$

and

$$\theta_2 - \theta_1 = \frac{Y_\pm}{X_\pm} \cos \delta_1 \cos \left( \frac{\Delta_{31}}{2} \right).$$

Notice that it is the degeneracy in matter though it looks like the one in vacuum [7].

Two T± trajectories may overlap for a distance shorter than those in Fig. 4, which would result in a mixed-sign degeneracy as in the case of CP measurement mentioned before. However, one can show that in the case of T-violation measurement there is no more ambiguity in $\delta$ once $\theta_{13}$ is given;

$$\delta_2 = \delta_1$$

and

$$\cos \delta_1 = -\frac{X_+ - X_-}{2Y_+ \cos \left( \frac{\Delta_{31}}{2} \right)}$$

apart from the computable higher order correction due to the matter effect. This is obvious from Fig. 4(b). A fuller treatment of the parameter degeneracy with T-violation measurement will be reported elsewhere [15].

5. Discussion and conclusions

First, it is worth pointing out that in matter the sensitivity to the CP- or T-violating phase, $\delta$, is larger for the T-violating pair ($\nu_\alpha \rightarrow \nu_\beta$, $\nu_\beta \rightarrow \nu_\alpha$) than for the CP-violating pair ($\nu_\alpha \rightarrow \nu_\beta$, $\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta$) of processes assuming $\Delta m^2_{31} > 0$. For $\Delta m^2_{31} < 0$ one should use the following T-violating pair, ($\nu_\alpha \rightarrow \bar{\nu}_\beta$, $\bar{\nu}_\beta \rightarrow \nu_\alpha$). This can be seen by comparing the size of the T ellipses to that of the CP ellipses in all of the previous diagrams. Unfortunately the experimental challenges associated with performing a T-violating experiment have yet to be overcome.

Due to the existence of the T-CP relationships for neutrino oscillations it is instructive and useful to add the T-violating trajectories to the CP-violating trajectories first proposed by Minakata and Nunokawa. The size and position of the T-violating ellipses can be easily estimated using simple arguments including the effects of matter and from them the CP-violating ellipses can be estimated. Thus, the combination of the CP and T trajectories form a unified picture of the CP–matter interplay in neutrino oscillations.

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Neutrinoless double beta decay with \( R \)-parity violation

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Abstract

We consider recently observed neutrinoless double beta decay in the context of the minimal supersymmetric standard model with \( R \)-parity violating couplings \( \lambda' \). We observe that most of the current experimental bounds on the \( R \)-parity violating couplings do not exclude the possibility that the neutrinoless double beta decay is caused by \( R \)-parity violation. But if we consider \( K^-/\overline{K} \) oscillation, we observe that we have to make the \( R \)-parity violating couplings generation-dependent to accommodate with the observed neutrinoless double beta decay. And furthermore, we need some mechanism to cancel the contribution to \( K^-/\overline{K} \) mixing from a large \( R \)-parity violating coupling. We realized this cancellation by assuming that the first- and the second-generation of quark sector do not couple with the first-generation lepton sector by \( R \)-parity violating couplings except the term \( W = \lambda'_{111} L_1 Q_1 D^c_1 \), which is responsible for the observed neutrinoless double beta decay. 

\( \frac{1}{2} \)

\[ T_{1/2}^{0\nu} = (0.8–18.3) \times 10^{25} \text{ yr.} \]

This means that, lepton number is broken in nature. In the Standard Model (SM), lepton number is conserved, and this evidence becomes signature for physics beyond the SM.

We can realize lepton number violation in the \( R \)-parity violating Minimal Supersymmetric Standard Model (MSSM) (for reviews, see [2]). The \( R \)-parity violating couplings are:

\[ W = \lambda_{ijk} L_i L_j E^c_k + \lambda'_{ijk} L_i Q_j D^c_k \]

+ \( \lambda''_{ijk} U^c_i D^c_j D^c_k \). (2)

These terms violates lepton number and baryon number simultaneously, and thus lead to rapid proton decay. So we must forbid some or all of these terms. Usually, to achieve that, a \( Z_2 \)-symmetry called as “\( R \)-parity” is imposed. \( R \)-parity is defined as:

\[ R_p = (-1)^{3B + L + 2S}, \]

where \( B \) is the baryon number of the particle, \( L \) is the lepton number of the particle and \( S \) is the spin of the particle. If we impose \( R \)-parity, all the couplings in Eq. (2) are forbidden, and no dangerous phenomena occur.
But there is another possibility. $Z_3$-symmetry is anomaly-free discrete gauge symmetry, and can protect proton from rapid decay [3]. This symmetry forbids baryon number violation, but allows lepton-number violation. So it is worthwhile to investigate the lepton-number violating phenomena as resultants of a $Z_3$-symmetry [4]. The charge assignment of this $Z_3$-symmetry is shown in Table 1.

<table>
<thead>
<tr>
<th>Charge</th>
<th>$Q$</th>
<th>$U^c$</th>
<th>$D^c$</th>
<th>$L$</th>
<th>$E^c$</th>
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<td>Particle</td>
<td>$\alpha^2$</td>
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Table 1
Charge assignment under the discrete gauge symmetry. $\alpha^3 = 1$

Neutrinoless double beta decay was considered in the context of the MSSM with lepton-number violating $R$-parity [5,6]. Detailed calculations for the neutrinoless double beta decay rate including nuclear matrix elements was done in [6]. When only $\lambda'$ couplings are considered, the Feynman diagrams contributing to the neutrinoless double beta decay are drawn in Fig. 1. Since squark- and gluino-mediated process dominates, we drop the contribution from neutralino- and slepton-exchange diagrams [6].

Following Ref. [6], the recent result yields following constraints:

$$1.6 \times 10^{-4} \left( \frac{m_{\tilde{q}}}{100 \text{ GeV}} \right)^2 \left( \frac{m_{\tilde{g}}}{100 \text{ GeV}} \right)^{1/2}$$

Fig. 1. The processes relevant for neutrinoless double beta decay.
where \( m_{\tilde{d}_R} = m_{\tilde{\mu}^c} = m_{\tilde{g}} \).

By scanning the parameter region \( 100 \text{ GeV} < m_{\tilde{g}} < 2000 \text{ GeV} \), \( 200 \text{ GeV} < m_{\tilde{\mu}^c} < 2000 \text{ GeV} \), we make a contour plot of the allowed values of \( \lambda'_{111} \). It is shown in Fig. 2. Here we have conservatively adopted \( m_{\tilde{g}} > 200 \text{ GeV} \). This figure shows the allowed region of \( m_{\tilde{g}} \) and \( m_{\tilde{\mu}^c} \) for given values of \( \lambda' \). We can see that as \( \lambda' \) couplings become smaller, the allowed region of \( m_{\tilde{g}} \) and \( m_{\tilde{\mu}^c} \) is lowered. This is because if squark and gluino masses are heavy, their contribution to the neutrinoless double beta decay becomes small.

It is interesting to compare the combined constraint on \( \lambda' \) vs. squark and gluino masses obtained here, with those from other experimental results. There are many experimental results which can constrain \( \lambda' \). Hereafter, we study them in detail.

For example, the existence of \( R \)-parity violation leads to a violation of the universality of quark and lepton couplings to the \( W \) boson. In the quark sector, the \( R \)-parity violating couplings \( \lambda'_{ijk}L_iQ_jD^c_k \) gives an additional contribution to the quark semileptonic decay (e.g., in nuclear \( \beta \) decay) like muon decay. The effective coupling becomes:

\[
g \frac{g^2}{8m_W^2}[V_{ud} + r'_{11k}(\tilde{d}^k_R)].
\]

where \( r'_{11k} \) is defined as:

\[
r'_{11k}(\tilde{d}) = \frac{m_W^2|\lambda'_{11k}|^2}{g^2m_{\tilde{d}}^2}.
\]

The CKM matrix elements are experimentally determined from the ratio of the \( \mathcal{O} \rightarrow q e\nu \) to \( \mu \rightarrow \nu e \) partial widths. The experimental value is related to theoretical quantities by:

\[
|V_{ud}|^2_{\text{exp}} = \frac{|V_{ud} + r'_{11k}(\tilde{d}^k_R)|^2}{|1 + r_{12k}(\tilde{f}^k_R)|^2},
\]

where \( r_{12k} \) is defined like \( r'_{11k} \). A comparison with the experimental value:

\[
\sum_j |V_{udj}|^2_{\text{exp}} = 0.9979 \pm 0.0021
\]

yields the limit [7]:

\[
|\lambda'_{11k}| < 0.03 \left( \frac{m_{\tilde{g}}^2}{100 \text{ GeV}} \right),
\]

at the \( 2\sigma \) level.

This does not exclude the possibility that \( R \)-parity violation is responsible for the neutrinoless double beta decay. For example, for \( m_{\tilde{d}_R} = m_{\tilde{\mu}^c} = 500 \text{ GeV} \), this limit becomes \( |\lambda'_{11k}| < 0.15 \), which is compatible with the allowed values of \( \lambda'_{111} \) shown in Fig. 2.

The decay rate of pion into electron and muon is also changed in the presence of the \( R \)-parity violating couplings. The ratio \( R_\pi \equiv \Gamma(\pi \rightarrow e\nu)/\Gamma(\pi \rightarrow \mu\nu) \) is

\[
\frac{R_\pi(\text{expt})}{R_\pi(\text{SM})} = 0.991 \pm 0.18.
\]

\( R \)-parity violation gives an effective contribution to \( R_\pi \) [7]:

\[
R_\pi = R_\pi(\text{SM}) \left[ 1 + \frac{2}{V_{ud}} \left( r'_{11k}(\tilde{d}^k_R) - r'_{12k}(\tilde{f}^k_R) \right) \right].
\]

The experimental value (10) set upper limit on the \( R \)-parity violating couplings as:

\[
|\lambda'_{11k}| < 0.05 \left( \frac{m_{\tilde{g}}^2}{100 \text{ GeV}} \right).
\]

This is weaker limit compared to Eq. (9), thus we can neglect this limit in this study.

The decay \( K^+ \rightarrow \pi^0\nu\bar{\nu} \) is also modified in the presence of the \( R \)-parity violating couplings [8].
obtain:

\[
\frac{\Gamma[K^+ \rightarrow \pi^+ \nu_i \bar{\nu}_j]}{\Gamma[K^+ \rightarrow \pi^0 \nu_e \bar{\nu}_e]} = \left( \frac{|\lambda'_{ijk}|^2}{4G_F m^3_{\tilde{g}} 100 \text{ GeV}} \right)^2 \frac{1}{|V_{ij}^* V_{jk}^*|^2}.
\]

(13)

So using \( B(K^+ \rightarrow \pi^+ \nu \bar{\nu}) \lesssim 4.4 \times 10^{-10} \) [9] and \( B(K^+ \rightarrow \pi^0 \nu_e \bar{\nu}_e) = 0.0482 \) [10], we obtain the constraint [8,9].

\[
|\lambda'_{ijk}| < 0.0056 \left( \frac{m_{\tilde{g}}}{100 \text{ GeV}} \right)^2.
\]

(14)

This constraint is stronger. For example, take \( m_{\tilde{g}} = 900 \text{ GeV} \), then \( \lambda'_{111} < 0.05 \). From Fig. 2 we can see that gluino mass is constrained in the region:

\[
m_{\tilde{g}} \lesssim 1100 \text{ GeV}.
\]

(15)

Other experiments, like \( K-\bar{K} \) oscillation, and \( B-\bar{B} \) oscillation give stronger limits on the lepton number violating couplings [11]. But their limit always contain products of two \( \lambda' \). Thus we cannot state strongly that we can derive upper limit on \( \lambda'_{111} \). For example, \( K-\bar{K} \) oscillation gives [11]:

\[
\text{Re} \left[ \sum_{i,j,f} \left( \frac{100 \text{ GeV}}{m_{\tilde{g}}} \right)^2 \lambda'_{ij2} \lambda'_{ij1} V_{ij1}^* V_{ij2}^* \right] < 4.5 \times 10^{-9}.
\]

(16)

So we cannot extract the information of \( \lambda'_{111} \) from \( K-\bar{K} \) oscillation. Of course if we assume generation-independence of the \( \lambda' \) couplings, we can estimate:

\[
\lambda'_{111} \lesssim 10^{-4} \left( \frac{m_{\tilde{g}}}{100 \text{ GeV}} \right).
\]

(17)

As we can see from Fig. 2, this is so strong constraint that we cannot explain the observed neutrinoless double beta decay if we impose this constraint. So we can say that if the observed neutrinoless double beta decay is truly the result of \( R \)-parity violation, the \( \lambda' \) couplings are not generation-independent.

But there still exists a non-trivial problem that how such a large \( \lambda'_{111} \) coupling can be consistent with the stringent bound from \( K-\bar{K} \) mixing (Eq. (16)). We should make such a large coupling be cancelled by some mechanism to accommodate with the stringent bound from \( K-\bar{K} \) mixing. One way is to assume that

\[
\lambda'_{112} = \lambda'_{121} = \lambda'_{122} = 0.
\]

(18)

In this case, the contribution from \( \lambda'_{111} \) to Eq. (16) becomes

\[
\text{Re} \left[ \lambda'_{111} \lambda'_{132} V_{31}^* V_{12} \right] < 4.5 \times 10^{-9}.
\]

(19)

We substitute \( |V_{31}| \sim 0.003, |V_{12}| \sim 0.22 \) and \( \lambda'_{111} \sim 0.005 \) into (19), then we get

\[
\lambda'_{132} < 1.4 \times 10^{-3}.
\]

(20)

which is moderate value compared to \( \lambda'_{111} = 5 \times 10^{-3} \). So we conclude that the large value of \( \lambda'_{111} \) can be consistent with the stringent bound from \( K-\bar{K} \) mixing.

To summarize, we consider the neutrinoless double beta decay in the context of the Minimal Supersymmetric Standard Model with the lepton-number violating \( R \)-parity couplings. We observe that most of the current experiments do not exclude the possibility that \( R \)-parity violation is the source of the observed neutrinoless double beta decay. But if the \( R \)-parity violating couplings are generation-independent, the constraint on \( K-\bar{K} \) oscillation excludes this possibility. Generation-dependency and some mechanism to cancel a large \( \lambda'_{111} \) coupling contribution to \( K-\bar{K} \) oscillation is needed. We realized this cancellation by assuming \( \lambda'_{112} = \lambda'_{121} = \lambda'_{122} = 0 \), namely, the first- and the second-generation of quark sector do not couple with the first-generation lepton sector by the \( R \)-parity violating couplings, except the coupling which is responsible for the neutrinoless double beta decay, \( \lambda'_{111} \).

**Note added**

After the submission of this Letter, we learned from Dr. Liu that they considered within the framework of \( R \)-parity violating supersymmetry, the sleptons play a partial role in electroweak symmetry breaking. The scalar neutrinos get non-vanishing vacuum expectation values (VEVs). These non-zero VEVs break the family symmetry (say, \( Z_3 \)) naturally. This breaking of the family symmetry may result in the realistic pattern of the fermion masses [12,13]. To be specific, Refs. [12,13] proposed that the muon mass originates from the sneutrino VEVs, whereas tau from Higgs. Neutrino masses are discussed in Ref. [14]. Especially in [14], they have obtained an electron-neutrino Majorana mass to be around 0.1 eV.
And we also learned from Dr. Dedes that the bounds on all $R$-parity violating couplings have been collated and updated in their paper [15].

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A method for analysing the jet azimuthal anisotropy in ultrarelativistic heavy ion collisions

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Abstract

The azimuthal anisotropy of jet spectra due to energy loss of jet partons in azimuthally non-symmetric volume of dense quark–gluon matter is considered for semi-central nuclear interactions at collider energies. We develop the techniques for event-by-event analysing the jet azimuthal anisotropy using particle and energy elliptic flow, and suggest a method for calculation of coefficient of jet azimuthal anisotropy without reconstruction of nuclear reaction plane.

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Keywords: Jet energy loss; Azimuthal anisotropy; Elliptic flow; Relativistic nuclear collisions

1. Introduction

High-$p_T$ jet production and other hard processes are considered as the promising tools for studying properties of hot matter created in heavy ion collisions at RHIC and LHC. The challenging problem is the behaviour of colour charge in quark–gluon environment associated with the coherence pattern of the medium-induced radiation. It results in interesting non-linear phenomena, in particular, the dependence of radiative energy loss per unit distance $dE/dx$ along the total distance traversed (see review [1] and references therein). In a number of papers [2–5] authors analysed the azimuthal anisotropy of high-$p_T$ hadron spectra in semi-central nuclear collisions at RHIC due to partonic energy loss in azimuthally non-symmetric volume of quark–gluon plasma. At LHC energy, when the inclusive cross section for hard jet production on $E_T \sim 100$ GeV scale is large enough to study the impact parameter dependence of such processes [6], one can hope to observe the similar effect for hadronic jet itself [7]. In particular, CMS experiment at LHC [8] will be able to provide jet reconstruction and adequate measurement of impact parameter of nuclear collision using calorimetric information [9].

It is important to notice that the coherent Landau–Pomeranchuk–Migdal radiation induces a strong dependence of the radiative energy loss of a jet (but not a leading parton) on the jet angular cone size [10–13]. It means that the medium-induced radiation will, in the first place, soften particle energy distributions inside the jet, increase the multiplicity of secondary
particles, and to a lesser degree affect the total jet energy. On the other hand, collisional energy loss turns out to be practically independent of jet cone size, because the bulk of “thermal” particles knocked out of the dense matter by elastic scatterings fly away in almost transverse direction relative to the jet axis [11].

The methodical advantage of azimuthal jet observables is obvious: one needs to reconstruct only azimuthal position of jet, but not the total jet energy. It can be done more easily and with high accuracy, while the reconstruction of the jet energy is more ambiguous task [9]. However, the performance of inclusive analysis of jet production as a function of azimuthal angle requires event-by-event measurement of the reaction plane angle. The summarized in papers [14] present methods for determination of the reaction plane angle are applicable for studying anisotropic flow of soft and semi-hard particles in current heavy ion dedicated experiments at SPS [15] and RHIC [16], and, in principle, might be also used at LHC [7].

In this work we suggest a method to calculate the coefficient of jet azimuthal anisotropy without reconstruction of nuclear reaction plane. In some sense, it represents the development and generalization of the well-known method for measuring azimuthal anisotropy of particle flow considered originally in a number of works (see [14,17,18], for instance).

2. Event-by-event analysis of azimuthal correlations

Let us remind the essence of techniques [14,17] for measuring azimuthal elliptic anisotropy of particle distribution, which can be written in the form

\[
\frac{dN}{d\varphi} = \frac{N_0}{2\pi} \left[ 1 + 2v_2 \cos 2(\varphi - \psi_R) \right].
\]

\[N_0 = \int_{-\pi}^{\pi} d\varphi \frac{dN}{d\varphi}, \quad (1)\]

Knowing the nuclear reaction plane angle \(\psi_R\) allows one to determine the coefficient \(v_2\) of azimuthal anisotropy of particle flow as an average (over particles) cosine of 2\(\varphi\):

\[
\langle \cos 2(\varphi - \psi_R) \rangle = \frac{1}{N_0} \int_{-\pi}^{\pi} d\varphi \cos 2(\varphi - \psi_R) \frac{dN}{d\varphi} = v_2. \quad (2)
\]

However, in the case when there are no other correlations of particles except those due to flow (or such other correlations can be neglected), the coefficient of azimuthal anisotropy can be determined using two-particle azimuthal correlator without the event plane angle \(\psi_R\):

\[
\langle \cos 2(\varphi_1 - \varphi_2) \rangle = \frac{1}{N_0^2} \int_{-\pi}^{\pi} d\varphi_1 \int_{-\pi}^{\pi} d\varphi_2 \cos 2(\varphi_1 - \varphi_2) \frac{d^2N}{d\varphi_1 d\varphi_2} = \frac{1}{N_0^2} \int_{-\pi}^{\pi} d\varphi_1 \int_{-\pi}^{\pi} d\varphi_2 \cos 2(\varphi_1 - \varphi_2) \frac{dN_1}{d\varphi_1} \frac{dN_2}{d\varphi_2} = v_2^2. \quad (3)
\]

Here one should to note that it is safe to neglect non-flow correlations only if the coefficient of azimuthal anisotropy \(v_2\) is much larger than \(1/\sqrt{N_0}\). In reality, this condition is not always satisfied. A new method, based on a cumulant expansion of multi-particle azimuthal correlations, which allows measurements of much smaller values of azimuthal anisotropies, down to \(1/N_0\), has been worked out recently in paper [19]. This method automatically eliminates the major systematic errors, which are due to azimuthal asymmetries in the detector acceptance, and in principle could be used in further improvements of our approach.

Let us consider now the event with high-\(p_T\) jet (di-jet) production, the distribution of jets over azimuthal angle relatively to the reaction plane being described well by the elliptic form [7],

\[
\frac{dN^{jet}}{d\varphi} = \frac{N_0^{jet}}{2\pi} \left[ 1 + 2v_2^{jet} \cos 2(\varphi - \psi_R) \right].
\]

\[N_0^{jet} = \int_{-\pi}^{\pi} d\varphi \frac{dN^{jet}}{d\varphi}, \quad (4)\]
where the coefficient of jet azimuthal anisotropy $v_2^\text{jet}$ is determined as an average over all events cosine of $2\varphi$,

$$
\langle \cos 2(\varphi - \psi_R) \rangle_{\text{event}}
= \frac{1}{N_0^\text{jet}} \int_{-\pi}^{\pi} d\varphi \cos 2(\varphi - \psi_R) \frac{dN^\text{jet}}{d\varphi}
= v_2^\text{jet}.
$$

(5)

One can calculate the correlator between the azimuthal position of jet axis $\psi_{\text{jet}}$ and the angles of particles, which are not incorporated in the jet(s). The value of this correlator is related to the elliptic coefficients $v_2$ and $v_2^{\text{jet}}$ as

$$
\langle \cos 2(\psi_{\text{jet}} - \varphi) \rangle_{\text{event}}
= \frac{1}{N_0^\text{jet}} \int_{-\pi}^{\pi} d\varphi \cos 2(\psi_{\text{jet}} - \varphi) \frac{dN^\text{jet}}{d\varphi} \frac{dN}{d\psi} v_2^\text{jet} v_2
= v_2^\text{jet} v_2.
$$

(6)

Using Eq. (3) and intermediate result in Eq. (6) (after averaging over particles $\cos 2(\psi_{\text{jet}} - \varphi)$ reduces to $v_2 \cos 2(\psi_{\text{jet}} - \psi_R)$) we derive the formula for computing absolute value of the coefficient of jet azimuthal anisotropy (without reconstruction of sign of $v_2^\text{jet}$):

$$
v_2^\text{jet} = \frac{\langle \cos 2(\psi_{\text{jet}} - \varphi) \rangle}{\langle \cos 2(\psi - \psi_R) \rangle} \langle \cos 2(\psi - \psi_R) \rangle_{\text{event}}.
$$

(7)

This formula does not require the direct determination of reaction plane angle $\psi_R$. The brackets $\langle \rangle$ represent the averaging over particles (not incorporated in the jet) in a given event, while the brackets $\langle \rangle_{\text{event}}$ the averaging over events.

The formula (7) can be generalized by introducing as weights the particle momenta,

$$
v_2^\text{jet}(p) = \frac{\langle \cos 2(\psi_{\text{jet}} - \varphi) p_T(\varphi) \rangle}{\sqrt{\langle \cos 2(\psi - \psi_R) p_T(\varphi) p_T(\varphi) \rangle}}.
$$

(8)

In this case the brackets $\langle \rangle$ denote the averaging over angles and transverse momenta of particles. The other modification of (8),

$$
v_2^\text{jet}(E) = \frac{\langle \cos 2(\psi_{\text{jet}} - \varphi) E(\varphi) \rangle}{\sqrt{\langle \cos 2(\psi - \psi_R) E(\varphi) E(\varphi) \rangle}}
$$

(9)

$E_i(\psi_i)$ being energy deposition in calorimetric ring $i$ of position $\psi_i$) allows one using calorimetric measurements (9) for the determination of jet azimuthal anisotropy, in particular, under condition of CMS experiment at LHC.

3. Numerical simulation and discussion

In order to illustrate the applicability of method presented for real physical situation, we consider the following model.

3.1. Jets

The initial jet distributions in nucleon–nucleon sub-collisions at $\sqrt{s} = 5.5$ TeV have been generated using PYTHA_5.7 generator [20]. We simulated the rescattering and energy loss of jets in gluon-dominated plasma, created initially in nuclear overlap zone in Pb–Pb collisions at different impact parameters. For details of this model one can refer to our previous papers [6,7]. Essentially for our consideration here is that in non-central collisions the azimuthal distribution of jets is approximated well by the elliptic form (4). In the model the coefficient of jet azimuthal anisotropy increases almost linearly with the growth of impact parameter $b$ and becomes maximum at $b \sim 1.2 R_A$, where $R_A$ is the nucleus radius. After that $v_2^\text{jet}$ drops rapidly with increasing $b$: this is the domain of impact parameter values, where the effect of decreasing energy loss due to reducing effective transverse size of the dense zone and initial energy density of the medium is crucial and not compensated more by stronger non-symmetry of the volume. Other important feature is that the jet azimuthal anisotropy
decreases with increasing jet energy, because the energy dependence of medium-induced loss is rather weak. Finally, the kinematical cuts on jet transverse energy and rapidity has been applied: $E_T^{\text{jet}} > 100$ GeV and $|\gamma^{\text{jet}}| < 1.5$. After this dijet event is superimposed on Pb–Pb event containing anisotropic flow.

### 3.2. Particle flow

Anisotropic flow data at RHIC [16] can be described well by hydrodynamical models for semi-central collisions and $p_T$ up to $\sim 2$ GeV/c [21], while the majority of microscopical Monte Carlo models underestimate flow effects (however, see [22]). One can expect that the hydrodynamical model can be applied to estimation of particle flow effects at LHC, may be extending this approach to even higher $p_T$ values. On the other hand, at collider energies one more reason for anisotropic flow in relatively high-$p_T$ domain can arise: the sensitivity of semi-hard particles to the azimuthal asymmetry of reaction volume under the condition that the major part of them are the products of in-medium radiated gluons or parent hard partons [2,3]. Of course, for more detailed simulation, one has to take into account the interplay between hydro flow and semi-hard particle flow. However, here we restrict our consideration to using the simple hydrodynamical Monte Carlo code [23] giving hadron (charged and neutral pion, kaon and proton) spectrum as a superposition of the thermal distribution and collective flow. For the fixed in the model “freeze-out” parameters—temperature $T_f = 140$ MeV, collective longitudinal rapidity $Y_T^{\text{max}} = 5$ and collective transverse rapidity $Y_T^{\text{max}} = 1$—we get average hadron transverse momentum $\langle p_T^h \rangle = 0.55$ GeV/c. We set the Poisson multiplicity distribution and take into account the impact parameter dependence of multiplicity in a simple way, just suggesting that the mean multiplicity of particles is proportional to the nuclear overlap function.\(^2\) \((dN/dy)/(b) \propto T_{AA}(b)\). In the framework of this model, anisotropic flow can be introduced on the assumption that the spatial ellipticity of “freeze-out” region,

$$\epsilon = \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle}, \quad (10)$$

is directly related to the initial spatial ellipticity of nuclear overlap zone, $\epsilon_0 = b/2R_A$. Such “scaling” allows one to avoid using additional parameters and, at the same time, results in introducing elliptic anisotropy of particle and energy flow due to dependence of effective transverse size of “freeze-out” region $R_f$ on azimuthal angle of “hadronic liquid” element $\Phi$:

$$R_f(b) = R_f(b = 0) \min \left\{ \sqrt{1 - \epsilon_0^2 \sin^2 \Phi} + \epsilon_0 \cos \Phi, \sqrt{1 - \epsilon_0^2 \sin^2 \Phi} - \epsilon_0 \cos \Phi \right\}. \quad (11)$$

Obtained in such a way azimuthal distribution of particles is described well by the elliptic form (1) for the domain of reasonable impact parameter values. Note that in the framework of present Letter we do not aim at detailed description of azimuthal hadronic spectra and its comparison with experimental data (in principle, fit of the data could be performed using, for example, $\epsilon_0$ in (11) as a parameter). In order to investigate the reliability of the method, we just need here to introduce the elliptic anisotropy of energy flow in Monte Carlo event generator.

### 3.3. Energy flow

To be specified, we consider the example of CMS detector at LHC collider [8]. The central (“barrel”) part of the CMS calorimetric system will cover the pseudo-rapidity region $|\eta| < 1.5$, the segmentation of electromagnetic and hadron calorimeters being $\Delta \eta \times \Delta \phi = 0.0174 \times 0.0174$ and $\Delta \eta \times \Delta \phi = 0.0872 \times 0.0872$, respectively [8]. In order roughly to reproduce the real experimental situation (not including real detector effects, but just assuming calorimeter hermeticity), we apply formula (9) to integrated with the increasing multiplicity, for weaker $b$-dependences one can obtain even better resolution for $\epsilon_0^2$.\(^2\)
over rapidity energy deposition $E_i(\phi_i)$ of generated particles in 72 rings (according to the number of rings in the hadron calorimeter) covering full azimuth.

Fig. 1 shows the $b$-dependence of “theoretical” value of $v_{jet}^2$ (calculated including collisional and radiative energy loss), and $v_{jet}^2$ determined by the method (9) for $dN^\pm/dy(y = 0, b = 0) = 3000$ (dotted curve) and 6000 (dashed curve).

To conclude this section, let us discuss the influence of different factors on results for $v_{jet}^2$ determined with the present method. First of all, note that the theoretical absolute value of $v_{jet}^2$ and its $b$-distribution are, of course, very model-dependent. For example, the effect of jet energy loss and corresponding azimuthal anisotropy are sensitive to the defined jet cone size [10–13]. Since we obtained here the energy loss only on the partonic level, for the finite jet cone size $\theta_0 \neq 0$ the effect should be less pronounced. Another reason for reducing jet azimuthal anisotropy can be sharp transverse collective expansion in events with large impact parameter values [5]. However, the relative accuracy of $v_{jet}^2$ determination does not show any significant dependence on its absolute value. The reason for this is the following. The relative error for the method (7) using particle flow can be roughly estimated as a sum of two terms, which are proportional to $(v_{jet}^2 N_{\text{event}})^{-1}$ and $(v_{part}^2 N_{\text{part}})^{-1}$, respectively ($v_{part}^2$ is the coefficient of particle azimuthal anisotropy and $N_{\text{part}}$ is the mean multiplicity in the event). If the number of events is large enough, $N_{\text{event}} \gg 1$, the main influence would be expected due to the second term, which does not depend on $v_{jet}^2$. This is also the reason why the relative error becomes significant at low values of both the coefficient of particle azimuthal anisotropy and the particle multiplicity. The same conclusion will be valid for the method (9) using energy flow, if the condition $N_{\text{part}} \gg N_{\text{ring}}$ is fulfilled.

Let us also note that in a real experimental situation, the pattern of azimuthal anisotropy can be more complicated due to non-flow correlations, finite accuracy of impact parameter determination, detector resolution effects, etc. On the other hand, one

---

Note that the applicability of hydrodynamical model to very peripheral collisions is unclear. Moreover, the edge effects near the surface of the nucleus, impact parameter dependence of nuclear parton structure functions, early transverse expansion of the system and other potentially important phenomena for such collisions are beyond our consideration here.

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*As it has been shown in recent work [5], in the extreme scenario assuming instant transverse expansion of the medium, the geometrical anisotropy is strongly reduced, and the anisotropy of particles (at $p_T \gtrsim 2–10$ GeV/c under RHIC conditions) may drop below the observable level. In principle, this effect can have influence also on jet anisotropy at LHC conditions. Let us just mention that the possibility for realization of such scenario for transverse expansion is not obvious. Within original Bjorken approach [25] the rarefaction wave front moves with the sound velocity $c_s = \sqrt{\partial p/\partial \varepsilon}$ and the whole volume of the fluid appears to be involved in three-dimensional expansion in the time of order $\tau \sim \tau_0 + R_A/c_s$.}
can try to improve accuracy of this technique considering, for example, the correlation between two equal multiplicity sub-events [14], or using results of work [19].

4. Conclusions

The strong interest is springing up to the azimuthal correlation measurements in ultrarelativistic heavy ion collisions. One of the main reasons is that the rescattering and energy loss of hard partons in azimuthally non-symmetric volume of dense quark–gluon matter can result in visible azimuthal anisotropy of high-\(p_T\) hadrons at RHIC and high-\(E_T\) jets at LHC.

In jet case, the methodical advantage of azimuthal observables is that one needs to reconstruct only azimuthal position of jet without measuring total jet energy. One of the ways to perform the inclusive analysis of jet production as a function of azimuthal angle is event-by-event determination of the nuclear reaction plane angle. In the present Letter we suggest the method for measurement of jet azimuthal anisotropy coefficients without reconstruction of the event plane. This technique is based on the calculation of correlations between the azimuthal position of jet axis and the angles of particles (not incorporated in the jet), azimuthal distribution of jets being described by the elliptic form. The method is generalized by introducing as weights the particle momenta or energy deposition in the calorimetric rings. In the latter case, we have illustrated the reliability of the present method in real physical situation under LHC conditions. The accuracy of the method improves with increasing multiplicity and particle (energy) flow azimuthal anisotropy, and is practically independent of the absolute values of azimuthal anisotropy of the jet itself.

To summarize, we believe that the present techniques may be useful for future data analysis in heavy ion collider experiments.

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References


Energy and centrality dependences of charged multiplicity density in relativistic nuclear collisions

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Abstract

Using a hadron and string cascade model, JPCIAE, the energy and centrality dependences of charged particle pseudorapidity density in relativistic nuclear collisions were studied. Within the framework of this model, both the relativistic p + ¯p experimental data and the PHOBOS and PHENIX Au + Au data at √s_{NN} = 130 GeV could be reproduced fairly well without retuning the model parameters. The predictions for full RHIC energy Au + Au collisions and for Pb + Pb collisions at the ALICE energy were also given. We computed participant nucleon distributions using different methods. It was found that the number of participant nucleons is not a well defined variable both experimentally and theoretically. Therefore, it may be inappropriate to use the charged particle pseudorapidity density per participant pair as a function of the number of participant nucleons for distinguishing various theoretical models.

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The main focus of the Relativistic Heavy-Ion Collider (RHIC) at the Brookhaven National Laboratory (BNL) is to explore the phase transition related to the quark deconfinement and the chiral symmetry restoration. The first available experimental data were the energy dependence of charged particle pseudorapidity density in central Au + Au collisions at √s_{NN} = 56 and 130 GeV from the PHOBOS Collaboration [1]. Soon later, the PHENIX Collaboration published their data of centrality dependence of the charged particle pseudorapidity density in Au + Au collisions at √s_{NN} = 130 GeV [2].

It has been predicted that the rare high charged multiplicity in the final state of relativistic nucleus–nucleus collisions might indicate the formation of the Quark-Gluon-Plasma (QGP) phase in the early stage of collisions [3–5]. In Ref. [6], the centrality depen-
dence of the charged multiplicity has been further proposed to provide information on the relative importance of soft versus hard processes in particle production and therefore to provide a means of distinguishing various theoretical models.

The pQCD calculation with assumption of gluon saturation [7,8] (later referred to as the EKRT model) was first used to study the centrality dependence of the charged particle pseudorapidity density at RHIC. The conventional eikonal approach and the high density QCD (later referred to as the KN model) [9] were also used to investigate the centrality dependence and these two methods surprisingly obtained almost identical centrality dependence. Recently, authors in [10] reported their study of the same issue from the Dual Parton Model. It was found that the experimental observation, the charged particle pseudorapidity density per participant pair slightly increasing with the number of participant nucleons, was reproduced by [6,9, 10], but contradicted the results of [8].

In this Letter, a hadron and string cascade model, JPCIAE [11], was employed to study this issue further. The JPCIAE model was developed based on PYTHIA [12]. In JPCIAE, the nucleons in a colliding nucleus are distributed according to the Woods–Saxon distribution. Each nucleon is given a beam momentum in $z$ direction and zero initial momentum in $x$ and $y$ directions. After the construction of an initial particle list, the collision time of each colliding pair is calculated under the requirement that the least approaching distance of the colliding pair along their straight line trajectory (since the mean field is not assumed) should be smaller than $\sqrt{\sigma_{\text{tot}}/\pi}$. Here $\sigma_{\text{tot}}$ refers to the total cross section. The nucleon–nucleon collision with the least collision time is selected from the initial collision list to perform the first collision. After the first collision, both the particle list and the collision list are updated. Now the collision list may consist of not only nucleon–nucleon collisions, but also collisions among the produced particles and the nucleons. The next collision is selected from the new collision list and the processes above are repeated until the collision list is empty.

For each collision pair, if its CMS energy is larger than a given cut, we assume that strings are formed after the collision and PYTHIA is used to deal with particle production. Otherwise, the collision is treated as a two-body collision [13–15]. The cut ($\approx 4$ GeV in the program) was chosen by observing that JPCIAE correctly reproduces charged multiplicity distributions in $AA$ collisions [11]. An important feature of JPCIAE at relativistic energies is that QCD parton–parton scatterings are included through PYTHIA, which causes charged particle yields to increase with collision energy as well as centrality since mini-jet production rates increase with energy and number of collisions suffered by participant nucleons. The mini-jet production as implemented in PYTHIA has been successfully tested in $p + p$ collisions at SPS energies [16]. It should be noted that the JPCIAE model is not a simple superposition of nucleon–nucleon collisions since the rescattering among participant nucleons, spectator nucleons, and produced particles is taken into account. We refer to [11] for more details of the JPCIAE model.

Since the number of participant nucleons, $N_{\text{part}}$, plays a crucial role in the presentation of PHOBOS or PHENIX data we first make a study on $N_{\text{part}}$. As the direct measurement of $N_{\text{part}}$ is not available, in the fixed target experiments the number of participant nucleons from the projectile nucleus with atomic number $A$, for instance, is estimated by

$$N_{\text{part}} = A \left(1 - \frac{E_{\text{ZDC}}}{E_{\text{beam}}} \right).$$

(1)

where $E_{\text{ZDC}}$ refers to the energy deposited in the Zero Degree Calorimeter (ZDC), dominated by the energy deposition from the projectile spectator nucleons, and $E_{\text{beam}}$ is the kinetic energy of beam [17]. However, in the collider experiments, in order to obtain $N_{\text{part}}$ one has to relate the measurements to the Monte Carlo simulations. In PHENIX, for instance, simulations for the response of the Beam–Beam Counter and the ZDC were used to calculate $N_{\text{part}}$ via the Glauber model [2]. In PHOBOS, $N_{\text{part}}$ is derived by relating HI-JING simulations to the signals in the paddle counter [18]. Therefore, $N_{\text{part}}$ here is a model-dependent variable.

Theoretically, the number of participant nucleons in a collision of $A + B$ at impact parameter $b$ can be estimated in different ways:

1. In the geometry method [19] $N_{\text{part}}(b)$ reads

$$N_{\text{part}}(b) = N_{\text{part}}^A(b) + N_{\text{part}}^B(b),$$

(2)
In the dynamical simulation of
\( A + N \) reactions, the inelastic nucleon–nucleon cross section at RHIC energies and through
\( A \) nucleus \( N \) and nucleus \( B \) respectively, and the nuclear density is normalized to the nuclear density of the projectile (target) nucleus, respectively. Not only is the theoretical uncertainty related to the calculation of \( N_{\text{part}}(b) \) in each of the above models large but also the definitions of participant nucleons or spectator nucleons are different among each other. We give only a necessary description for the following dynamical simulations mentioned in this Letter:

1. In the Glauber model, \( N_{\text{part}}(b) \) is calculated through

\[
N_{\text{part}}^A(b) = \rho_A \int dV \\
\times \theta\left( R_A - \left( x^2 + (b - y)^2 + z^2 \right)^{1/2} \right) \\
\times \theta\left( R_B - \left( x^2 + y^2 \right)^{1/2} \right),
\]

\[
N_{\text{part}}^B(b) = \rho_B \int dV \\
\times \theta\left( R_B - \left( x^2 + y^2 + z^2 \right)^{1/2} \right) \\
\times \theta\left( R_A - \left( x^2 + (b - y)^2 \right)^{1/2} \right),
\]

where \( \theta(x) = 0 \) if \( x < 0 \) and \( \theta(x) = 1 \) otherwise, \( R_A \) and \( \rho_A(R_B \text{ and } \rho_B) \) are the radius and nuclear density of the projectile (target) nucleus, respectively, and the nuclear density is normalized to the atomic number.

2. In the Glauber model, \( N_{\text{part}}(b) \) is calculated through

\[
N_{\text{part}}(b) = \int d^2s \\
\times T_A(\vec{b} - \vec{s}) \left[ 1 - \exp\left( -\sigma_{\text{in}}T_B(\vec{s}) \right) \right] \\
+ \int d^2s T_B(\vec{s}) \left[ 1 - \exp\left( -\sigma_{\text{in}}T_A(\vec{b} - \vec{s}) \right) \right],
\]

where \( \sigma_{\text{in}} \approx 40 \text{ mb} \) is the inelastic nucleon–nucleon cross section at RHIC energies and \( T_A \) (\( T_B \)) refers to the nuclear thickness function of nucleus \( A \) (\( B \)) and is normalized to \( A \) (\( B \)) [8].

3. In the dynamical simulation of \( A + B \) collisions, \( N_{\text{part}}(b) \) can be estimated via counting the participant or spectator nucleons and averaging over events simulated at a given impact parameter \( b \). However, there are multifarious in simulating models such as: FRITIOF [20], VENUS [21], HIJING [22], JPCIAE [11], UrQMD [23], and AMPT [24], etc. Not only is the theoretical uncertainty related to the calculation of \( N_{\text{part}}(b) \) in each of the above models large but also the definitions of participant nucleons or spectator nucleons are different among each other. We give only a necessary description for the following dynamical simulations mentioned in this Letter:

- In FRITIOF [20] and HIJING [22], the wounded nucleons, i.e., nucleons which suffer at least one inelastic collision, are counted and identified as \( N_{\text{part}}(b) \). It should be pointed out here that in FRITIOF and HIJING the produced particles from the string fragmentation do not have rescattering.
- In JPCIAE, we have counted the nucleons involved in at least one inelastic nucleon–nucleon collision with string excitation and identified them as \( N_{\text{part}}(b) \).
- In AMPT [25], the spectator nucleons, \( N_{\text{spec}}(b) \), are counted in the final state without rescattering and \( N_{\text{part}}(b) \) is then calculated through

\[
N_{\text{part}}(b) = (A + B) - N_{\text{spec}}(b).
\]

The spectator nucleons here refer to the nucleons with zero transverse momentum and beam energy in the final state of the AMPT simulation.

In the PHENIX or PHOBOS experiment, centrality bin was defined by the cut in the particle multiplicity distribution and expressed as the percentage of geometrical (total) cross section [2], \( g \). However, in the theoretical calculation it is more convenient to define the centrality by impact parameter \( b \). In order to compare the experimental data of centrality dependence with the theoretical results a relation between \( g \) and \( b \) is required, which is obtained from the definition of the geometrical cross section, that reads

\[
b = \sqrt{g} b_{\text{max}}, \quad b_{\text{max}} = R_A + R_B,
\]

where \( R_A = 1.12A^{1/3} + 0.54 \text{ fm} \), for instance, is the radius of nucleus \( A \) in Woods–Saxon nuclear density distribution. However, \( R_{\text{Au}} = 7.5 \text{ fm} \) is taken in our calculations since this value (corresponding to \( \sigma_{\text{geo}} = 7.2 b \)) was used in [2] to extract the number of participant nucleons in \( \text{Au} + \text{Au} \) collisions. We give in Table 1 the mapping between the centrality bin \( (g \text{ bin}) \) selected from Table 1 in [2] and the \( b \) bin according to Eq. (7), and the averaged impact parameter \( \bar{b} \) over the \( b \) bin according to the \( \bar{b}^2 \) law.

<table>
<thead>
<tr>
<th>Centrality bin</th>
<th>Bin of ( b ) (fm)</th>
<th>( \bar{b} ) (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>below 6(^a)</td>
<td>3.67</td>
<td>2.45</td>
</tr>
<tr>
<td>10–15</td>
<td>4.74–5.81</td>
<td>5.29</td>
</tr>
<tr>
<td>25–30</td>
<td>7.50–8.22</td>
<td>7.87</td>
</tr>
<tr>
<td>30–35</td>
<td>8.22–8.87</td>
<td>8.55</td>
</tr>
<tr>
<td>35–40</td>
<td>8.87–9.49</td>
<td>9.18</td>
</tr>
<tr>
<td>45–50</td>
<td>10.1–10.6</td>
<td>10.4</td>
</tr>
</tbody>
</table>

\(^a\) Below 5 for PHENIX.
Fig. 1. The energy dependence of the charged particle pseudorapidity density at mid-pseudorapidity in relativistic $p + \bar{p}$ and central $A + A$ collisions.

As the experimental data were averaged over events in each $g$ bin the theoretical results, to be compared with the experimental data, are also averaged over the corresponding $b$ bin. We denote the number of participant nucleons after averaging over $g$ or $b$ bin as $\langle N_{\text{part}} \rangle$ later.

In Fig. 1(a) the experimental data of charged particle pseudorapidity density per participant pair at mid-pseudorapidity in relativistic $p + \bar{p}$ (open triangles and rhombuses with error bar) and in central $A + A$ collisions (open circles with error bar for Pb + Pb at SPS and full circles with error bar for Au + Au at RHIC) [1] were compared with the results of the JPCIAE model (full stars for $p + \bar{p}$, open squares for Au + Au collisions at $\sqrt{s_{\text{nn}}} = 56, 130$ and 200 GeV, and full squares for Pb + Pb at $\sqrt{s_{\text{nn}}} = 17.3$ and 5500 GeV). In addition, the results from other models were plotted as follows: the dashed and dotted-dash curves are from the HIJING model [6] with and without jet quenching, respectively, the solid curve is from the EKRT model [8]. The EKRT results were obtained from [6] directly, except that the EKRT result for Au + Au collisions at $\sqrt{s_{\text{nn}}} = 5500$ GeV was taken from [26]. Fig. 1(b) is the same as Fig. 1(a), but the vertical coordinate here is the charged particle pseudorapidity density itself. One knows from Fig. 1 that both the data of $p + \bar{p}$ and $A + A$ collisions at relativistic energies are also reproduced fairly well by the JPCIAE model.

Fig. 2. The number of participant nucleons ($N_{\text{part}}$) as a function of the percentage of total cross section.

In Fig. 2, $\langle N_{\text{part}} \rangle$, extracted by PHENIX [2] and PHOBOS [18] from Au + Au collisions at $\sqrt{s_{\text{nn}}} =$
130 GeV were compared with the model computations. The horizontal axis in Fig. 2 is the percentage of geometrical cross section, $g$, and each $g$ bin is represented by its middle point for convenience in plotting (the same for Fig. 3). The solid and open circles with error bar in Fig. 2 were PHENIX and PHOBOS results, respectively. $(N_{\text{part}})$ from the geometry method, open stars, were from the geometric $N_{\text{part}}(b)$ Eqs. (2)–(4) after averaging over $b$ sampled randomly in each $b$ bin due to the $b^2$ law. Similarly, the FRITIOF, JPCIAE, and AMPT [25] results, full stars, full triangle-ups, and full triangle-downs in Fig. 2, were obtained averaging over events simulated for $b$ sampled randomly in each $b$ bin due to the $b^2$ law, respectively.

In the EKRT model, the curves of $N_{\text{part}}(b)$ vs. $b/R_A$ were given and $N_{\text{part}}$ is calculated by the Glauber method with $R_A = 1.12 A^{1/3} - 0.86 A^{-1/3}$ [8]. The open triangle-ups in Fig. 2 are the EKRT model results taken from the $\sqrt{s_{\text{NN}}} = 130$ GeV one of those curves according to $b$ in Table 1. Originally, HIJING $N_{\text{part}}$ [27] were calculated for individual $b$, the HIJING points, open squares, in Fig. 2 were plotted after relating $b$ to $g$ according to Eq. (7). The full squares in Fig. 2 were the results of the KN model taken from Table 1 in [9] under the centrality bins of 0–6, 10–20, 20–30, 30–40, and 40–50%, respectively.

In the KN model, $(N_{\text{part}})$ was the result of $N_{\text{part}}(b)$ from the Glauber model after averaging over centrality bin. In this approach, the particle multiplicity and impact parameter were related by a Gaussian distribution with parameters fixed via fitting the PHOBOS charged multiplicity distribution. As proved in [28], such kind of average is approximately equivalent to the average method based on Eq. (7). From Fig. 2 one knows that the discrepancies among the PHENIX or PHOBOS and the model results are visible.

![Figure 3](image-url)  
Fig. 3. The charged particle pseudorapidity density at mid-pseudorapidity in $Au+Au$ collisions at $\sqrt{s_{\text{NN}}} = 130$ GeV as a function of the percentage of total cross section.

The charged particle pseudorapidity density at mid-pseudorapidity in $Au+Au$ collisions at $\sqrt{s_{\text{NN}}} = 130$ GeV as a function of the percentage of geometrical cross section was given in Fig. 3. In this figure, the full and open circles with error bar are the PHENIX [2] and PHOBOS [18] data, respectively. The full and open triangles, respectively, are the JPCIAE results with and without rescattering. One knows from Fig. 3 that the rescattering only leads to a few percent increase in the charged multiplicity although rescattering might enhance yields of strangeness, $\Xi^- + \Omega^-$ for instance, by a couple of times.

In Fig. 4(a) we compared the PHENIX data of charged particle mid-pseudorapidity density per participant pair (full circles with shaded area of systematic errors) [2] with the results of the JPCIAE model (full triangles) and the results of other models (obtained from [2] directly): HIJING (the dotted curve), the KN model (the solid curve), and EKRT (the dashed curve). One sees that except EKRT, three other models predict an increase of $(dN_{\text{ch}}/d\eta)|_{\eta=0} / (0.5 \langle N_{\text{part}} \rangle)$ as a function of $(N_{\text{part}})$ though the theoretical results seem to underestimate the PHENIX data. Such an increase can be understood in JPCIAE as a result of increasing hard parton scatterings per participant nucleon. Fig. 4(b) compared the PHENIX data to the results of single $dN_{\text{ch}}/d\eta|_{\eta=0}$ from JPCIAE normalized by the $(N_{\text{part}})$ from different models (taken from the corresponding curve in Fig. 2 at the middle point of $g$ bins for KN, HIJING, and EKRT models): full squares, $(N_{\text{part}})$ from the KN model, open squares from HIJING, open triangles from EKRT, full stars from FRITIOF, and full triangles from JPCIAE. One sees from Fig. 4(b) that starting from the charged particle mid-pseudorapidity density obtained in JPCIAE, but using $(N_{\text{part}})$ from different models, the corresponding results of $(dN_{\text{ch}}/d\eta|_{\eta=0}) / (0.5 \langle N_{\text{part}} \rangle)$ are different visibly among each other, in peripheral col-
Fig. 4. The charged particle pseudorapidity density per participant pair at mid-pseudorapidity in Au + Au collisions at $\sqrt{s_{\text{NN}}}$ = 130 GeV as a function of the number of participant nucleons, $\langle N_{\text{part}} \rangle$.

Collisions especially. Therefore, it might be inappropriate using $(dN_{\text{ch}}/d\eta|_{\eta=0})/(0.5 \langle N_{\text{part}} \rangle)$ as a function of $\langle N_{\text{part}} \rangle$ to distinguish various theoretical models for particle production since $\langle N_{\text{part}} \rangle$ is not a well defined physical variable.

In summary, we used the hadron and string cascade model, JPCIAE, to investigate the energy and centrality dependences of charged particle pseudorapidity density at mid-pseudorapidity in relativistic $p + \bar{p}$ and $A + A$ collisions. Both the relativistic $p + \bar{p}$ experimental data and the PHOBOS and PHENIX data of Au + Au collisions at RHIC energies could be reproduced fairly well within the framework of the JPCIAE model without retuning any parameters. The JPCIAE model predictions for full RHIC energy Au + Au collisions and for Pb + Pb collisions at the ALICE energy were also given. This study shows that since $\langle N_{\text{part}} \rangle$ is not a well defined physical variable both experimentally and theoretically it may be hard to use charged particle pseudorapidity density per participant pair at mid-pseudorapidity as a function of $\langle N_{\text{part}} \rangle$ to distinguish various theoretical models for particle production.

Additionally, in the refereeing process of this Letter, PHOBOS reported the results of charged particle pseudorapidity density at mid-pseudorapidity: $dN_{\text{ch}}/d\eta|_{\eta=0} = 650 \pm 35$ and $(dN_{\text{ch}}/d\eta|_{\eta=0})/(0.5 \langle N_{\text{part}} \rangle) = 3.78 \pm 0.25$ in 6% most central Au + Au collisions at $\sqrt{s_{\text{NN}}}$ = 200 GeV [29], the corresponding results from the JPCIAE model are 660.6 and 3.83, respectively.

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Energy criterion to select the behavior of dynamical masses in technicolor models

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Abstract

We propose a quite general ansatz for the dynamical mass in technicolor models. We impose on this ansatz the condition that it should lead to the deepest minimum of energy. This criterion selects a particular form of the technifermion self energy.

In the standard model of elementary particles the fermion and gauge boson masses are generated due to the interaction of these particles with elementary Higgs scalar bosons. Despite the success there are some points in the model as, for instance, the enormous range of masses between the lightest and heaviest fermions and other peculiarities that could be better explained with the introduction of new fields and symmetries. One of the possibilities in this direction is the substitution of elementary Higgs bosons by composite ones in the scheme named technicolor [1].

The beautiful characteristics of technicolor (TC) as well as its problems were clearly listed recently by Lane [2]. Most of the technicolor problems may be related to the dynamics of the theory as described in Ref. [2]. Although technicolor is a non-Abelian gauge theory it is not necessarily similar to QCD, and if we cannot even say that QCD is fully understood up to now, it is perfectly reasonable to realize the enormous work that is needed to abstract from the fermionic spectrum the underlying technicolor dynamics.

The many attempts to build a realistic model of dynamically generated fermion masses are reviewed in Refs. [1,2]. Most of the work in this area try to find the TC dynamics dealing with the particle content of the theory in order to obtain a technifermion self energy that does not lead to phenomenological problems as in the scheme known as walking technicolor [3]. The idea of this scheme is quite simple. First, remember that the expression for the technifermion self energy is proportional to $\Sigma(p^2)_{TC} \propto (\langle \bar{\psi}\psi \rangle_{TC}/p^2)(p^2/\Lambda_{TC}^2)^{-\gamma^*}$, where $\langle \bar{\psi}\psi \rangle_{TC}$ is the TC condensate and $\gamma^*$ its anomalous dimension. Second, depending on the behavior of the anomalous dimension we obtain different behaviors for $\Sigma(p^2)_{TC}$. A large anomalous dimension may solve the problems in TC models. In principle, we could deal with many different models, varying fermion representations and particle content, finding different expres-
sions for $\Sigma(p^2)_{\text{TC}}$ and testing them phenomenologically, i.e., obtaining the fermion mass spectra without any conflict with experiment.

In this Letter we will introduce as one ansatz a quite general expression for the technifermion self energy. We will discuss its general properties without paying attention to any group structure and will verify when this ansatz imply that we have the tightest composite scalar boson, or the deepest minimum of energy of the theory. In principle, if we are able to find the most probable expression for the technifermion self energy based on a general criterion we will only need to find the right theory that lead to this formal expression.

In order to establish a quite general ansatz for $\Sigma(p^2)_{\text{TC}}$ we go back to early work on the phenomenology of chiral symmetry breaking in QCD, not because we shall assume that TC is similar to QCD, but because the knowledge about solutions of the Schwinger–Dyson equations for the fermion propagator in the QCD case will help us to find our general ansatz. This equation as a function of the fermion and gauge boson propagators ($S$ and $D$, respectively) is given by

$$S^{-1}(p) = \tilde{p} - \frac{4}{3} \int \frac{d^4q}{(2\pi)^4} \Gamma_\mu S(q)\Gamma_\nu(p,q) \times g^2 D^{\mu\nu}(p-q),$$

and has two asymptotic solutions [4]

$$\Sigma_I(-p^2) = \mu \left[ 1 + bg^2(\mu^2) \ln(-p^2/\mu^2) \right]^{-\gamma},$$

$$\Sigma_R(-p^2) = \frac{\mu}{-p^2} \left[ 1 + bg^2(\mu^2) \ln(-p^2/\mu^2) \right]^\gamma,$$

which are named, respectively, as irregular and regular solutions, where $\mu$ is the dynamical fermion mass ($\approx \langle \bar{\psi} \psi \rangle^{1/3}$), and $\gamma = 3c/16\pi^2b$, with $c$ given by

$$c = \frac{1}{2} \left[ C_2(R_1) + C_2(R_2) - C_2(R_3) \right],$$

where $C_2(R_i)$ is the Casimir operator for the fermions in the representations $R_1$ and $R_2$ that condensate in the representation $R_3$, and $b$ is the coefficient of $g^4$ in the $\beta$ function. Only the solution $\Sigma_R(p)$ is compatible with OPE [4] and is consistent with asymptotic freedom. The $\Sigma_I$ solution can only be understood if the theory has an explicit breaking of the chiral symmetry.

These solutions show naturally the extreme forms that we are looking for. One obeys asymptotic freedom ($\Sigma_R(p)$), appears in the case of a perturbative anomalous dimension, and lead to the known TC problems. Any other form of self energy decaying faster than $1/p^2$ is not interesting phenomenologically (i.e., it is worse than $\Sigma_R(p)$). The other solution ($\Sigma_I(p)$) could only be understood if suitable new interactions are assumed to be relevant at the scale of the cutoff of Eq. (1), or if, alternatively, the ultraviolet cutoff is eliminated altogether as assumed in the model of Ref. [5]. The only restriction on this solution is $\gamma > 1/2$ [4], and if we consider the formal equivalence between the solution of the Schwinger–Dyson equation with the Bethe–Salpeter one for pseudoscalar bound states, the above restriction indicates the condition for wave function normalization of the Goldstone bosons.

Considering the above discussion we can formulate the following ansatz for $\Sigma(p^2)_{\text{TC}}$

$$\Sigma(-p^2)_{\text{TC}} = \mu \left( \frac{\mu^2}{-p^2} \right)^\alpha \left[ 1 + bg^2(\mu^2) \ln(-p^2/\mu^2) \right]^{-\beta \cos(\alpha \pi)},$$

(4)

This choice interpolates between $\Sigma_R(p)$ and $\Sigma_I(p)$. When $\alpha \to 1$ we reproduce Eq. (3) with $\beta = \gamma$, and when $\alpha \to 0$ the solution of explicit chiral breaking is obtained. As far as we know there is not any solution that has been discussed in the TC literature that cannot be represented by Eq. (4).

We can now discuss which are the basic conditions that Eq. (4) should satisfy. Since $\Sigma(p)_{\text{TC}}$ is also equivalent to the solution of the Bethe–Salpeter equation for the scalar sector of technibosons, we could impose that the theory should be stable when it forms the tightest bound states. This condition is the same as saying that $\Sigma(-p^2)_{\text{TC}}$ must minimize the vacuum energy (or the vacuum expectation value of the TC condensate).

To compute the vacuum energy for the technifermion self energy we can make use of the effective potential for composite operators which is given by [6]

$$V(S, D) = -t \int \frac{d^4p}{(2\pi)^4} \text{Tr}(\ln S_0^{-1}S - S_0^{-1}S + 1) + V_2(S, D),$$

(5)

where $S$ and $D$ are the complete propagators of fermions and gauge bosons, respectively; $S_0$ and $D_0$, the corresponding bare propagators.
\( V_2(S, D) \) is the sum of all two-particle irreducible vacuum diagrams, depicted in Fig. 1, and the equation
\[
\frac{\delta V}{\delta S} = 0,
\]
gives the SDE for fermions. We are not considering the contributions to the vacuum due to gauge and ghosts loops, because we are interested only in the vacuum value of the fermionic operator.

We can represent \( V_2(S, D) \) analytically in the Hartree–Fock approximation by
\[
t V_2(S, D) = \frac{1}{2} \text{Tr}(\Gamma S \Gamma S D)
\]
where \( \Gamma \) is the fermion proper vertex. In Eq. (7) we have not written the gauge and Lorentz indices, as well as the momentum integrals.

We want to determine the vacuum expectation value for technifermion self energy. Therefore it is better to compute the vacuum energy density, which is given by the effective potential calculated at minimum subtracted by its perturbative part, which does not contribute to dynamical mass generation [6,7]
\[
\langle \Omega \rangle = V_{\text{min}}(S, D) - V_{\text{min}}(S_p, D_p),
\]
where \( S_p \) is the perturbative counterpart of \( S \). \( V_{\text{min}}(S, D) \) is obtained substituting the SDE Eq. (6) back into Eq. (5), and in the chiral limit \( S_p = S_0 \). The complete fermion propagator \( S \) is related to the free propagator by
\[
S^{-1} = S_0^{-1} - \Sigma,
\]
where \( S_0 = i/\not{p} \).

Choosing Landau gauge and working in the Euclidean space \((P^2 = -p^2)\) we find that \( V_{\text{min}} \) is equal to [8]
\[
V_{\text{min}} = 2N \int \frac{d^4 p}{(2\pi)^4} \left[ -\ln \left( \frac{p^2 + \Sigma^2}{p^2} \right) + \frac{\Sigma^2}{p^2 + \Sigma^2} \right],
\]
where \( N \) is the number of technicolors (techniquarks are in the fundamental representation of \( SU(N) \)).

Since we are interested in the vacuum value \( \langle \Omega \rangle \), and, particularly, in its leading term, we can expand \( V_{\text{min}} \) in powers of \( \Sigma/P \) to obtain
\[
\langle \Omega \rangle \simeq -N \int \frac{d^4 p}{(2\pi)^4} \frac{\Sigma^4}{p^4}.
\]

To perform the integral in Eq. (11), with the self energy given by Eq. (4), it is helpful to use a particular Mellin transform [9]
\[
\left[ 1 + \kappa \ln \frac{\mu^2}{\mu^2} \right]^{-\epsilon} = \frac{1}{\Gamma(\epsilon)} \int_0^\infty d\sigma e^{-\sigma} \left( \frac{\mu^2}{\mu^2} \right)^{-\sigma \epsilon - 1}.
\]

Before we calculate the expression of the vacuum energy for Eq. (4) it is instructive to show the result for Eqs. (2) and (3), which are
\[
\langle \Omega \rangle_I \simeq -\frac{\mu d^4}{64\pi^2 b} \frac{N}{g^2} \left[ 1 + b g^2 (\mu^2) \ln (A^2/\mu^2) \right]^{-4\epsilon},
\]
\[
\langle \Omega \rangle_R \simeq -\frac{\mu^4 N}{64\pi^2} \left( \frac{\mu^2}{\Lambda^2} \right) \left[ 1 + b g^2 (\mu^2) \ln (A^2/\mu^2) \right]^{4\epsilon},
\]
where \( d = \frac{1}{V_{\text{g}} - 1} \mu^2 \), \( g^2 = g^2 (\Lambda^2) \) and \( A \) is an infrared cutoff momentum. The natural value of this infrared cutoff is \( A = \mu \) [8]. Therefore,
\[
\langle \Omega \rangle_I \simeq -\frac{\mu d^4}{64\pi^2 b} \frac{N}{g^2},
\]
\[
\langle \Omega \rangle_R \simeq -\frac{\mu^4 N}{64\pi^2}.
\]

It is obvious that the values of the vacuum energy for the ansatz of Eq. (4) will also interpolate between the ones of Eqs. (15) and (16).

There are some interesting remarks about this result. Only a solution behaving logarithmically as \( \Sigma_I \) will give a vacuum energy proportional to \( 1/g^2 \), any other solution falling off faster (as \( 1/p \) at some power) will produce a vacuum energy independent of \( g \). This is a consequence of the almost convergent behavior of Eq. (11). It is possible to verify that the vacuum energy (11) is related to the fermion condensate \( \langle \Omega \rangle \propto \langle \psi \psi \rangle \) (see, for instance, Ref. [10]), consequently this one will also be proportional to \( 1/g^2 \). Therefore, for this solution an argument by Gupta and Quinn [11] could imply that this solution cannot be predicted by

\[\text{Fig. 1. Diagrammatic expansion for the effective potential.}\]
we obtain the perturbative power counting of the OPE series. Conversely proportional to the coupling constant destroys OPE, because the vacuum value of an operator in-278

energy using Eq. (4), and we can foresee two possible behaviors: (a) the ansatz lead to some intricate behav-


can now proceed to the calculation of the vacuum en-

shall restrict ourselves to the scaling law $g$ could be expected in walking TC theories. Here we

this does not happen in QCD [12] and this is also what

defined

we have

and the first one is the only minimum.

Due to the form of Eq. (4) it is better to write

Eq. (11) in the following form:

We will present our analysis of $\langle \Omega \rangle$ in the different regions of the parameter $\alpha$. We start with the case $\alpha \simeq 0$, where we can make the expansion

then Eq. (19) can be put in the following form:

\begin{equation}
\langle \Omega \rangle_0 \simeq -\frac{\mu^4 N}{16\pi^2 a} \frac{1}{\Gamma(\delta)} \int_0^\infty dz \, z^{\delta-1} e^{-z} \left[ 1 - \frac{4\alpha}{az} + \cdots \right].
\end{equation}

Retaining only the first two terms after integration and using properties of Gamma functions we can write

\begin{equation}
\langle \Omega \rangle_0 \simeq -\frac{\mu^4 N}{16\pi^2 a} \frac{1}{\Gamma(\delta)} \left[ \frac{1}{(\delta - 1)} - \frac{4\alpha}{a} \frac{1}{(\delta - 1)(\delta - 2) + \cdots} \right].
\end{equation}

In this case we have $\delta \simeq 4\beta$. The coefficient $a = b g^2$ can be roughly estimated around of one, $b g^2 \simeq 1$, because for many groups the coefficient $b$ is always smaller that one. For example, in the case of $SU(N)$ and $N = 4$, we have $b = 0.067$ and if we assume the scaling law $\frac{g^2}{\lambda^2} \approx 1$, we obtain $b g^2 \approx 0.85$. For $N = 8$ we have $b g^2 \approx 2.0$. Therefore, we can define

\begin{equation}
\Phi(\Omega) \equiv \frac{64\pi^2 \langle \Omega \rangle_0}{\mu^3 N} \times \left[ \frac{1}{(\delta - 1)} - \frac{4\alpha}{a} \frac{1}{(\delta - 1)(\delta - 2) + \cdots} \right].
\end{equation}

This result is plotted in Fig. 2 as a function of $\alpha$ and $\beta$. In this figure we see that the deeper minimum happens for the smallest value of $\alpha$ and $\beta$ ($= 1/2$);

In the case when $\alpha \simeq 1$ we have

\begin{equation}
\langle \Omega \rangle_1 \simeq -\frac{\mu^4 N}{64\pi^2 a} \left[ \frac{1}{\alpha} + \frac{\beta}{\alpha} + \cdots \right].
\end{equation}

Now $\beta \cos(\alpha \pi) \approx -\beta$, and we define

\begin{equation}
\Gamma(\Omega) \equiv \frac{64\pi^2 \langle \Omega \rangle_1}{\mu^3 N} = -\frac{1}{\alpha} \left[ 1 + \frac{\beta}{\alpha} + \cdots \right].
\end{equation}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Behavior of $\Phi(\Omega) = \frac{64\pi^2 \langle \Omega \rangle_0}{\mu^3 N}$ plotted in terms of the $\alpha$ and $\beta$ parameters.}
\end{figure}
This case is depicted in Fig. 3. Note that in Fig. 3 the region of large $\alpha$ is one of maximum energy. The surface decreases smoothly towards small $\alpha$ changing gradually its minima from large $\beta$ to small $\beta$.

Note that Eq. (19) can also be solved without recurring to expansions in terms of the Wittaker functions $(W_{a,b}(x))$ and the result is equal to

$$\langle \Omega \rangle^\prime \simeq -\frac{\mu^4}{16\pi^2a} \left( \frac{4\alpha}{a} \right)^{\frac{3}{2}} e^{2\alpha/a} W_{\frac{1}{2},\frac{1}{2}}(4\alpha/a) \right].$$

However, this solution is valid only for positive $\beta \cos(\alpha \pi)$. Expanding this solution for $\alpha \approx 0$ we recover the result for $\langle \Omega \rangle_0$.

Our result shows that the deepest minimum of energy happens for a solution behaving like

$$\Sigma_{TC}(-p^2) \simeq \mu \left[ 1 + b\gamma^2(\mu^2) \ln(-p^2/\mu^2) \right]^{-\gamma},$$

where $\gamma = 3c/16\pi^2b$ has the smallest possible value ($\approx 1/2$).

To conclude we can remember that the expression of Eq. (26) is indeed the one that solve many of the TC problems. Unfortunately, we verified only one criterion that the solution must obey, from this one to the construction of realistic models (and test of the criterion) there is still a lot of work to do.

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Vector manifestation in hot matter

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Abstract

Based on the hidden local symmetry (HLS) Lagrangian as an effective field theory of QCD, we find that the chiral symmetry restoration for hot QCD can be realized through the vector manifestation where the ρ meson becomes massless degenerate with π as the chiral partner. This is done by including, in addition to the hadronic thermal effects due to the π- and ρ-loops, the intrinsic temperature dependences of the parameters of the HLS Lagrangian through the matching of the HLS with the underlying QCD. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

Vector meson mass in hot and/or dense matter is one of the most interesting physical quantities in studying the hot and/or dense QCD where the chiral symmetry is expected to be restored (for reviews, see, e.g., Refs. [1–4]). The BNL Relativistic Heavy Ion Collider (RHIC) has started to measure several effects in hot and/or dense matter. Especially, the light vector meson mass is important for analysing the dilepton spectra in RHIC. In Refs. [2,5] it was proposed that the ρ-meson mass scales like the pion decay constant and vanishes at the chiral phase transition point in hot and/or dense matter.

To study the ρ mass in hot matter it is useful to use models including the ρ meson. Among several such models we use the model based on the hidden local symmetry (HLS) [6] which successfully reproduces the phenomena of ρ−π system at zero temperature. The HLS model is a natural extension of the nonlinear sigma model, and reduces to it in the low-energy region. It was shown [7] that the HLS model is equivalent to other models for vector mesons at tree level. We should stress here that, as first pointed by Georgi [8] and developed further in Refs. [9–12], thanks to the gauge symmetry in the HLS model, we can perform a systematic loop expansion including the vector mesons in addition to the pseudoscalar mesons.

Several groups [4,13,14] studied the ρ mass in hot matter using the HLS model. Most of them included only the thermal effect of π and dropped that of ρ itself. In Ref. [14], the first application of the systematic chiral perturbation with HLS [8–12] in hot matter was made. There hadronic thermal effects of ρ and π were included at one loop and the ρ mass was shown to increase with temperature T at low temperature.
In the analysis done in Ref. [14] the parameters of the Lagrangian at $T = 0$ were used by assuming no temperature dependences of them. When we naively extrapolate the results in Ref. [14] to the critical temperature, the resultant axialvector and vector current correlators do not agree with each other. Disagreement between these correlators is obviously inconsistent with the chiral restoration in QCD. However, the parameters of the HLS Lagrangian should be determined by the underlying QCD. As was shown in Ref. [11], the bare parameters of the HLS Lagrangian defined at the matching scale $\Lambda$ for $N_f = 3$ at $T = 0$ are determined by matching the HLS with the underlying QCD at $\Lambda$ through the Wilsonian matching conditions: This was done by matching the current correlators by the OPE at non-zero temperature. The resultant axialvector and vector current correlators in the HLS with those derived by the operator product expansion (OPE) in QCD. Since the current correlators by the OPE at non-zero temperature depend on the temperature (see, e.g., Refs. [15, 16]), the application of the Wilsonian matching to the hot matter calculation implies that the bare parameters of the HLS do depend on the temperature, which we call the intrinsic temperature dependences in contrast to the hadronic thermal effects. We stress here that the above disagreement between the current correlators is cured by including the intrinsic temperature dependences.

In Ref. [17], on the other hand, the vector manifestation (VM) is proposed as a new pattern of the Wigner realization of chiral symmetry, in which the chiral symmetry is restored at the critical point by the massless degenerate pion (and its flavor partners) and the $\rho$ meson (and its flavor partners) as the chiral partner, in sharp contrast to the traditional manifestation à la linear sigma model where the symmetry is restored by the degenerate pion and the scalar meson. It was shown that VM actually takes place in the large $N_f$ QCD through the Wilsonian matching. Since the VM is a general property in the chiral restoration when the HLS can be matched with the underlying QCD at the critical point, it was then suggested [3,17,18] that the VM may be applied to the chiral restoration in hot and/or dense matter.

In this Letter, we demonstrate that the VM can in fact occur in the chiral symmetry restoration in hot matter, using the HLS as an effective field theory of QCD. Here we determine the intrinsic temperature dependences of the bare parameters of the HLS through the Wilsonian matching in hot matter, and convert them to the intrinsic temperature dependences of the on-shell parameters by including the quantum effects through the Wilsonian RGEs for the HLS parameters [11,19]. Then, we separately include the hadronic thermal effects to obtain physical quantities by explicitly calculating the $\pi$- and $\mu$-thermal loops.

2. Hidden local symmetry

Let us first describe the HLS model based on the $G_{\text{global}} \times H_{\text{local}}$ symmetry, where $G = SU(N_f)_L \times SU(N_f)_R$ is the global chiral symmetry and $H = SU(N_f)_V$ is the HLS. The basic quantities are the gauge boson $\rho_{\mu}$ and two variables

$$\xi_{L,R}(x) \rightarrow \tilde{\xi}_{L,R}(x) = h(x)\xi_{L,R}(x)\xi_{L,R}^T,$$

where $h(x) \in H_{\text{local}}$ and $\xi_{L,R} \in G_{\text{global}}$. The covariant derivatives of $\xi_{L,R}$ are defined by

$$D_{\mu} \xi_L = \partial_{\mu} \xi_L - ig\rho_{\mu} \xi_L + i\xi_L \mathcal{L}_{\mu},$$
$$D_{\mu} \xi_R = \partial_{\mu} \xi_R - ig\rho_{\mu} \xi_R + i\xi_R \mathcal{L}_{\mu},$$

where $g$ is the HLS gauge coupling and $\mathcal{L}_{\mu}$ and $\mathcal{R}_{\mu}$ denote the external gauge fields gauging the $G_{\text{global}}$ symmetry.

The HLS Lagrangian is given by [6]

$$\mathcal{L} = F^2_\pi [\tilde{\alpha}_{\perp} \tilde{\alpha}_{\perp}^T] + F^2_\sigma [\tilde{\alpha}_{\parallel} \tilde{\alpha}_{\parallel}^T] + \mathcal{L}_{\text{kin}}(\rho_{\mu}),$$

where $\mathcal{L}_{\text{kin}}(\rho_{\mu})$ denotes the kinetic term of $\rho_{\mu}$ and

$$\tilde{\alpha}_{\perp,\parallel} = (D_{\mu} \xi_R \cdot \xi_L + D_{\mu} \xi_L \cdot \xi_R)/(2i).$$

When the kinetic term $\mathcal{L}_{\text{kin}}(\rho_{\mu})$ is ignored in the low-energy region, the second term of Eq. (4) vanishes by integrating out $\rho_{\mu}$ and only the first term remains. Then, the HLS model is reduced to the nonlinear sigma model based on $G/H$.

Note that this $\sigma$ is different with the scalar meson in the linear sigma model.
At zero temperature $T = 0$, it was shown [8,10] that, thanks to the gauge symmetry in the HLS, we can perform the systematic loop expansion including the vector meson. Here the expansion parameter is a ratio of the $\rho$ meson mass to the chiral symmetry breaking scale $\Lambda_{\chi}$ [8] in addition to the ratio of the momentum $p$ to $\Lambda_{\chi}$ as used in the ordinary chiral perturbation theory. By assigning $O(\rho)$ to the HLS gauge coupling $g$ [8,10], the Lagrangian in Eq. (4) is counted as $O(p^2)$, and one-loop quantum corrections obtained from the Lagrangian are counted as $O(p^4)$.

Due to quantum corrections, three parameters $F_\pi$, $F_\sigma$ and $g$ are renormalized at one-loop level, and depend on the renormalization scale $\mu$ [9,11,19]. Furthermore, at non-zero temperature $T > 0$, these parameters have the intrinsic temperature dependences. We write both dependences explicitly as $F_\pi(\mu; T)$ and $g(\mu; T)$.

To avoid confusion, we use $f_\pi$ for the physical decay constant of $\pi$, and $F_\pi$ for the parameter of the Lagrangian. Similarly, $M_\rho$ denotes the parameter of the Lagrangian and $m_\rho$ the $\rho$ mass. For calculating the hadronic thermal corrections it is convenient to adopt the on-shell renormalization scheme at $T = 0$ as in Ref. [14]. Below, we use the following abbreviated notations:

$$F_\pi = F_\pi(\mu = 0; T),$$
$$g = g(\mu = M_\rho(T); T),$$
$$a = a(\mu = M_\rho(T); T),$$

where $M_\rho$ is determined from the on-shell condition:

$$M_\rho^2 = M_\rho(T)^2$$
$$= a(\mu = M_\rho(T); T) g(T)^2 (\mu = M_\rho(T); T)$$
$$\times F_\pi^2 (\mu = M_\rho(T); T).$$

Then, the parameter $M_\rho$ in this Letter is renormalized in such a way that it becomes the pole mass at $T = 0$.

3. Hadronic thermal corrections

Here we summarize the hadronic thermal effects to the decay constant of $\pi$ and the $\rho$ mass shown in Ref. [14] where the temperature $T$ is assigned to be of $O(\rho)$ following Ref. [20].

The decay constant of $\pi$ is defined through the longitudinal component of the axialvector current correlator at the low energy limit [21]. The hadronic thermal corrections from $\pi$ and $\rho$ are summarized as [14]

$$f_{\pi}^2(T) = F_{\pi}^2 - \frac{N_f}{2\pi^2} \left[ I_2 - a J_1 + \frac{a}{3M_\rho^2}(I_4 - J_1^4) \right],$$

where $I_n$ and $J_m^n (n, m$: integer) are defined as

$$I_n = \frac{1}{\omega^n} \int_0^{\infty} \frac{k^{n-1}}{e^{\omega/T} - 1} d\omega,$$

$$J_m^n = \frac{1}{\omega^n} \int_0^{\infty} \frac{1}{e^{\omega/T} - 1} \omega^m d\omega,$$

with $\omega = \sqrt{k^2 + M_\rho^2}$. When we consider the low temperature region $T \ll M_\rho$ in Eq. (8), only the $I_2$ term remains:

$$f_{\pi}^2(T) \approx F_{\pi}^2 - \frac{N_f}{2\pi^2} I_2 = F_{\pi}^2 - \frac{N_f T^2}{12},$$

which is consistent with the result in Ref. [20].

We estimate the critical temperature by naively extrapolating the above result to the higher temperature without including the intrinsic temperature dependences. The critical temperature for $N_f = 3$ is approximated as

$$T_{c}^{(\text{had})} \approx \sqrt{12N_f} \, f_\pi(T = 0)$$
$$= 2 f_\pi(0) \approx 180 \text{ MeV}. \quad (10)$$

In Ref. [14] $m_\rho$ is defined by the pole of the longitudinal $\rho$ propagator at rest frame:

$$m_\rho^2(T) = M_\rho^2 - \text{Re} \, \Pi_L^\rho(p_0 = M_\rho, \vec{p} = 0; T),$$

where $\text{Re} \, \Pi_L^\rho$ denotes the real part of the longitudinal component of the $\rho$ two-point function at one-loop level. Inside the one-loop correction $\text{Re} \, \Pi_L^\rho$ we replaced $m_\rho$ by $M_\rho$, since the difference is of higher order. The resultant thermal corrections are summarized as [14]

$$m_\rho^2(T) = M_\rho^2 - \frac{N_f}{2\pi^2} \frac{g^2}{12} \left[ \frac{a}{2} + \frac{5}{4} J_1^4 - \frac{33}{16} M_\rho^2 F_3^2 \right],$$

(13)

The renormalization scale $\mu$ and the temperature $T$ are independent of each other in the present approach.
where \( J_1^2 \) is defined in Eq. (9), and \( \overline{F}_3^n \) and \( \overline{G}_n \) are defined as

\[
\overline{F}_3^n \equiv \int_0^\infty \frac{d k}{P} \frac{1}{e^{\alpha k} - 1} \frac{4 k^n}{\alpha (4 \omega - M_\rho^2)},
\]

\[
\overline{G}_n \equiv \int_0^\infty \frac{d k}{P} \frac{k^{n-1}}{e^{\alpha k} - 1} \frac{4 k^2}{4 \omega^2 - M_\rho^2}, \tag{14}
\]

with \( P \) denoting the principal part. From this expression it was shown [14] that there is no \( T^2 \) term in the low temperature region consistently with the result in Ref. [22].

4. Intrinsic temperature dependences

Let us now include the intrinsic temperature dependences of \( F_\pi, a \) and \( g \) and \( M_\rho^2 = a g^2 F_\pi^2 \) appearing in Eqs. (8) and (13). To do that, we first determine the bare parameters defined at the matching scale \( \Lambda \) by extending the Wilsonian matching [11], which was originally proposed for \( T = 0 \), to non-zero temperature.

We should note that, for the validity of the expansion parameter \( M_\rho/\Lambda \) must be smaller than the chiral symmetry breaking scale \( \Lambda \). We match the axialvector and vector current correlators in the HLS with those derived in the OPE for QCD at non-zero temperature (see, e.g., Refs. [15,16]). The correlators in the HLS around the matching scale \( \Lambda \) are well described by the same forms as those at \( T = 0 \) [11] with the bare parameters having the intrinsic temperature dependences:

\[
\Pi_{A}^{(\text{HLS})}(Q^2) = \frac{F_\pi^2(\Lambda; T)}{Q^2} - 2 z_2(\Lambda; T),
\]

\[
\Pi_{V}^{(\text{HLS})}(Q^2) = \frac{F_\rho^2(\Lambda; T)[1 - 2 g^2(\Lambda; T)z_3(\Lambda; T)]}{M_\rho^2(\Lambda; T) + Q^2} - 2 z_1(\Lambda; T), \tag{15}
\]

where \( M_\rho^2(\Lambda; T) \equiv g^2(\Lambda; T) F_\rho^2(\Lambda; T) \) is the bare \( \rho \) mass, and \( z_1, z_2, z_3(\Lambda; T) \) are the bare coefficient parameters of the relevant \( O(p^4) \) terms [10–12].

Matching the above correlators with those by the OPE in the same way as done for \( T = 0 \) [11], we determine the bare parameters including the intrinsic temperature dependences, which are then converted into those of the on-shell parameters through the Wilsonian RGEs [11,19]. As a result, the parameters appearing in the hadronic thermal corrections have the intrinsic temperature dependences. In this way we include both the intrinsic and hadronic thermal effects together into the physical quantities.

5. Vector manifestation

Now, we study the chiral restoration in hot matter. Here we assume that the chiral broken phase is in the confining phase, i.e., the critical temperature \( T_c \) for chiral phase transition is not larger than the critical temperature for confinement–deconfinement phase transition, and the hadronic picture is valid. When the symmetry is completely restored, the HLS is not applicable. We approach to the critical temperature from the broken phase where the HLS is applicable. At the moment we assume that the expansion parameter \( M_\rho/\Lambda \) is small near \( T_c \). It turns out that it is actually small since \( M_\rho \rightarrow 0 \) when \( T \rightarrow T_c \), as we will show below. We first consider the Wilsonian matching at the critical temperature \( T_c \) for \( N_f = 3 \) with assuming that \( \langle \bar{q}q \rangle \) approaches to 0 continuously for \( T \rightarrow T_c \). In such a case, the axialvector and vector current correlators by the OPE approach to each other, and agree at \( T_c \). Then through the Wilsonian matching we require that the correlators in Eq. (15) agree with each other. As was shown in Ref. [17] for large \( N_f \) chiral restoration, this agreement is satisfied if the following conditions are met:

\[
g(\Lambda; T) \rightarrow 0 \quad \text{as} \quad T \rightarrow T_c, \quad a(\Lambda; T) \rightarrow 1, \quad z_1(\Lambda; T) \rightarrow z_2(\Lambda; T) \rightarrow 0. \tag{16}
\]

As we explained above, the conditions for the bare parameters \( g(\Lambda; T_c) = 0 \) and \( a(\Lambda; T_c) = 1 \) are converted into the conditions for the on-shell parame-

---

4 It is known that there is no Ginzburg–Landau type phase transition for \( N_f = 3 \) (see, e.g., Refs. [1,2]). There may still be a possibility of non-Ginzburg–Landau type continuous phase transition such as the conformal phase transition [23]. When the Wilsonian matching can be applicable for \( N_f = 2 \), the VM should occur.

5 We should note that we can take \( T \rightarrow T_c \) limit with \( \Lambda \) fixed in Eq. (16) since \( \Lambda \) and \( T \) in the bare parameters of the HLS are independent of each other.
terms through the Wilsonian RGEs. Since \( g = 0 \) and \( a = 1 \) are separately the fixed points of the RGEs for \( g \) and \( a \) \[19\], the on-shell parameters also satisfy \((g, a) = (0, 1)\), and thus \( M_\rho = 0 \).

Let us include the hadronic thermal effects to obtain the \( \rho \) pole mass. Here we extrapolate the result in Eq. (13) to the higher temperature with including the intrinsic temperature dependences of the parameters. Noting that \( G_2^\pi \rightarrow \pi^2 T^2/6 \), \( J_1^\pi \rightarrow \pi^2 T^2/6 \) and \( M_\rho^2 f_\pi^2 \rightarrow 0 \) for \( M_\rho \rightarrow 0 \), Eq. (13) for \( M_\rho \ll T \) reduces to

\[
m_\rho^2(T) = M_\rho^2 + g^2 N_f \frac{15 - a^2 \pi^2}{12 - 4} T^2.
\]

Since \( a \simeq 1 \) near the restoration point, the second term is positive. Then the \( \rho \) pole mass \( m_\rho \) is bigger than the parameter \( M_\rho \) due to the hadronic thermal corrections. Nevertheless, the intrinsic temperature dependence determined by the Wilsonian matching requires that the \( \rho \) becomes massless at the critical temperature:

\[
m_\rho^2(T) \rightarrow 0, \quad T \rightarrow T_c.
\]

since the first term in Eq. (17) vanishes as \( M_\rho \rightarrow 0 \), and the second term also vanishes since \( g \rightarrow 0 \) for \( T \rightarrow T_c \). This implies that, as was suggested in Refs. [3, 17, 18], \( \text{the vector manifestation (VM)} \) actually occurs at the critical temperature. This is the main result of this Letter, which is consistent with the picture shown in Refs. [2, 3, 5, 18]. We should stress here that the above \( m_\rho(T) \) is the \( \rho \) pole mass, which is important for analyzing the dilepton spectra in RHIC experiment. It is noted [17] that although conditions for \( g(A; T) \) and \( a(A; T) \) in Eq. (16) coincide with the Georgi’s vector limit [8], the VM \( (f_\pi \rightarrow 0) \) should be distinguished from Georgi’s vector realization [8].

6. Critical temperature

Let us determine the critical temperature. For \( T > 0 \) the thermal averages of the Lorentz non-scalar operators such as \( \hat{q} \gamma_i B_{\mu \nu} q \) exist in the OPE [16]. Since these are smaller than the main term \( 1 + \alpha_s/\pi \), we expect that they give only small corrections to the value of \( T_c \), and neglect them here. Then, the Wilsonian matching condition to determine the bare parameter \( F_\pi(A; T_c) \) is obtained from that in Eq. (4.5) of Ref. [11] by taking \( \langle \bar{q} q \rangle = 0 \) and including a possible temperature dependence of the gluonic condensate:

\[
F_\pi^2(A; T_c) = \frac{1}{8\pi^2} \left[ 1 + \frac{\alpha_s}{\pi} + \frac{2\pi^2}{3} \frac{\langle G_{\mu \nu} G^{\mu \nu} \rangle_{T_c}}{\Lambda^4} \right] - \frac{N_f}{4}.
\]

(19)

The on-shell parameter \( F_\pi(0; T_c) \) is determined through the Wilsonian RGE [11, 19] for \( F_\pi \) with taking \( (g, a) = (0, 1) \). As for large \( N_f \) [17, 19], the result is given by

\[
F_\pi^2(0; T_c) = \frac{N_f}{2(4\pi)^2}.
\]

(20)

We need to include the hadronic thermal effects to obtain the relation between the parameter \( F_\pi(0; T_c) \) and the order parameter \( f_\pi(T_c) \). Here we extrapolate the hadronic thermal effect shown in Eq. (8) to higher temperature with including the intrinsic thermal effect. Then, taking \( M_\rho \rightarrow 0 \) and \( a \rightarrow 1 \) in Eq. (8), we obtain

\[
0 = F_\pi^2(T_c) = \frac{N_f}{24} T_c^2 - \frac{N_f}{24} T_c^2
\]

(21)

Here we should note that the coefficient of \( T^2 \) in the second term is a half of that in Eq. (10) which is an approximate form for \( T \ll M_\rho \) taken with assuming that the \( \rho \) does not become light. On the other hand, here the factor \( 1/2 \) appears from the contribution of \( \sigma \) (longitudinal \( \rho \)) which becomes the real NG boson at \( T = T_c \) due to the VM where the chiral restoration in QCD predicts \( a \rightarrow 1 \) and \( g \rightarrow 0 \) for \( T \rightarrow T_c \). From Eq. (21) together with Eqs. (19) and (20), \( T_c \) is expressed as

\[
T_c = \sqrt{\frac{24}{N_f} F_\pi(0; T_c)} = \sqrt{\frac{3A^2}{N_f \pi^2}} \times \left[ 1 + \frac{\alpha_s}{\pi} + \frac{2\pi^2}{3} \frac{\langle G_{\mu \nu} G^{\mu \nu} \rangle_{T_c}}{\Lambda^4} - \frac{N_f}{4} \right]^{1/2}.
\]

(22)

We estimate the value of \( T_c \) for \( N_f = 3 \). The value of the gluonic condensate near phase transition point becomes about half of that at \( T = 0 \) [3, 24], so we use \( \langle \frac{1}{2} F_{\mu \nu} F^{\mu \nu} \rangle_{T_c} = 0.006 \text{ GeV}^4 \) obtained by multiplying the value at \( T = 0 \) shown in Ref. [25] by \( 1/2 \). For
the value of the QCD scale $\Lambda_{QCD}$ we use $\Lambda_{QCD} = 400$ MeV as a typical example. For this value of $\Lambda_{QCD}$, it was shown [11] that the choice of $\Lambda = 1.1$ GeV provides the predictions in good agreement with experiment at $T = 0$. However, the matching scale may have the temperature dependence. In the present analysis we use $\Lambda = 0.8, 0.9, 1.0$ and $1.1$ GeV, and determine $T_c$ from Eq. (22). We show the resultant values in Table 1.

We note that the estimated values of $T_c$ in Table 1 are larger than that in Eq. (11) which is obtained by naively extrapolating the temperature dependence from the hadronic thermal effects without including the intrinsic temperature dependences. This is because the extra factor $1/2$ appears in the second term in Eq. (21) compared with that in Eq. (11). As we stressed below Eq. (21), the factor $1/2$ comes from the contribution of $\sigma$ (longitudinal $\rho$) which becomes massless at the chiral restoration point.

7. Summary and discussions

To conclude, by imposing the Wilsonian matching of the HLS with the underlying QCD at the critical temperature, where the chiral symmetry restoration takes place, the vector manifestation (VM) necessarily occurs: the vector meson mass becomes zero. Accordingly, the light vector meson gives a large thermal correction to the pion decay constant, and the value of the critical temperature becomes larger than the value estimated by including only the $\pi$ thermal effect. The result that the vector meson becomes light near the critical temperature is consistent with the picture shown in Refs. [2,3,5,18].

Several comments are in order.

As shown in Ref. [17], in the VM only the longitudinal $\rho$ couples to the vector current near the critical point, and the transverse $\rho$ is decoupled from it. The $A_1$ in the VM is resolved and/or decoupled from the axialvector current near $T_c$, since there is no contribution in the vector current correlator to be matched with the axialvector correlator. We expect that the scalar meson is also resolved and/or decoupled near $T_c$ since it in the VM is in the same representation as the $A_1$ is. We also expect that excited mesons are also resolved and/or decoupled.

The estimated values of $T_c$ shown in Table 1 as well as $T_c^{\text{had}}$ in Eq. (11) may be changed by higher order hadronic thermal effects, as in the chiral perturbation analysis [27]. On the other hand, the VM at $T_c$ is governed by the fixed point and not changed by higher order effects.

The parameter $M_\rho^2$ in Eq. (13) presumably has an intrinsic temperature dependence proportional to $T^2$ through the Wilsonian matching. Since we studied the intrinsic dependences only at $T_c$, we cannot definitely argue how $m_\rho(T)$ falls in $T_c$. However, we think that $g^2(\Lambda; T)$ vanishes as $\langle \bar{q}q \rangle^2_{T_c}$ near $T_c$ in the VM. If $\langle \bar{q}q \rangle^2_T$ falls as $(1 - T^2/T_c^2)$ near $T_c$, then the $\rho$ pole mass $m_\rho^2(T)$ as well as the parameter $M_\rho^2(T)$ vanishes as $(1 - T^2/T_c^2)$ which seems to agree with the behavior of $f_\pi^2(T)$. In such a case the scaling property in the VM may be consistent with the Brown–Rho scaling $m_\rho(T)/m_\rho(0) \sim f_\pi(T)/f_\pi(0)$ [5].

Although we concentrated on the hot matter calculation in this Letter, the present approach can be applied to the general hot and/or dense matter calculation.

At present, there are no clear lattice data for the $\rho$ pole mass in hot matter. Our result here will be checked by lattice analyses in future. [28]

In this Letter we performed our analysis at the chiral limit. We need to include the explicit chiral symmetry breaking effect from the current quark masses when we apply the present analysis to the real QCD. In such a case, we need the Wilsonian matching conditions with including non-zero quark mass which have not yet been established. Here we expect that the qualitative structure obtained in the present analysis will not be changed by the inclusion of the current quark masses.

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Table 1

Estimated values of the critical temperature $T_c$ for several choices of the value of the matching scale $\Lambda$. Units of $\Lambda$ and $T_c$ are GeV

<table>
<thead>
<tr>
<th>$\Lambda$</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
<th>1.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_c$</td>
<td>0.21</td>
<td>0.22</td>
<td>0.23</td>
<td>0.25</td>
</tr>
</tbody>
</table>

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6 This value of $\Lambda_{QCD}$ is within the range of values estimated in Ref. [26]: $\Lambda_{\overline{MS}}^{(3)} = 297 \sim 457$ MeV.
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References

Interpretation of the Wigner energy as due to RPA correlations

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Abstract

In a schematic model with equidistant fourfold degenerate single-nucleon levels, a conventional isovector pairing force and a symmetry force, the RPA correlation energy rises almost linearly with the isospin \( T \), thus producing a Wigner term in accordance with the empirical proportionality of the symmetry energy to \( T(T+1) \).

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Nearly symmetric nuclei have an extra binding, the so-called Wigner energy, that is not described by the quadratic symmetry term in the semi-empirical mass formula [1]. This is explained in various ways in the literature. Counting the even bonds among supermultiplet degenerate nucleon orbitals, Wigner estimates that the isospin-dependent part of the interaction energy in the ground state of a doubly even nucleus is proportional to \( T(T + 4) \), where \( T \) is the isospin [2]. Talmi proves that for seniority-conserving forces such as the pairing force acting in a single \( j \)-shell, this part of the interaction energy is proportional to \( T(T + 1) \) [3], and Bohr and Mottelson point out that this isospin-dependence arises in general from an interaction proportional to the scalar product of the isospins of the interacting nucleons [4]. A symmetry energy proportional to \( T(T + 1) \) also fits the empirical masses well [5]. Myers and Swiatecki attribute the extra binding of nearly symmetric nuclei to the interaction of neutrons and protons in overlapping orbitals [6]. Shell model calculations with realistic forces are successful in reproducing the measured binding energies [7], whereas with Skyrme forces, no Wigner term appears in Hartree–Fock–Bogolyubov calculations and only a small one in Hartree–Fock calculations [8]. However, by invoking a particular isoscalar pairing force that breaks geometric symmetries, Satula and Wyss obtain a significant Wigner energy in approximately number-projected Bogolyubov calculations [9]. In finite-temperature Bogolyubov calculations with a Yamaguchi force, Röpke et al. get at low temperature in the local density approximation a contribution from neutron–proton pairing to the binding energies in the \( A = 40 \) isobaric chain with a maximum at \( N - Z = -1 \) and a quite irregular dependence on the asymmetry [10].

The RPA is the leading order correction to the Hartree–Fock–Bogolyubov approximation. I have therefore calculated the RPA correlation energy from
the schematic Hamiltonian
\[ H = H_0 - GP^1 \cdot P + \frac{\kappa}{2} T^2, \]
\[ H_0 = \sum_{k\sigma\tau} \epsilon_k a^\dagger_{k\sigma\tau} a_{k\sigma\tau}. \]

In this expression, the index \( k\sigma\tau \) labels orthonormal nucleon orbitals, and \( a_{k\sigma\tau} \) are the corresponding annihilation operators. \( k\sigma \) takes the values \( k \) and \( \tilde{k} \) so that the orbital \( \tilde{k} \) is obtained from the orbital \( k \) by time-reversal, and \( \tau \) is ‘n’ for a neutron orbital or ‘p’ for a proton orbital. \( P \) is the pair annihilation isovector, and \( T \) denotes the total isospin. The former has the coordinates
\[ P_x = \frac{-P_n + P_p}{\sqrt{2}}, \quad P_y = \frac{-i(P_n + P_p)}{\sqrt{2}}, \]
\[ P_z = P_{np}, \quad P_t = \sum_k a^\dagger_{k\tau} a_{k\tau}, \]
\[ P_{np} = \sum_k a^\dagger_{kp} a_{kn} + a^\dagger_{kn} a_{kp}. \]

The single-nucleon energy \( \epsilon_k \) takes \( \Omega \) equidistant values separated by \( \eta \), and \( G \) and \( \kappa \) are coupling constants.

To describe states with a given number \( \Lambda_v = \sum_{k\sigma\tau} a^\dagger_{k\sigma\tau} a_{k\sigma\tau} \) of valence nucleons and a given isospin, I employ the Routhian
\[ R = H - \lambda \Lambda_v - \mu T_z. \]

It is just for convenience that \( T_z \) is chosen here as the isospin-coordinate to be constrained. Since \( H \) is isobarically invariant, one could equivalently constrain the projection of \( T \) on any axis in isospace. Following Marshalek [11] I base the RPA on the Hartree–Bogolyubov (not Fock) self-consistent state derived from \( R \). This is the Bogolyubov vacuum that minimizes \( E_0 = \lambda \langle \Lambda_v \rangle - \mu \langle T_z \rangle \), where
\[ E_0 = \langle H_0 \rangle - \frac{\langle |\Delta|^2 \rangle}{G} + \frac{\kappa}{2} \langle T^2 \rangle, \quad \Delta = -G(P). \]

At the minimum one has \( \langle T_z \rangle = \langle T_v \rangle = 0 \). For large values of \( \mu \) a product of neutron and proton BCS states is expected. Since this state is invariant under the transformation \( \exp(-i \pi (\Lambda_v/2 + T_z)) \), I enforce this symmetry, which entails \( \Delta = 0 \). I furthermore assume that both gaps \( \Delta_n = (-\Delta_x + i \Delta_y)/\sqrt{2} \) and \( \Delta_p = (\Delta_x + i \Delta_y)/\sqrt{2} \) are positive, as may always be achieved by a transformation of the form
\[ \exp(-i (\xi A_v + \chi T_z)). \]

When \( \lambda \) is placed midway between the lowest and the highest \( \epsilon_k \), one then gets \( \langle A_v \rangle = 2\Omega \) and \( \Delta_n = \Delta_p = \Delta \) for any value of \( \mu \) due to the equidistant single-nucleon spectrum. To speed up the calculation, I keep \( \Delta \) rather than \( G \) fixed with the variation of \( \mu \). Then \( G \) varies in the case considered by less than 0.6%.

With quasinucleon annihilation operators \( \alpha_i = \sum_{k\sigma\tau} (u_{i,k\sigma\tau} a^\dagger_{k\sigma\tau} + v_{i,k\sigma\tau} a_{k\sigma\tau}) \) defined by
\[ [\alpha_i, R_m] = E_i \alpha_i, \quad E_i > 0, \quad \{ \alpha_i, \alpha^\dagger_j \} = \delta_{ij}, \]
\[ R_m = H_0 + \Delta (P_n + P^\dagger_n + P_p + P^\dagger_p) + \kappa \langle T_z \rangle T_z - \lambda A_v - \mu T_z, \]
the RPA Routhian \( \tilde{R} \) is obtained by truncating to second order the boson expansion of \( R \) that results from making in the expressions for \( H_0, \quad P, \quad T \) and \( A_v \) the substitutions
\[ \alpha_j \alpha_i = b_{ij} + \cdots, \quad \alpha_i^\dagger \alpha_j^\dagger = \sum_l b^\dagger_{il} b_{jl}, \]
where the boson annihilation operators \( b_{ij} = -b_{ji} \) satisfy \( [b_{ij}, b_{km}] = 0 \) and \( [b_{ij}, b^\dagger_{km}] = \delta_{ij} \delta_{km} - \delta_{im} \delta_{kj} \).

The normal mode annihilation operators \( B_v = \sum_{l < j} (\phi_{v,ij} b_{ij} + \psi_{v,ij} b^\dagger_{ij}) \) are then given by
\[ [B_v, \tilde{R}] = \omega_v B_v, \quad \omega_v > 0, \quad \{ B_v, B^\dagger_v \} = \delta_{v^\prime v}, \]
and the ground state energy is [11] \( E = E_0 + E_2 \) with
\[ E_2 = \sum_{k\tau \tau'} (-G |\alpha^\dagger_{k\tau} \alpha_{k\tau'} T|)^2 + \frac{\kappa}{2} |\alpha^\dagger_{k\tau} \alpha_{k\tau'} T|^2 \]
\[ + \frac{1}{2} \left( \sum_v \omega_v - \sum_{i < j} [b_{ij}, [\tilde{R}, b^\dagger_{ij}]]) \right). \]

It may be noticed that when the terms with four quasinucleon annihilation operators or four quasinucleon creation operators are removed from the expression for the Hamiltonian \( H \) so that the Bogolyubov vacuum becomes an eigenstate of \( H \) (when the operators \( A_v \) and \( T_z \) are replaced by their eigenvalues in the relation \( H = R + \lambda A_v + \mu T_z \)), then only the first term in the expression for \( E_2 \) survives, and \( E_0 + E_2 \) becomes an exact expression for the ground state energy. Therefore this expression derived from the boson expansion is expected to provide a reliable approximation even
though most of the normal modes are not much different from two-quasinucleon excitations.

The implications of the symmetries of this model for the normal modes are discussed by Ginočcc and Wesener [12]. Two Goldstone modes result from the commutation relations \([A_v, R] = [I_z, R] = 0\). Furthermore, since \([T_+, R] = \mu T_+\) one normal mode has the frequency \(\mu\). Its annihilation operator is the linear boson part of \(T_+/\sqrt{2(T_+)}\), and it becomes a Goldstone mode in the limit \(\mu \to 0\). The degree of freedom of this mode is the direction of the isospin. In particular the isospin quantum numbers \(M_T = T = (T_z)\) may be assigned to the ground state of the Routhian \(R\). The Goldstone modes contribute with the frequency zero to the expression for the second order energy \(E_2\).

The parameters of the calculation are chosen so as to simulate the \(A = 48\) isobaric chain: \(\Omega = 24\), \(\eta = 2.1\) MeV, \(\Delta = 1.7\) MeV, \(\kappa = 1.2\) MeV. The result is shown in Fig. 1. \(E_0 - E_{0,T=0}\) depends essentially quadratically on \(T\). It is in fact given in a very good approximation by the expression \(E_0 - E_{0,T=0} = \frac{1}{2}(\eta + \kappa)T^2\) obtained for \(\Delta = 0\). The almost exact quadratic \(T\)-dependence of \(E_0 - E_{0,T=0}\) is seen also indirectly from the linearity of \(\mu = dE_0/dT + 2(\Delta/G)^2dG/dT\), where the second term is negligible. \(E_2 - E_{2,T=0}\) shows a different behaviour. It rises for \(T \approx 0\) linearly with \(T\) and is in fact in this limit equal to \(\mu/2\). The linearity thus stems from the single term in the expression for \(E_2\) which represents the zero-point energy of the normal mode with the frequency \(\mu\), or, in other words, from the quantal fluctuation of the isospin.

The rest of the second order energy \(E_2\) is fairly independent of \(T\). This suggests that the other normal modes are to a large extent independent of the isorotational degree of freedom, or, stated otherwise, that the iso-rotation is highly collective. The deformation underlying this collectivity is in the pair field [13]. Thus with \(\Delta_2 = 0\) is the isovector \(\Delta\) perpendicular to the iso-rotational axis. So it breaks the isorotational invariance with respect to this axis, and a collective iso-rotation can arise. On the other hand the contribution to \(E_2\) from the non-collective modes varies from \(T = 0\) to \(T = 4\) by almost 2 MeV, so the RPA correlation energy should be taken into account in a detailed comparison of the results of Hartree–Fock–Bogolyubov and Hartree–Fock calculations mentioned in the introduction.

With \(E_0 - E_{0,T=0} = \frac{1}{2}(\eta + \kappa)T^2\) and \(E_2 - E_{2,T=0} = \frac{1}{2}\mu = \frac{1}{2}dE_0/dT = \frac{1}{2}(\eta + \kappa)T\) we have altogether \(E - E_{T=0} = \frac{1}{2}(\eta + \kappa)T(T + 1)\), that is, we get the \(T\)-dependence of the symmetry energy found in the data. The Hartree–Fock–Bogolyubov energy expectation value includes only the first sum in the expression for \(E_2\) [11]. This sum is cancelled to a large extent by parts of the second term in the second sum. For the contributions from the symmetry potential energy \(\frac{3}{2}T^2\) this cancellation is in fact exact. For \(\Delta > 0\) the first sum in the expression for \(E_2\) is a function of \(T\) analytic and even at \(T = 0\). For \(\Delta = 0\) it equals, however, \(\frac{3}{2}T\). Thus it produces in the absence of pairing a Wigner term, albeit with only \(\kappa/(\eta + \kappa) \approx 35\%\) of the full value. Although the forces there are different, this may explain the experience with Hartree–Fock–Bogolyubov and Hartree–Fock calculations mentioned in the introduction.

A Wigner term corresponding to the term \(\frac{3}{2}T\) in the present model is actually the only one that may be derived from arguments like those in Ref. [2–4], which are based on the form of the residual two-nucleon interaction. It is remarkable that with a deformation one gets also a term \(\frac{3}{2}T\) corresponding to the “kinetic” part \(\frac{3}{2}T^2\) of the symmetry energy.

The symmetry potential energy \(\frac{3}{2}T^2\) of the present model differs from an interaction potential energy \(\kappa \sum_{i<j} t_i \cdot t_j\), where the index \(i\) or \(j\) labels the individual nucleons, by the term \(\frac{3}{2}\kappa A\), which depends
only on the number $A_v$ of valence nucleons. This interaction favours isoscalar nucleon pairs. It is included in the Hamiltonian in order to get a realistic symmetry energy coefficient. It is in fact well known and discussed in detail by for example Bohr and Mottelson [4] that the kinetic term accounts for only a part of the empirical symmetry energy coefficient. Bohr and Mottelson consider a contribution to the single-nucleon potential energy corresponding to a two-nucleon potential of the form $\kappa \mathbf{t}_1 \cdot \mathbf{t}_2$. Thus the form of the symmetry potential energy of the present model is adopted from their discussion. This symmetry force is inessential, however, for the formation of the Wigner energy, which results, as it was seen, from the collectivity of the iso-rotation due to the isobaric non-invariance of the isovector pair field.

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References

Anatomy of the lattice magnetic monopoles

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Abstract

We study the Abelian and non-Abelian action density near the monopole in the maximal Abelian gauge of SU(2) lattice gauge theory. We find that the non-Abelian action density near the monopoles belonging to the percolating cluster decreases when we approach the monopole center. Our estimate of the monopole radius is $R_{\text{mon}} \approx 0.04$ fm. © 2002 Elsevier Science B.V.

1. Introduction

Confinement of color in QCD implies that the color field of external quarks is squeezed into a tube connecting the quarks (provided that the distance between the quarks is large enough). Similarly, the ordinary magnetic field cannot penetrate superconductors and the dual superconductor model of confinement [1] makes this analogy manifest. The model assumes condensation of magnetic monopoles in QCD, similar to the condensation of charged Cooper pairs in superconductor. The monopole confinement mechanism is confirmed in SU(2) lattice gauge theory by many numerical calculations, for a recent review see, e.g., [2]. Microscopically, the condensation of the monopoles can be understood as percolation of a monopole cluster.

And, indeed, it was observed that there exists always a big percolating cluster which is responsible for confinement [3]. Since the percolating cluster may seemingly have any size we will call it infrared (IR). On the other hand, there are also many small, or ultraviolet (UV) clusters which are usually viewed as lattice artifacts [3].

Although there is a lot of data on the lattice monopole the understanding of the monopole dynamics in terms of the continuum theory is far from being complete at the moment. Qualitatively, there are two ways of looking at the monopoles in non-Abelian theories. First, one can think in terms of an analogy with the ‘t Hooft–Polyakov monopoles [4] which are classical solutions to the Yang–Mills equations with a triplet of matter fields. However, there are no matter fields in QCD. As a result, one rather changes the strategy of...
defining the monopoles [5]. Namely, they can be defined as purely topological defects, with no direct relation to the density of the non-Abelian action. According to the original idea of Ref. [5] in case of SU(2) gauge group one can choose any vector in color space and (partially) fix the gauge by rotating the vector to the third direction. Such gauge fixing fails when all the components of the vector vanish at some point. The crucial observation is that vanishing of a vector gives three conditions which in the \( D = 4 \) case define line-like defects, that is the monopole trajectories. The success of the monopole confinement model depends in fact on the particular choice of the gauge. The observation might imply that a purely topological definition, devoid of any dynamic content is not in fact adequate. The so-called maximal Abelian gauge and the corresponding projection [2] turns out to be the most carefully studied and very successful. Since the Abelian projection emphasizes the role of the Abelian-like field configurations this might be an indication that at large distances the lattice monopoles are similar to the Abelian or Dirac monopoles.

To get insight into the dynamics of the lattice monopoles we will concentrate here on measuring the full non-Abelian and Abelian actions at the centers of the monopoles. Actually, this kind of measurements have been reported earlier [6]. Namely, it was shown that the non-Abelian action on the plaquettes close to the monopole trajectories. It is easy to realize, however, that if this were true at arbitrary small distances the monopoles would be strongly suppressed by the action factor and could not condense, see, e.g., [7]. In this Letter we report on the measurements which demonstrate for the first time that the above mentioned excess of the action goes down for smaller lattice spacing, or larger \( \beta \). A crucial novel point is that we distinguish between the monopoles belonging to the UV and IR clusters and the statement on the decreasing of the action refers to the IR monopoles only. In this sense, the structure of the IR and UV monopoles turns out to be different and one can say that the monopoles in the IR clusters are condensed due to their special anatomy.

The separation of the monopole ensemble on IR and UV clusters is unambiguous for large enough lattices. The distribution of the cluster lengths clearly shows [3] that each monopole configuration contains typically one large IR cluster and a lot of small UV clusters separated by clearly observed gap. Thus we do not need to introduce any artificial mass scale to distinguish between IR and UV clusters.

The most important question to be addressed here is the estimation of the size of the monopoles. Our definition of the monopole size will be described in Section 3. As we shall see, the size of the monopole turns out to be rather small numerically. This observation supports speculations on the existence of a numerically large mass scale in the non-perturbative physics, see, e.g., [8,9] and references therein. On the other hand, our results show that the size of the Abelian monopoles is much smaller than the distance between the monopoles. Thus the monopole cores are not overlapping and the system can be tractable as a dilute gas.

In the next section we will summarize the current views on the anatomy of the monopoles. In Section 3 we present our data and discuss their implications. It occurs that due to the finite size the monopoles in gluodynamics are condensed at any value of the bare coupling. In compact QED, on the other hand, where monopoles are point-like, the critical coupling, separating confinement and deconfinement phases, exists.

### 2. Monopoles on the lattice and in the continuum

Let us first remind the reader the backbone of the theory of the monopole condensation in the compact photodynamics [10]. In this case the monopoles are classical solutions, the same as the original monopoles of Dirac [11]. The radial magnetic field of the monopole is similar to the electric field of a point-like charge, \( |\mathbf{H}| \sim 1/e^2 r^2 \), where \( e \) is the electric charge and the factor \( 1/e^2 \) appears because of the Dirac quantization condition. The corresponding energy is ultraviolet-divergent:

\[
\epsilon_{\text{mon}} \sim \int d^3 x \mathbf{H}^2 \sim \frac{1}{e^2 a}, \tag{1}
\]

where \( \mathbf{H} \) is the magnetic field, \( a \) is the lattice spacing which provides an ultraviolet cut off. Note that the Dirac string does not contribute to the energy (1) because of the compactness of the \( U(1) \). Otherwise it would result in a quadratically divergent term (for further details and references see [12]). Eq. (1)
implies that the probability to find a monopole trajectory of length $L$ is suppressed by the action as $\exp[-\text{const} \cdot L/(e^2 \cdot a)]$. This suppression can be overcome, however, for $e^2 \sim 1$ by the entropy factor. Indeed, on the hypercubic lattice the number $N$ of trajectories of the length $L$ grows exponentially, $N \sim \exp(\ln 7 \cdot L/a)$, where the constant $\ln 7$ is of pure geometrical origin. Since the self-energy (1) can be found with all the coefficients fixed the equating of the entropy and action factors provides a quantitative means to find the value of $e^2_{\text{crit}}$. A detailed quantitative analysis along these lines as well as further references can be found in [13].

In case of the gluodynamics, we choose the maximal Abelian gauge which is defined through maximization of the functional $R[U] = \sum_i \text{Tr}[\sigma_i U_i^\dagger \sigma_i U_i]$ over all gauge transformations $U_i^{\dagger} = \Omega_i^* U_i \Omega_i$. Moreover, in the standard parameterization of the link matrix

$$U_i = \begin{pmatrix} \cos \varphi_i e^{i \theta_i} & \sin \varphi_i e^{-i \theta_i} \\ -\sin \varphi_i e^{i \theta_i} & \cos \varphi_i e^{i \theta_i} \end{pmatrix},$$

(2)

$\theta, \chi \in [-\pi, +\pi)$, $\varphi \in [0, \pi)$, the functional $R$ can be rewritten as: $R[U] = \sum_l \text{cos} \varphi_l$.

Thus, the maximization of $R$ corresponds to the maximization of the absolute values of the diagonal elements of the link matrix (2). Since the $SU(2)$ plaquette action is $\beta \sum_i \text{Tr} U_i$, at large values of $\beta$ the link matrices are close to unit matrix up to gauge transformations. Thus, at large values of $\beta$ in the maximal Abelian gauge $\cos \varphi_l$ are close to unity, the angles $\varphi_l$ are small and the $SU(2)$ plaquette action has the form:

$$\beta S = \beta \left[ \cos \theta_P \prod_{i=1}^{4} \cos \varphi_i + O(\sin \varphi_i) \right].$$

(3)

The projected action $S_{\text{Abel}}$ is defined by putting $\varphi_i = 0$. In the form (3) the projected action closely resembles the action of the compact electrodynamics:

$$\beta S_{U(1)} S_{\text{QED}} = \beta S_{U(1)} \cos \theta_P.$$

(4)

However, if we would try to transfer the picture with Abelian monopoles directly onto the non-Abelian case, the conclusion would be that there is no monopole condensation in gluodynamics. Indeed, because of the asymptotic freedom $g^2(a) \rightarrow 0$ if $a \rightarrow 0$. Thus, one substantiates the dual superconductor model of the confinement with dynamical considerations like the following. Let us start increasing the lattice spacing $\alpha \rightarrow 0$. Then the corresponding effective coupling $g^2$ grows according to the renormalization equations. The same coupling governs any of the $U(1)$ subgroups and once $g^2$ reaches the value where the $U(1)$ monopoles condense (see above) the condensation occurs in the non-Abelian theory as well. In this way one readily understands that there exists one monopole per volume of order $(\Lambda_{\text{QCD}})^{-3}$ so that the monopoles survive in the continuum limit. However, since the running of the coupling is a pure quantum effect there is no much hope to explicitly match a quasi-classical, Abelian-like field configuration at large distances with perturbative-vacuum fluctuations at short distances.

Instead, we can think in terms of a phenomenological expansion of the action density inside the monopoles. Since the action density is measured in lattice units it is convenient to consider an expansion of the form:

$$\left( |F_{\mu \nu}^a|^2 \right)_{\text{monopole center}} - \left( |F_{\mu \nu}^a|^2 \right)_{\text{average}}$$

$$= \frac{1}{g^2(a)^4} \left( \sum_{k=0}^{\infty} c_k g^{2k}(a^2) + \sum_{n=1}^{\infty} b_n a^n \right).$$

(5)

Then the theoretical expectation is that all ultraviolet-divergent pieces vanish, $c_k = 0$. Indeed, otherwise we would have point-like objects beyond the ordinary gluons, in direct contradiction with the asymptotic freedom. Thus, only terms of order $\exp(-\text{const}/g^2(a))$ or powers of $a$ are allowed in the r.h.s. of Eq. (5). Moreover, we expect that the series actually starts with the $a^4$ term. Indeed, the monopole field is of order $(F_{\mu \nu}^a)_{\text{mon}} \sim A_{\text{QCD}}^2$ as discussed above. The perturbative fields, on the other hand, are of order $a^{-2}$. However, there is no reason to expect any interference between the perturbative and monopole contributions, at least upon the averaging. Thus the excess of the action near the monopoles is to vanish proportional to $a^4$ if measured in the lattice units.

The prediction of the $a^4$ behavior holds in the academic limit $a \rightarrow 0$. It is a different matter of course how close to this limit the existing lattices are. In the next section we will present first indications that the
excess, as measured in the lattice units, decreases with the decreasing lattice size. However, it is too early to claim that the excess is vanishing fast at $a \to 0$. In this sense the measurements presented in this Letter can be considered as a first step in studying the monopole anatomy.

3. Numerical results

We have performed measurements of the full non-Abelian action, $S_{\text{mon}}^{SU(2)}$, on the plaquettes closest to the monopole trajectory. The simulations have been done using Wilson action on lattices $12^4$ for $\beta = 2.27, 2.3, 2.33, 2.35, 2.38, 2.4, 16^4$ for $\beta = 2.45, 20^4$ for $\beta = 2.5, 2.55$ and $28^4$ for $\beta = 2.6$. We thus kept our physical volume $\gtrsim 1.5$ fm. We made 20 measurements on $12^4$ and $20^4$ lattices, 15 measurements on $16^4$ lattice and 20 measurements on $28^4$ lattice. The full non-Abelian action, $S_{\text{mon}}^{SU(2)}$, on the plaquettes closest to the monopole trajectory have been measured. The error analysis has been carried out with bootstrap and jackknife methods. Both methods gave consistent estimates for statistical errors.

To fix MA gauge the simulated annealing (SA) algorithm was employed. It is known that this algorithm is vital for reducing the uncertainty due to Gribov copy effects in the gauge non-invariant observables computed in MA gauge [14]. Our SA algorithm implementation is essentially the same as described in [14] with the exception that we increased the total number of SA sweeps up to 2000. To further reduce bias due to Gribov copies we made gauge fixing for 5 randomly generated gauge copies for every Monte Carlo configuration. Only the copy with the maximal value of the gauge fixing functional $R[U]$ has been used to compute our observables. To estimate the residual effect of the Gribov copies we compared results obtained with different numbers of gauge copies $N_{\text{cop}}$ in the range from 1 to 5. We have found systematic albeit weak dependence of our observables on the number of gauge copies. The difference between results with $N_{\text{cop}} = 1$ and $N_{\text{cop}} = 5$ was less than our statistical errors. At the same time the difference from results obtained with the iterative gauge fixing algorithm was an order of magnitude larger than statistical errors.

While measuring $S_{\text{mon}}$ on the plaquettes closest to the monopole trajectory, we discriminate between

![Fig. 1. The dependence of excess of the non-Abelian action, $\overline{S}$, on the distance to the monopole, $a/2$, for all monopoles (circles), monopoles from IR clusters (boxes) and monopoles from UV clusters (triangles). The dashed line is ln7. The error bars are within the symbols for most of points.](image)

the monopoles belonging to the IR and UV clusters. Our lattices are of the physical size $\gtrsim 1.5$ fm, i.e., large enough for most of observables. On the other hand it has been found out in [3] that essentially larger volume is necessary to have only one large IR cluster. Some of our lattices are not large enough and we find on them not one but a few large clusters. We believe that combining all these clusters one gets the set of infrared monopoles. This conjecture has been confirmed recently [15]. In the present work we considered only monopoles from the largest cluster as IR monopoles. This introduces some systematic uncertainty into results for UV clusters, namely the corresponding non-Abelian action is underestimated. At the same time this uncertainty does not influence our main conclusions.

In Fig. 1 we show the dependence of $\overline{S} = 6\beta(S_{\text{mon}} - \overline{S})$ on the half of the lattice spacing $a/2$. The factor $6\beta$ is introduced here to make convenient the comparison of the action and the entropy factors. The explanation of the scale of the horizontal axis, i.e., $a/2$, is the following. Since $\langle S_{\text{mon}} \rangle$ is measured on the plaquettes which are faces of the cube dual to the monopole cur-

\[ \text{To define the lattice distance in Fermi, we find the correspondence between the bare charge and lattice spacing by fixing the value of the string tension } \sigma = 440 \text{ MeV and using the numerical data for the string tension in lattice units, } \sigma \cdot a^2, \text{ see [16].} \]
rent, this corresponds to measuring the average field strength at the distance $a/2$ from the monopole center. Note that the excess of the action is dominated by the closest plaquettes. In this figure we compare the action on the plaquettes nearest to the monopole center with the $\ln 7$. As is mentioned in Section 2, the $\ln 7$ is a geometrical constant determining the monopole entropy. The action in the lattice units for the percolating monopoles should not exceed $\ln 7$, see, e.g., [7,13] and references therein.

Our main observation is that $\overline{S}$ for the monopole belonging to the IR cluster decreases when we approach the monopole center. Moreover, it is below $\ln 7$ for all data in agreement with percolation condition discussed above. On the other hand, the action for the UV monopoles is increasing and exceeds $\ln 7$ in agreement with the fact that these clusters are not percolating. Thus for the first time we demonstrate by direct computation of the action density that the percolation condition works in $SU(2)$ gluodynamics: percolating monopoles carry action density (in lattice units) less than $\ln 7$, while for non-percolating monopoles the action density is above this value. Note, that the excess of the action near all monopoles behaves similarly to the leading term in the monopole action [17]. This term is proportional to the length of the monopole trajectory.

The action density distribution on Fig. 1 has the same physical meaning as the action profile of the ’t Hooft–Polyakov monopole classical solution [4]. However, the monopoles studied in the present Letter are of a purely quantum origin.

The results of the calculation of the Abelian action near the monopole, $\overline{S}_{\text{Abel}} = 6\beta (\overline{S}_{\text{mon}}^{\text{Abel}} - \overline{S}_{\text{Abel}})$, are presented in Fig. 2. Unlike the full non-Abelian action, the Abelian action associated with the monopoles, $\overline{S}_{\text{Abel}}$, for monopoles belonging to IR and UV clusters is approximately the same. Moreover, it increases when one approaches the center of monopole. There is no known explanation of this effect. A comparison of Figs. 1 and 2 shows that the role of the off-diagonal degrees of freedom seem to compensate the divergent contribution into the monopole energy from the Abelian part of the gluon fields. Thus the monopoles in the maximal Abelian projection look like ’t Hooft–Polyakov monopoles which are not singular at the origin. Note that the monopoles in compact QED are singular Dirac monopoles.

Now we discuss the size of the IR monopole. It occurs that the fit of the data in Fig. 1 by the function $C_0 + C_1 \exp\left(-R^2/(R_{\text{mon}}^2)\right)$, $R = a/2$, can be performed with high quality, $\chi^2/N_{\text{DOF}} = 0.26$. The values of fit parameters are: $C_0 = 1.706(5)$, $C_1 = -0.63(2)$, $R_{\text{mon}} = 0.041(1)$ fm. This fit is shown in Fig. 1 by solid line. Thus our estimation for the monopole radius is $R_{\text{mon}} \approx 0.04$ fm. Of course, the definition of the monopole radius is not unique, but we believe that all reasonable definitions give the monopole radius of the same order. Note that in Ref. [7] it was found that the monopole condensation starts for monopoles approximately of the same physical size as $R_{\text{mon}}$ determined in the present Letter. Up to now we did not study the scaling behaviour of the monopole radius. Such calculations are possible if we surround monopoles by cubes of various size and we measure the action on the faces of these cubes. This study is now in progress.

Our data give support to the following picture. At the large enough distances from the monopoles gauge field is Abelian-like and approximation Eq. (3) works well. At short distances the non-Abelian nature of the monopoles is manifest and while the Abelian part of the action grows up the total action decreases.

To summarize, we have shown that the phenomenon of the monopole condensation in the lattice gluodynamics is due to a special anatomy of the monopoles belonging to the IR cluster. On the other hand, in the limit $a \rightarrow 0$ one would expect much faster vanishing of the excess of the action than it was observed so
far. Since the theoretical prediction on the vanishing of the excess of the action at small \(a\) seems very reliable (see the preceding section) the results obtained are to be rather interpreted in terms of various scales of the non-perturbative physics. Indeed, the average distance between nearest monopoles in the IR cluster is about 0.5 fm, as can be extracted from the data in Refs. [3, 15]. Now we observe for the first time that the excess of the action goes down when we approach the monopole center. The corresponding radius turns out to be small numerically, \(R_{\text{mon}} \approx 0.04\) fm. Moreover, even a smaller scale might emerge in future. Indeed, in the limit \(a \to 0\) we should have the \(a^4\) behavior for the excess of the action which is not yet in sight at present. Thus, there appears a hierarchy of scales all of which are formally of the same order, \(\sim \Lambda_{\text{QCD}}\). Note that existence of such hierarchies has already been conjectured on various grounds. First, a great variety of scales is manifested through QCD sum rules [8]. Further evidence has been accumulated via various lattice measurements, see [9] and references therein. In particular, very recently the relevance of the scale of order 2 GeV was revealed through the measurements of the \(\langle A^2 \rangle\) vacuum condensate. This scale would roughly correspond to the monopole radius \(R_{\text{mon}} \approx 0.04\) fm which we are observing.

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Note that this implies that monopoles form a gas rather than a liquid since the monopole cores are most probably not interacting due to the separation of scales.

References

Massless flows between minimal $W$ models

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Abstract

We study the renormalisation group flows between minimal $W$ models by means of a new set of nonlinear integral equations which provide access to the effective central charge of both unitary and nonunitary models. We show that the scaling function associated to the nonunitary models is a nonmonotonic function of the system size. © 2002 Elsevier Science B.V. All rights reserved.

1. A recent study of the renormalisation group flows between nonunitary minimal models revealed an unexpected behaviour for the groundstate energy $E_0(R)$, in that it was a nonmonotonic function of the system size $R$ [1]. The nonmonotonicity was illustrated using the finite-size scaling function $c_{\text{eff}}(r)$, which up to the bulk term is proportional to the groundstate energy

$$E_0(R) = E_{\text{bulk}}(M, R) - \frac{\pi c_{\text{eff}}(r)}{6R}, \quad r = MR,$$

where $M$ is the so-called crossover scale (the mass in massive theories). As the system size goes to zero $c_{\text{eff}}(r)$ becomes the effective central charge

$$\lim_{r \to 0} c_{\text{eff}}(r) = c - 24\Delta_0.$$  

We denote the actual central charge by $c$ while $\Delta_0$ is the conformal dimension of the lowest primary field of the UV CFT.

The effective central charge and the central charge of the unitary minimal models coincide, and according to Zamolodchikov’s $c$-theorem [2] there exists a function $\tilde{c}$ which is monotonic. However, apart from the UV and IR points at which $\tilde{c}$ equals the central charge of the relevant CFT, it is not clear if there is any connection with $E_0(r)$. Nevertheless the groundstate energy of the unitary models is always monotonic. Analogously it had been thought that the groundstate energy of the nonunitary models would also be monotonic, but the results of [1] and [3–6] provide a number of counter examples. In this Letter we study a further set of perturbed conformal field theories, demonstrating that $c_{\text{eff}}(r)$ behaves nonmonotonically for the majority of nonunitary models.

We consider the minimal models $W_{N}^{p,q}$ based on one of the simply laced Lie algebras $G = A_{n-1}, D_n, E_6, E_7, E_8$ [7]. The models are specified by two coprime integers $p$ and $q$ with $p > h$, in terms of which the central charge and the effective central charge are

$$c = N \left( 1 - \frac{h(h+1)(p-q)^2}{pq} \right),$$

$$c_{\text{eff}} = N \left( 1 - \frac{h(h+1)}{pq} \right).$$

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Here $N$ denotes the rank of the algebra and $h$ the dual Coxeter number. The primary fields $\Phi_{\Omega,\Omega'}$ are labelled by a pair of weights $\Omega,\Omega'$ which satisfy
\[
\theta \cdot \Omega \leq q, \quad \theta \cdot \Omega' \leq p,
\]
where $\theta$ is the highest root of $G$ [8]. All models have a primary field $\Phi_{\text{adj}}$ that has weights $\Omega' = \Omega_{\text{id}}$ and $\Omega = \Omega_{\text{adj}}$ corresponding to the trivial and adjoint representations of $G$, respectively. It has conformal dimensions
\[
\Delta = \tilde{\Delta} = \frac{(q-p)h}{q},
\]
and is relevant for all $p, q$ such that $q > p$. Formally denoting the action of the unperturbed CFT $\mathcal{A}_{\text{CFT}}$, that of the perturbed model may be written
\[
\mathcal{A} = \mathcal{A}_{\text{CFT}} + \lambda \int d^2x \Phi_{\text{adj}},
\]
reproducing for $A_1$ the well-known $\phi_{13}$ perturbations of the Virasoro minimal models.

Depending on the sign of the coupling constant $\lambda$, the perturbation either leads to a massive quantum field theory, or it induces a ‘massless flow’ into a conformal field theory with smaller effective central charge. One of the standard methods of studying the groundstate energy of both types of model is the thermodynamic Bethe ansatz. The result is a set of coupled nonlinear integral equations (the TBA equations), whose solution provides direct access to $c_{\text{eff}}(r)$ at all values of $r$.

The unitary $W_{\theta}^{p,p+1}$ can alternatively be described as $G_N^p \times G_N^{k+1}$ coset models at $k = p - h$, in terms of which the perturbing operator $\Phi_{\text{adj}}$ is usually known as $\Phi_{\text{id,id,adj}}$. TBA equations describing the evolution of the effective central charge between these coset models, and therefore the unitary $W_G$ models, are already known [9–11], and they verify the conjectured pattern of flows [8,12]:
\[
W_{\theta}^{p,p+1} + \Phi_{\text{adj}} \rightarrow W_{\theta}^{p,p-1}. \tag{7}
\]
By dropping the restriction $q = p + 1$, we may also consider flows originating from the much larger class of nonunitary $W_G$ minimal models, which also have a description as a coset but at fractional level $G_N^{p/(q-p)h} \times G_N^{1/q} / G_N^{1/(q-p)h+1}$. Analogous to the known behaviour of the $\phi_{13}$ perturbations of the Virasoro models [13–16], it is natural to suppose that the nonunitary generalisation of (7) will be
\[
W_{\theta}^{p,q} + \Phi_{\text{adj}} \rightarrow W_{\theta}^{2p-q,p}. \tag{8}
\]
TBA equations describing massless flows from these nonunitary models are not yet known. Instead, motivated by [1,6], we propose a different type of nonlinear integral equation (NLIE) whose solution provides access to the effective central charge of both unitary and nonunitary minimal models. The equations can be found in Section 2, and are tested in Section 3. In Sections 4 and 5 we extract some exact results and make a comparison with ultraviolet and infrared perturbation theory. The connection between the massive models and the Gross–Neveu models and other comments can be found in Section 6.

2. Our starting point is a set of nonlinear integral equations that encode the groundstate energy of the imaginary-coupled simply laced affine Toda field theories. Our interest in these theories lies in the fact that for values of the Toda coupling constant $\beta^2 = p/(p+1)$ the theory can be consistently restricted to the massive $\Phi_{\text{adj}}$ perturbation of the unitary $W_{\phi}^{p,p+1}$ minimal models [17–22]. Moreover, the massive $\phi_{13}$ perturbation of the nonunitary $W_A^{p,q}$ minimal models can be obtained from the sine-Gordon model by tuning the coupling to $\beta^2 = p/q$. A similar result may be true for the nonunitary minimal models based on the other simply laced Lie algebras [23], and at the level of the NLIES we do find the choice $\beta^2 = p/q$ yields both unitary and nonunitary perturbed models.

The NLIES describing the groundstate energy of the massive imaginary-coupled Toda field theories were first obtained in [24,25], and have appeared in a different context in [26]. The effective central charge is defined in terms of $N$ functions which satisfy a set of coupled equations
\[
f^{(a)}(\theta) = -\frac{i}{2} m_a r e^{i\theta} + i n \sum_{b=1}^{C} c_{ab}^{-1} a_b
\]
\[
+ 2i \sum_{b=1}^{N} \left[ \int d\theta' \varphi_{ab}(\theta - \theta') \right] \times 3m \ln \left( 1 + e^{f^{(b)}(\theta')} \right). \tag{9}
\]
The integration contour $\mathcal{C}$ runs just below the real axis while $r$ is built from the lightest mass $M_a$ of the theory and the cylinder size $R$ via $r = M R$. We have set $m_a = M_a / M$, where each mass $M_a$ is associated to a node of the Dynkin diagram via the labelling of [27], and is such that $(M_1, M_2, \ldots, M_N)$ forms an eigenvector of the Cartan matrix with eigenvalue $4 \sin^2 (\pi / 2h)$. Our particular normalisation is given in Table 1. The kernel functions

$$
\varphi_{ab}(\theta) = \int_0^\infty \frac{dk}{2\pi} \frac{e^{i k \theta}}{k} \left( \sum_{a,b} \frac{\sinh(\frac{\pi}{2} k \xi)}{\sinh(\frac{\pi}{2} k) \cosh(\frac{\pi}{2} k)} C^{-1}_{ab}(k) \right)
$$

are written in terms of the ‘deformed’ Cartan matrix $C_{ab}(k)$, which is equal to 2 if $a = b$, and $-1 / \cosh(\pi k / h)$ if nodes $a$ and $b$ of the relevant Dynkin diagram are connected. Note that at $k = 0$ it reduces to the standard Cartan matrix $C_{ab}$. The exact effective central charge may be determined using

$$
c_{\text{eff}}(r) = \frac{6 r}{\pi^2} \sum_{a=1}^N m_a
\times \left[ \int \frac{d\theta}{\mathcal{C}} \sinh(\theta) \Im \ln(1 + e^{f_{ab}(\theta)}) \right].
$$

Due to the nonlinear nature of (9) and (11), $c_{\text{eff}}(r)$ is usually obtained by solving the equations numerically. However, like the TBA equations, the NLIEs can be exactly evaluated at the ultraviolet point [28], the result for the above equations being [25]

$$
c_{\text{eff}}(0) = N - \frac{3 \xi}{\xi + 1} \sum_{a,b=1}^N C^{-1}_{ab} \alpha_a \alpha_b.
$$

The effective central charge of the massive $\Phi_{\text{adj}}$ perturbation of $W G^{p,q}_N$ is obtained by setting the parameter $\xi$ and the twists $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ to

$$
\xi = p / (q - p), \quad \alpha = (2/p, 2/p, \ldots, 2/p).
$$

The affine Toda coupling constant $\beta^2$ is related to $\xi$ via $\beta^2 = \xi / (\xi + 1)$, and the above ensures $\beta^2 = p / q$. The choice of $\alpha$ is motivated by [29,30] for the $A_1$ related models, and the prescription given in [25] which yields the central charge of the $W G$ minimal models rather than $c_{\text{eff}}(r)$. As a first check we insert (13) into (12), simplify using $12 \sum_{a,b=1}^N C^{-1}_{ab} = Nh(h + 1)$, and recover the expected UV effective central charge (3).

In accordance with (8) we modify the massive equations to interpolate from a model with $\xi = p / (q - p)$ to one with $\xi' = (2p - q) / (p - q)$, that is $\xi' = \xi - 1$. Motivated by [1,6], we associate two functions $f^{(a)}_R$ and $f^{(a)}_L$ to each node of the Dynkin diagram, introduce new kernels and twists such that the functions satisfy

$$
f^{(m)}_R(\theta) = -i \frac{e^\theta}{2 m_a r} + i \int \frac{d\theta}{\mathcal{C}} \sinh(\theta) \Im \ln(1 + e^f_{ab}(\theta))
\times \left[ \int d\theta' \varphi_{ab}(\theta - \theta') \times \Im \ln(1 + e^{f^l_{ab}(\theta')} -f^l_{ab}(\theta')) \right].
$$

0:00
\[ f_L^{(m)}(\theta) = -\frac{i}{2} m_a r e^{-\theta} - i\pi \sum_{b=1}^{N} C^{-1}_{ab} \alpha'_b \]

\[ -2i \sum_{b=1}^{N} \left[ \int d\theta' \phi_{ab}(\theta - \theta') \right. \]
\[ \times \Im \ln \left( 1 + e^{-f_L(\theta')} \right) \]
\[ \left. + \int d\theta' \chi_{ab}(\theta - \theta') \right] \]
\[ \times \Im \ln \left( 1 + e^{-f_L(\theta')} \right) \]

(15)

and replace the formula for the effective central charge with

\[ c_{\text{eff}}(r) = -\frac{6r}{\pi} \sum_{a=1}^{N} m_a \left[ \int d\theta e^{\theta} \Im \ln \left( 1 + e^{-f_{\text{eff}}(\theta)} \right) \right] \]

\[ -\int d\theta e^{-\theta} \Im \ln \left( 1 + e^{-f_{\text{eff}}(\theta)} \right) \]

(16)

As explained in [1], we fix the kernel functions by considering the equations in the limits in which \( r \rightarrow 0 \) and \( r \rightarrow \infty \). In the far infrared, the massless equations coincide with the ultraviolet limit of the massive equations, with kernel \( \phi_{ab}(\theta) \) replaced by \( \phi_{ab}(\theta) \). Since this should describe a model with parameter \( \xi - 1 \) we set \( \phi_{ab}(\theta) \) to

\[ \phi_{ab}(\theta) \]
\[ = \int \frac{dk}{2\pi} e^{ik\theta} \]
\[ -\infty \]
\[ \times \left( \delta_{mt} - \frac{\sinh(\frac{\pi k}{2})}{\sinh(\frac{\pi}{2}(\xi - 1)k)} \cosh(\frac{\pi}{2}k) C^{-1}_{mt}(k) \right) \]

(17)

For very small \( r \), the massless equations should instead coincide with the ultraviolet limit of the massive equations with parameter \( \xi \). After some manipulations [1], the fourier transformed massive and massless equations can be directly compared. With the tilde denoting the fourier transformed functions, the equations will match provided \( \tilde{\chi}_{ab}(k) \) and \( \alpha'_a \) satisfy

\[ \tilde{\phi}_{ab}(k) = \tilde{\phi}_{ab}(k) \]
\[ + \sum_{c,d=1}^{N} \tilde{\chi}_{ac}(k) \left[ 1 - \tilde{\phi}(0) \right]^{-1}_{cd} \tilde{\chi}_{db}(k) \]

(18)

\[ \alpha_a = \alpha'_a + \sum_{c,d=1}^{N} \tilde{\chi}_{ac}(0) \left[ 1 - \tilde{\phi}(0) \right]^{-1}_{cd} \tilde{\chi}_{db}(0) \]

(19)

Inserting the expressions for \( \tilde{\phi}_{ab} \) and \( \tilde{\phi}_{ab} \) into (18) we find

\[ \left( \frac{\sinh(\frac{\pi k}{2})}{\sinh(\frac{\pi}{2}(\xi - 1)k)} \cosh(\frac{\pi}{2}k) \right) \]
\[ C^{-1}_{ab}(k) \]
\[ = \sum_{c,d=1}^{N} \tilde{\chi}_{ac}(k) C_{cd}(k) \tilde{\chi}_{db}(k) \]

(20)

After multiplying both sides by \( C_{fa}(k) \) and summing over the index \( a \), we can write the right-hand side as a square: \( (\tilde{\chi}(k) C(k))^2_{tb} \). Taking the square root and inverting the fourier transform yields

\[ \chi_{ab}(\theta) = \pm \int \frac{dk}{2\pi} e^{ik\theta} \]
\[ \frac{\sinh(\frac{\pi}{2}k)}{\sinh(\frac{\pi}{2}(\xi - 1)k)} \cosh(\frac{\pi}{2}k) \]
\[ \times C^{-1}_{ab}(k) \]

(21)

The above only fixes \( \chi_{ab}(\theta) \) up to a sign, but we find that choosing the negative sign results in an effective central charge consistent with a \( W_G \) minimal model, whereas the other choice does not yield a recognisable formula for \( c_{\text{eff}}(0) \). With the negative sign we find the new twists should be

\[ \alpha'_a = -\frac{N}{\xi - 1} \alpha_a \]

(22)

To avoid the pole in (22) at \( \xi = 1 \), and the poles in the kernels which cross the real axis as \( \xi \) falls below one, we only consider models with \( 2p > q \) and therefore \( \xi > 1 \). This is a sensible restriction since a flow of the form (8) with \( 2p < q \) would have an infrared CFT labelled by \((2p - q, p)\), the first of which is negative.

3. The massless NLIEs have ultraviolet and infrared values of \( c_{\text{eff}}(r) \) which exactly match those of the conjectured flow (8) provided we continue to use the massive prescription for \( \xi \) and \( \alpha \) (13). We find

\[ c_{\text{eff}}(0) = N \left( 1 - \frac{h(h+1)}{pq} \right) \]

\[ c_{\text{eff}}(\infty) = N \left( 1 - \frac{h(h+1)}{(2p - q)p} \right) \]

(23)

The massless flows naturally fall into families indexed by an integer \( J = q - p \):

\[ W_{G}^{p, p+J} + \Phi_{\text{adj}} \rightarrow W_{G}^{p, p-J} \]

(24)
At \( J = 1 \) there is a unique family corresponding to the flows between the unitary minimal models. For these models we tested the massless NLIEs against the TBA equations [10,11], typically finding very good agreement. As expected the effective central charge was consistently monotonic. For \( J > 1 \) there are \( \varphi(J) \) different families, each of which interpolates between nonunitary models (here \( \varphi \) denotes the Euler-

\[ e^{-J} \]

\[ \text{(25)} \]

\[ \text{(26)} \]

\[ \varphi \]

Solving the NLIEs for the nonunitary models, we found that \( c_{\text{eff}}(r) \) increases away from its UV value, undergoes a number of oscillations and then settles down to the predicted IR value. The nonmonotonic behaviour of two families of flows is illustrated in Figs. 1 and 2.

4. To strengthen the validity of the conjectured equations we extract a number of further predictions, which are then compared with results from ultraviolet and infrared conformal perturbation theory.

The UV groundstate energy of a CFT perturbed by a primary field \( \Phi \) of conformal dimension \( \Delta_{UV} \) is predicted to behave as [31,32]

\[ E_0(R) = R B(\lambda) - \frac{\pi c_{\text{pert}}(r)}{6 R} \]

\[ c_{\text{pert}}(r) = c_{\text{eff}}(0) + \sum_{j=1}^{\infty} C_j (\lambda R^y)^j, \]

where \( y = 2(1 - \Delta_{UV}) \) and the coefficients \( C_j \) are proportional to the connected correlation functions of the perturbing field on the plane. The action (6) implies \( \lambda \) and \( M \) must satisfy

\[ \lambda = \kappa M^y \]

for a dimensionless constant \( \kappa \), further implying that \( c_{\text{pert}} \) expands in powers of \( r^y \). On the other hand, the \( 2(\xi + 1)i\pi / h \) periodicity\(^1\) of the nonlinear integral equations suggests that \( c_{\text{eff}}(r) \) expands as a series in \( r^{2h/(1+\xi)} \), which will agree with \( c_{\text{pert}}(r) \) provided \( 2(1 - \Delta_{UV}) = 2h/(1 + \xi) \). Substituting \( p/(q - p) \) for \( \xi \), we find the NLIEs predict a value of \( \Delta_{UV} \) which exactly matches the conformal dimension of \( \Phi_{\text{adj}} \) (5).

The expansion

\[ c_{\text{eff}}(r) = c_{\text{eff}}(0) + B(r) + \sum_{j=1}^{\infty} C_j (r^y)^j, \]

differs from \( c_{\text{pert}}(r) \) by the bulk term \( B(r) \), but it may easily be extracted from the NLIEs. For this we need the leading asymptotics of the kernels as \( \theta \to -\infty \). The denominator of the inverse deformed Cartan

\(^1\) While the periodicity is easily extracted from the associated Bethe ansatz equations [25,26], it is less trivial to see directly from the NLIEs.
The bulk term corresponding to the massive perturbation of the unitary minimal models has been calculated by Fateev [33] in the context of the associated coset. By analytically continuing the coset parameter $k$ to rational values (set $k = \xi - h$), we find Fateev’s massive bulk term exactly coincides with ours for both unitary and nonunitary models.

The massless bulk terms are new, but we can make at least one concrete check before turning to numerics: the models with $p = h + 1$ and $q = h + 2$ correspond to the coset $G_N^1 \times G_N^2$, for which the massive and massless perturbations are known to coincide [9,34], as fortunately do our bulk terms at $\xi = h + 1$.

The bulk terms have a simple pole whenever $\xi + 1 = mh$ for some integer $m$, which should cancel against one of the terms in the infinite sum so that the expansion (27) continues to be regular [35–37]. In all cases the result, found by evaluating

$$\lim_{\xi = mh+1} \frac{B(r)}{r^2} = \frac{1}{r^2} - \frac{2m}{\pi \xi + \pi},$$

is a logarithmic term:

$$B_{\text{massive}}(r)\big|_{\xi = mh+1} = \frac{3v(G)}{\pi^2 m} r^2 \ln r,$$

$$B_{\text{massless}}(r)\big|_{\xi = mh+1} = \frac{3(-1)^m v(G)}{\pi^2 m} r^2 \ln r.$$  

Now we are in position to compare (25) with (27), apart from one remaining difficulty. The perturbative coefficients $C_j$ are usually hard to calculate, and instead it is easier to estimate, from the massless and massive NLIEs, respectively, the expansion coefficients $c_j$ and $\tilde{c}_j$. The perturbative coefficients $C_j$ do not depend on $\lambda$, and if we assume the mass and crossover scales $M$ are equal we should find

$$c_j = (-1)^j \tilde{c}_j.$$  

The table below shows the comparison of the massive and massless UV coefficients, found via the NLIEs.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$c_j$</th>
<th>$\tilde{c}_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.657142856</td>
<td>1.657142856</td>
</tr>
<tr>
<td>1</td>
<td>$-1.49526585$</td>
<td>$1.49526587$</td>
</tr>
<tr>
<td>2</td>
<td>0.01009599</td>
<td>0.01009595</td>
</tr>
<tr>
<td>3</td>
<td>$-0.0015271$</td>
<td>$0.0015278$</td>
</tr>
<tr>
<td>4</td>
<td>0.0000603</td>
<td>0.0000602</td>
</tr>
</tbody>
</table>

The table above shows the comparison of the massive and massless UV coefficients, found via the NLIEs.
We include a small sample of our numerics in Table 2.
Such good agreement provides an excellent check on both the massive and massless NLIEs, the associated bulk terms and the above assumption on $M$.

5. Close to the infrared fixed point the model is described by the action of the infrared CFT plus an infinite number of contributions from irrelevant operators, resulting in a theory which is unrenormalisable. However, by considering the contribution of a finite number of fields it is still possible to make a comparison with results from either NLIEs such as ours or from TBA equations [38,39]. We consider

$$A = A_{IR} + g \int \psi d^2 x + t \int T \overline{T} d^2 x + \cdots,$$  \hspace{1cm} (36)$$

where the (possibly missing) irrelevant field $\psi$ of dimension $\Delta_{IR}$ and $T \overline{T}$ of dimension 2 belong to the infrared CFT. The couplings are related to the crossover scale $M$ as

$$g = \kappa t M^{2 - 2\Delta_{IR}}, \hspace{1cm} t = \kappa t M^{-2}.$$  \hspace{1cm} (37)

The action implies $c_{eff}(r)$ has IR expansion

$$c_{eff}(r) \sim c_{eff}(\infty) + \sum_{j=1}^{\infty} g_j (\kappa t r)^{(2 - 2\Delta_{IR})j} + \sum_{j=1}^{\infty} t_j (\kappa t r)^{-2j} + \cdots.$$  \hspace{1cm} (38)

From the NLIE point of view [1] corrections to $c_{eff}(r)$ come from the $\theta \rightarrow -\infty$ asymptotic of $\chi_{ab}(\theta)$ given by (29), the $e^{\theta/(\xi - 1)}$ term generating a series of the form $r^{-2h/(\xi - 1)}$. Comparing with the CPT expansion (38) leads to the prediction $\Delta_{IR} = 1 + h/(\xi - 1)$, which can be identified with the conformal dimension of the primary field $\Phi_{adj}$ with weights $\Omega = \Omega_{adj}, \Omega' = \Omega_{adj}$.

We can also extract a prediction for $\kappa_t$ from the NLIEs. We start with the first two coefficients of the series generated by $T \overline{T}$ [39]:

$$t_1 = -\frac{\pi^3 c_{eff}(\infty)^2}{6}, \hspace{1cm} t_2 = \frac{\pi^6 c_{eff}(\infty)^3}{18}. \hspace{1cm} (39)$$

and compare them with the coefficients of $r^{-2}$ and $r^{-4}$ found from the NLIEs. Adapting the TBA argument [37], we use the $e^{\theta}$ term in the expansion of $\chi_{ab}(\theta)$ to find

$$c_{eff}(r) \sim c_{eff}(\infty) = \frac{2\pi^2}{3} c_{eff}(\infty) x_{11}^{(1)} r^{-2} + 2 c_{eff}(\infty) \left( \frac{2\pi^2}{3} x_{11}^{(1)} \right)^2 r^{-4} + \cdots.$$  \hspace{1cm} (40)

The explicit form

$$x_{11}^{(1)} r^{-2} = \frac{\sin^2(\frac{\pi}{\xi})}{\pi \sin^2(\frac{\pi}{\xi} - 1))v(G)} r^{-2}$$  \hspace{1cm} (41)

indicates a pole whenever $\xi - 1 = m'h$ for some integer $m'$. Evaluating as for the UV case we find (41) becomes

$$x_{11}^{(1)} r^{-2} |_{\xi = m'h + 1} = -\frac{2(-1)^{m'} \sin^2(\frac{\pi}{\xi})}{\pi^2 m'v(G)} r^{-2} \ln r.$$  \hspace{1cm} (42)

The infrared expansion coefficients have been obtained numerically for the models $W_{Ap,p}^{5,p+1} + \phi_{13}$, $p = 5, \ldots, 10$, in [37,40], and they show good agreement with our predictions, while for $W_{Ap,p}^{5,p+1}$ and $W_{D_{n+1}}^{2n+2,2n+3}$ we find agreement with the theoretical results of [37]. Finally, the effective central charge of the $\phi_{12}, \phi_{21}$ and $\phi_{15}$ perturbations of the Virasoro minimal models is exactly half that of certain $W_{A_2}$ models:

$$W_{A_2}^{2,p} + \phi_{12} \leftrightarrow W_{A_2}^{2,p} + \Phi_{adj},$$

$$W_{A_2}^{2,p} + \phi_{21} \leftrightarrow W_{A_2}^{2,p} + \Phi_{adj},$$

$$W_{A_2}^{2,p} + \phi_{15} \leftrightarrow W_{A_2}^{2,p} + \Phi_{adj}.$$  \hspace{1cm} (43)

Only the $\phi_{21}$ and $\phi_{15}$ perturbed models have a massless flow, and the IR expansion coefficients found in [1] match with those predicted above for the associated $W_{A_2}$ model. The correspondence actually works for any value of $\xi$ since the $\phi_{12}/\phi_{21}/\phi_{15}$ NLIE is based on the tadpole diagram $T_1$, which is related by folding to $A_2$.

Finally, comparing (40), (41) to (38), (39) yields the promised prediction for $\kappa_t$:

$$\kappa_t = \frac{4 \sin^2(\frac{\pi}{\xi})}{\pi^2 \sin^2(\frac{\pi}{\xi} - 1))v(G)}.$$  \hspace{1cm} (44)

6. We have shown that the function $c_{eff}(r)$ for nonunitary models interpolates from a CFT in the ultraviolet to an infrared CFT that has smaller effective
central charge, but in a nonmonotonic way. It is likely that there is a function, as yet unknown, which monotonically interpolates between the nonunitary CFTs and satisfies a ‘nonunitary $c$-theorem’.

We would like to make two further comments concerning the massive nonlinear integral equations. First, level-rank (or KNS) duality [23,41] relates two nonunitary minimal models

$$W_{n-1}^{p,n} = W_{p-n-1}^{p,n}, \quad n, p \text{ coprime}, \quad (45)$$

which both have a field $\Phi_{\text{adj}}$ with conformal dimension $\Delta = (pn - p - n^2)/p$, resulting in entirely equivalent perturbed theories. It is not at all obvious that the NLIEs based on $A_{2n}^{(2)}$, and those based on $A_{p-n-1}$, at the appropriate values of $\xi$ will produce the same value of $c_{\text{eff}}(r)$, but provided the normalisation of the lightest mass is such that the coupling $\lambda$ is the same for both models, our numerical studies confirm this. By equating the massive bulk terms we find

$$\sin(\pi/(p-n)) M_{[n-1]} = \sin(\pi/n) M_{[p-n-1]}, \quad (46)$$

the same result can also be deduced from the relation $\lambda = \kappa M^y$ given for the unitary minimal models in [33]. We can rule out a massless flow from $W_{n-1}^{p,n}$ via $\Phi_{\text{adj}}$ by considering the conjectured IR model $W_{n-1}^{2n-p,n}$, a sensible CFT if $2n - p > h$. Since $h = n$ this would require $p < n$, contradicting the assumption $n < p$.

As mentioned above, the massive $\phi_{15}$ and $\phi_{12}$ perturbations of the Virasoro models $W_{A_{n-1}^{(2)},p}$ (for odd $p$ greater than 4) are integrable, and the groundstate energy can be found using a single NLIE based on $A_{2}^{(2)}$ [42], with appropriate tuning of the free parameters. Thus by level-rank duality the models $W_{A_{p-3,n-1}^{(2),p}}$ have up to two extra massive perturbations in addition to $\Phi_{\text{adj}}$ whose groundstate energy can be described in terms of a nonlinear integral equation.

Second, the massive NLIEs encode the finite-size effects of a further set of models obtained by sending $\xi = k - h$ to infinity. In this limit the perturbed coset $G_{k} \times G_{N}/G_{N}^{k+1} + \Phi_{\text{adj}}$ becomes the $G$ Gross–Neveu model [22,43], with groundstate energy described in terms of an infinite number $(k.N)$ of coupled TBA equations. The NLIEs offer a clear advantage over the TBA as the $N$ equations can still be solved numerically at any value of $r$. The kernels

$$q_{ab}(\theta) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{i\theta k} \left( \delta_{ab} - \frac{e^{2\pi|k|/h}}{\cosh(\frac{\pi}{h}k)} c_{ab}^{-1}(k) \right) \quad (47)$$

form part of the prefactors of the associated S-matrices [44]. The perturbation is (almost) marginal and the ultraviolet expansion of the effective central charge no longer has a simple power series form (27). By studying the NLIEs (numerically and analytically) we hope to uncover the expected logarithmic corrections to $c_{\text{eff}}(r)$. If this approach is successful there are a number of two-dimensional sigma models which have an interpretation as a perturbed conformal field theory [44]. Since in all cases the groundstate energy is described by an infinite number of TBA equations, it would also be interesting to extend the current set of NLIEs to include such models.

Finally we remark that even though the groundstate energy of the perturbed nonunitary models is apparently real, recent results based on the $A_{2}^{(2)}$ models indicate the massive finite size spectrum may in general be complex [45]. It remains an open question to study the excited states of the massless models, perhaps via the massless NLIEs, to see if a similar result holds.

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Quantum gravity influences the black hole physics

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Abstract

The new state equations of thermal radiation is obtained by using the generalized uncertainty relation, in the context of quantum gravity. It is noticeable that the pressure of radiation becomes divergent when the system approaches a non-zero minimal length, which implies the prohibition against the singularity in the formation of black hole. Using the time-energy uncertainty, the corrections to the Schwarzkchild black hole thermodynamics are investigated. A negative and logarithmic correction to the Bekenstein–Hawking entropy is obtained. The mass loss rate of the black hole in the Planck realm is also discussed. © 2002 Elsevier Science B.V. All rights reserved.

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Using the semiclassical method, Hawking find that the particles can be emitted from a Schwarzkchild black hole due to the quantum effect of the field near the horizon \[1\]. This radiation satisfies the distribution of the Planckian spectrum with temperature

\[ T_H = \frac{1}{8\pi M}, \]  

(1)

where \( M \) the mass of the hole. According to the first law of black hole mechanics, the entropy of the hole can be easily identified as

\[ S = 4\pi M^2 = \frac{A}{4}, \]  

(2)

which proves the Bekenstein’s earlier argument \[2\]:

the black hole possesses the entropy (so-called “Bekenstein–Hawking entropy”) proportional to the area of the event horizon, \( A \). Some efforts \[3–9\] have been devoted to the quantum correction to the Bekenstein–Hawking entropy and a logarithmic term, \( \ln A \), is added to the right-hand side of Eq. (2). To the best of my knowledge, Eq. (1) has not been corrected. However, there are at least two reasons to change Eq. (1): firstly, if the thermodynamical relation \( dM = T dS \) is still maintained, the correction to the entropy leads to the correction to the temperature; secondly, according to the Stefan–Boltzmann law, the mass loss rate of the black hole is given by

\[ \frac{dM}{dt} = \frac{\sigma_0}{4} T_H^4 A \sim \frac{1}{M^2}, \]  

(3)

where \( \sigma_0 \) is the Stefan–Boltzmann constant, \( \pi^2/15 \) (in the units \( G = c = \hbar = k_B = 1 \)). Eq. (3) becomes divergent as \( M \to 0 \). This is hard to understand:
the emission still goes on even though the black hole vanishes, the vacuum becomes unstable? Although it is believed that the length of a hole is no longer less than the Planck length, the mechanism of such a brake on the fierce emission is still unclear. Therefore, Eqs. (1)–(3) should be corrected in the domain of quantum gravity.

Although the quantum theory of gravity is still not perfect, some efforts have shown that one of its consequences is generalization of Heisenberg uncertainty principle (see the review [10]). The simplest generalization of the position–momentum uncertainty relation (4); secondly, taking into account the existence of a minimal length can remove the singularity in the formation of black hole.

This Letter is organized as follows: firstly, we investigate the effect on the state equations of thermal radiation, due to the simplest generalization of uncertainty relation (4); secondly, taking into account the time–energy uncertainty, we derive the corrections to the temperature and the entropy of a Schwarzschild black hole by using the thermodynamical method. The emission rate of the hole is also discussed.

Let us investigate the photon gas in a spherical box with radius $R$. In order to compare with usual result, the Planck units is temporally ignored. We note that $k_B$ is the Boltzmann constant; the energy of photon $\epsilon = pc$, $c$ is the speed of light; the inverse temperature $\beta = 1/(k_B T)$. The photons are limited in the box, so the uncertainty of position $\Delta x \leq 2R$. From Eq. (4), the uncertainty of momentum satisfies

$$\Delta p \geq \frac{h}{2\lambda} \left[ \Delta x - \left( \Delta x^2 - 4\lambda \right)^{1/2} \right],$$  

then the uncertainty of position–momentum reads

$$\Delta x \Delta p \geq \frac{h}{2\lambda} \left[ \Delta x^2 - \Delta \left( \Delta x^2 - 4\lambda \right)^{1/2} \right] \geq \frac{2h}{\lambda} \left[ R^2 - R(R^2 - \lambda)^{1/2} \right] = \frac{2h}{\lambda} f(R, \lambda),$$  

where $\Delta x \leq 2R$. We have let

$$f(R, \lambda) = R^2 - R(R^2 - \lambda)^{1/2}.$$  

Looking at Eq. (6), we find that the Planck constant, $h$ in the standard Heisenberg principle, has been replaced by a new parameter $2h f/\lambda$. However, if $R^2 \gg \lambda$, Eq. (6) approximately goes back to the standard Heisenberg uncertainty relation

$$\Delta x \Delta p \geq h \left( 1 + \frac{\lambda}{4R^2} \right) \approx h.$$  

As shown in the next discussion, $\sqrt{\lambda}$ is of order of the Planck length, $10^{-33}$ cm. Therefore, for a macroscopic system with length 1 cm, $\lambda/R^2 \sim 10^{-66}$. Even for the usual microscopic scale (for example, the Compton wavelength of an electron, $10^{-11}$ cm), $\lambda/R^2 \sim 10^{-44}$. Thus the Heisenberg principle without correction is a good approximation in the usual realm. But we continue our discussion with Eq. (6) since we want to know something at the Planck scale.

As pointed out by the usual textbooks, the number of quantum states in the interval $d^3 p = dp_x dp_y dp_z$ is given by

$$\frac{V d^3 p}{(2\pi h)^3} = \frac{4\pi V p^2 dp}{(2\pi h)^3}.$$  

where $V$ is the volume of system. This can be understood as follows [16]: since the uncertainty relation $\Delta x \Delta p \sim 2\pi h$, one quantum state of a particle occupies a minimal “cell” with volume $(2\pi h)^3$ in the phase space, then the number of quantum state in the interval $d^3 x d^3 p = dx dy dz dp_x dp_y dp_z$ is $d^3 x d^3 p/(2\pi h)^3$. Doing the integration $\int dx dy dz$, we obtain Eq. (9). However, as shown by the generalized uncertainty relation (6), $h$ has been replaced by $2h f/\lambda$, in other words, the elementary scale in the phase space has been changed. Therefore, in the domain of the new uncertainty relation (6), the number of quantum state in the interval $d^3 p$ is given by substituting $2h f/\lambda$ for $h$,
that is
\[ dg(p) = 2\frac{4\pi V p^2 dp}{(4\pi \hbar f/\lambda)^3} = \left(\frac{\lambda}{2f}\right)^3 8\pi V p^2 dp \frac{1}{(2\pi \hbar)^3}. \]
\[ (10) \]
where we have taken into account the two directions of polarisation of the photon. Using the usual method, we obtain the free energy of the system
\[ F = \frac{1}{\beta} \int dg(p) \ln\left(1 - e^{-\beta\varepsilon}\right) \]
\[ = -\frac{1}{3} \left(\frac{\lambda}{2f}\right)^3 \frac{\pi^2 V}{15(h\bar{c})^3} \beta^{-4} = -\frac{1}{3} \left(\frac{\lambda}{2f}\right)^3 \sigma_0 T^4 V, \]
\[ (11) \]
where \( \sigma_0 \) is the Stefan–Boltzmann constant, \( \pi^2 k_B^4 / (15h^3\bar{c}^3) \). Correspondingly, the entropy and internal energy are given respectively
\[ S = k_B \beta^3 \frac{dF}{d\beta} = \frac{4}{3} \left(\frac{\lambda}{2f}\right)^3 \sigma_0 T^4 V, \]
\[ U = F + TS = \left(\frac{\lambda}{2f}\right)^3 \sigma_0 T^4 V. \]
\[ (12) \]
Comparing with the usual state equation, we see that only the coefficient is different. We redefine the Stefan–Boltzmann constant
\[ \sigma = \left(\frac{\lambda}{2f}\right)^3 \sigma_0 \]
\[ (13) \]
and write the entropy density and energy density as usual form
\[ s = \frac{U}{V} = \frac{4}{3} \sigma T^3, \quad u = \frac{S}{V} = \sigma T^4, \]
\[ (14) \]
where \( \sigma = \sigma(\lambda, R) \) is no longer a constant, which is related to the length of the system. On the macroscopic scale,
\[ \sigma = \left(1 - \frac{3\lambda}{4R^2}\right) \sigma_0, \]
\[ (15) \]
where the second term of the coefficient is due to the dynamical effect of the gravitational interactions of the photons. It contributes a negative correction to the energy density of the system and is associated with the length of the system, which is similar to the Casimir effect in the quantum field theory. We first give the following formulae
\[ \frac{d\sigma}{dV} = \frac{\sigma}{V} \left[\frac{R}{\sqrt{R^2 - \lambda}} - 1\right]. \]
\[ (16) \]
Substituting Eqs. (14) and (16) into the first law of thermodynamics as follows
\[ dU = T dS - p dV, \]
\[ (17) \]
we obtain the radiation pressure
\[ p = \frac{\sigma}{3} T^4 \frac{R}{\sqrt{R^2 - \lambda}}. \]
\[ (18) \]
We can derive its usual expression when neglecting the \( \lambda/R^2 \) term on the macroscopic scale. However, it is noticeable that the pressure becomes divergent when the system approaches the minimal length. This implies that the explosive pressure will withstand the star’s self gravitational collapse at the Planck scale, so the singularity is removed in the formation of black hole. This is compatible to the existence of a minimal length \( 2\sqrt{\lambda} \). Here we do not discuss the details of the star’s gravitational collapse, but it is believed that the quantum gravity will essentially change the usual state equations of matters to keep off the singularity.

The effects of the generalized uncertainty relation on the state density are also discussed in [13,14], but their arguments obviously differ from this Letter.

Let us discuss the thermodynamics of a Schwarzschild black hole. We turn to the units \( G = c = \hbar = k_B = 1 \). Considering the fluctuation of the horizon, the existence of a minimal length implies
\[ \Delta x = \sqrt{r^2 - \bar{r}^2} \geq 2\sqrt{\lambda}, \]
\[ (19) \]
then the change in the area of horizon reads
\[ \Delta A \geq 16\lambda\pi. \]
\[ (20) \]
This is consistent with the Bekenstein’s discrete area spectrum [19]. It is argued that the area of the horizon of a black hole is classical adiabatic invariant [17]. According to the Ehrenfest principle [18], a quantity that is adiabatic invariant corresponds to a quantum number and has a discrete spectrum. Bekenstein suggests that the area of the horizon is quantized as follows
\[ A = an, \quad n = 1, 2, \ldots, \]
\[ (21) \]
where \( \alpha \) is a constant, that is, the minimal element of area. From Eq. (20), we have \( \alpha = 16\lambda\pi \). In Ref. [20], Bekenstein argues that the area quantization will change the spectroscopy of the black hole. However,
the temperature and the entropy are supposed to be
the same as Eqs. (1) and (2). As shown by the
next discussion, it is an approximation only for the
large black hole. Considering a particle flees from a
Schwarzschild hole, it decreases the hole’s mass. Due
to Eqs. (20) or (21), the particle’s energy is not less
than a minimum m determined by
\[ M^2 - (M - m)^2 = \lambda, \]
\[ m = M - \sqrt{M^2 - \lambda}. \]  (22)
According to quantum mechanics [21], we can define
the characteristic time \( \tau \) as follows
\[ \tau^{-1} = \frac{dM/dt}{m} \]  (23)
which satisfies the time–energy uncertainty relation
[10]
\[ \tau \geq \frac{1}{m} + \lambda m \]  (24)
we obtain
\[ \frac{dM}{dt} = \tau^{-1} m \leq \frac{m^2}{1 + \lambda m^2}. \]  (25)
This is a universal bound to the mass loss rate
due to the quantum mechanics, which leads to the
conclusions essentially different from Eq. (3), at the
Planck scale. On the other hand, the mass loss rate
of the black hole satisfies the generalized Stefan–
Boltzmann law
\[ \frac{dM}{dt} = \frac{\mu A}{4} = \frac{\sigma}{4} T^4 A. \]  (26)
This is because the first equal sign is always valid. For
a large black hole, we have
\[ \sigma = \left( 1 + \frac{\lambda}{4R^2} \right)^{-3} \sigma_0 \approx \left( 1 - \frac{3\lambda}{16M^2} \right) \sigma_0. \]  (27)
\[ m \approx \frac{\lambda}{2M}. \]  (28)
When taking the upper bound of (25), we obtain
\[ T^{-4} = \frac{16\pi \sigma_0 M^4}{\lambda^2} \left( 1 - \frac{3\lambda - 4\lambda^3}{16M^2} \right). \]  (29)
We require that the leading term in the above equation
be consistent with the semiclassical result, \( T^{-1} = \)
\( 8\pi M \), and obtain
\[ \lambda = \sqrt{\frac{1}{3840\pi}}. \]  (30)
\[ \text{The minimal length} \ 2\sqrt{\lambda} \approx 0.2, \ \text{near the Planck length. We neglect the} \ \lambda^3 \ \text{term in (29) and obtain the} \ \text{inverse temperature} \]
\[ \beta = 8\pi M \left( 1 - \frac{3\lambda}{16M^2} \right)^{1/4} \approx 8\pi M \left( 1 - \frac{3\lambda}{64M^2} \right). \]  (31)
The entropy reads
\[ S = \beta dM = 4\pi M^2 - \lambda \frac{3\pi}{8} \ln M \]
\[ \approx \frac{A}{4} - \lambda \frac{3\pi}{16} \ln A. \]  (32)
The logarithmic correction accompanies the appearance
of \( \lambda \). It is due to the contribution of the \( \lambda/R^2 \) term
in Eq. (27), a result derived from the generalized un-
certainty principle. The negative correction decreases
the entropy of the black hole. This is consistent with
Carlip’s result except for the different coefficient [6].
From Eq. (4) we can understand this point: the in-
crease in the elementary scale of the phase space leads
to the decrease in the number of quantum states.

We turn to the final fate of the black hole emission.
at the final stage of emission, \( m \sim M \), then
\[ \frac{dM}{dt} \leq \frac{M^2}{1 + \lambda M^2} \]  (33)
which suggests that \( dM/dt \to 0 \), as \( M \to 0 \). It is
essentially different from Eq. (3). Certainly, due to
Eq. (22), the case of \( M = 0 \) is not allowed in the
existence of a black hole. When the last one mass
quantum is emitted, the mass loss rate of the black hole
approaches its maximum
\[ \frac{\lambda}{1 + \lambda^2} \]  (34)
at \( M = \sqrt{\lambda} \). After that, the black hole vanishes.
The significance of Eq. (33) is that there is no longer any
emission after the hole vanishes, in other words, the
gravitational vacuum is stable.

As shown by Eqs. (22) and (25), the emission rate
increases as the mass of the black hole decreases.
We do not find a reasonable mechanism to prevent
the black hole from emitting completely, at least in
the present frame of this Letter. It seems to result
in the information paradox. However, we notice that
one can define the temperature as Eq. (26) only in
the case \( M > \sqrt{\lambda} \), with the Stefan–Boltzmann law.
It is meaningless to define the temperature for the case $M < \sqrt{\lambda}$. This implies that the radiation from the black hole with mass $\sqrt{\lambda}$ is not thermal. The picture of the black hole emission is as follows: the radiation from the massive black hole is thermal; however, at the last stage of emission, the effect of quantum gravity becomes important and deforms the thermal nature of the radiation from the black hole. The black hole emits completely through quantum transition not by the thermal radiation. Such a picture allays the information paradox in a certain degree. Furthermore, the change in the temperature leads to a negative correction to the Bekenstein–Hawking entropy, which is consistent with Ref. [22]: compared with the usual black body spectrum, the black hole radiation is less entropic due to the Bekenstein’s discrete area spectrum. It implies that the spectral line from the hole carries a significant amount of information, the unitarity may be maintained.

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**References**

Quantum stabilization of compact space by extra fuzzy space

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Abstract

We investigate the quantized scalar field on the Kaluza–Klein spacetimes of $M^D \times T^d \times S_{FZ}$, where $M^D$ is the ordinary $D$-dimensional flat Minkowski spacetimes, $T^d$ is the $d$-dimensional commutative torus, and $S_{FZ}$ is a noncommutative fuzzy two-sphere with a fixed quantized radius. After evaluating the one-loop correction to the spectrum we use the mass-corrected term to compute the Casimir energy of the scalar field on the model spacetime. It is seen that, for some values of $D$ and $d$, the Casimir energy due to vacuum fluctuation in the model spacetimes could give rise a repulsive force to stabilize the commutative torus. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

Casimir effect is the vacuum fluctuation in a nontrivial geometry [1–3]. In the original paper [1], the corresponding change in the vacuum electric energy is found to attract the two perfectly conducting parallel plates. Appelquist and Chodos [4] have found that the attractive Casimir force could render the extra spaces in the Kaluza–Klein unified theory [5] to be sufficiently small to be unobserved. Since the attractiveness of the Casimir energy will push the size of the extra dimension down to the Planck scale, a natural cutoff scale of the linearized gravity was that presumably the dynamics of Planck scale, where the nonperturbative quantum gravity sets in, will stabilize the size of the extra dimensions.

In the string/M theory [6–9], a new available scale, noncommutativity $\theta_{ij}$, is found to be able to stabilize the size of the extra noncommutative spaces [10–12]. In the paper [10] Nam has tried to use the noncommutativity as a minimum scale to protect the collapse of the extra spaces. He use the one-loop Kaluza–Klein spectrum derived by Gomis, Mehen and Wise [13] to compute the one loop Casimir energy of an interacting scalar field in a compact noncommutative space of $M^{1,d} \times T_2^2$, where $1 + d$ dimensions are the ordinary flat Minkowski space and the extra two dimensions are noncommutative torus with noncommutativity $\theta$. The correction is found to contribute an attractive force and have a quantum instability. He therefore turns to investigate the case of vector field and find the repulsive force for $d > 5$.

In the paper [11] we followed the method of [13] to evaluate the one-loop correction to the spectrum of Kaluza–Klein system for the $\phi^3$ model on $M^{1,d} \times (T_2^2)^L$, where the extra dimensions are the $L$ two-dimensional
noncommutative tori. We evaluate the corrected Kaluza–Klein mass spectrum to compute the Casimir energy and found that, when $L > 1$ the Casimir energy due to the noncommutativity could give repulsive force to stabilize the extra noncommutative tori in the cases of $d = 4n - 2$, with $n$ a positive integral. This therefore suggests a possible stabilization mechanism for a scenario in superstring theory, where some of the extra dimensions are noncommutative.

The noncommutative fuzzy sphere, which can appear in the string/M theory [6–9], is known to correspond sphere D2-branes in string theory with background linear $B$-field [14]. It is found that, in the presence of constant RR three-form potential, the D0-branes can expand into a noncommutative fuzzy sphere configurations [15]. It also knows that the field theories on fuzzy sphere appear naturally from D-brane theory and matrix model with some backgrounds [16–19].

In the ordinary matrix model one could not find the fuzzy sphere solutions. However, adding a Chern–Simons term to the matrix model will enable us to describe the noncommutative fuzzy sphere as a classical solution. Comparing the energy in the various classical solutions one can find that the separated D0-branes will expand into a largest noncommutative fuzzy sphere to achieve minimum energy [14,15].

In this Letter we will study the Casimir effect on a spacetimes with an extra two-dimensional spaces of fuzzy sphere. Our initial motivation to investigate this problem is coming from the following two simple observations. First, the presence of noncommutativity is known to be able to stabilize the size of the extra noncommutative, at least in some cases. Second, the fuzzy sphere is known to be in the largest fuzzy sphere to achieve minimum energy, in which the radius of sphere has been fixed by the strength of the background linear $B$-field [14,15]. Therefore, it is natural to ask a question that when there exist a noncommutative fuzzy two spheres, as an extra space, could the vacuum fluctuation give rise the sufficient repulsive force to stabilize the commutative compact space?

Following this motivation we therefore consider the quantized scalar field on the model spacetimes of $M_{D,d,FZ} = M_D \times T^d \times S_{FZ}$, where $M_D$ is the ordinary $D$-dimensional flat Minkowski spacetimes, $T^d$ is the $d$-dimensional commutative torus, and $S_{FZ}$ is a noncommutative fuzzy two sphere with a fixed quantized radius. In the next section, after review the appearance of the fuzzy sphere in the matrix model with a Chern–Simons term, we then quantize the scalar field on the fuzzy sphere and set up the Feynman rule.

In Section 3, we extend the works of Gomis, Mehen and Wise [13] to evaluate the one-loop correction to the spectrum of Kaluza–Klein system for the $\phi^4$ model on the spacetimes of $M_{D,d,FZ}$. In Section 4, the obtained spectrum is used to compute the Casimir energy and then we analyze the associated Casimir effect. We find that when $D$ is an even integral and $D > 4$ then the Casimir energy could give repulsive force to stabilize the extra commutative torus if $d = 2n$, where $n$ is a odd integral.

Note that according to the brane-world scenario, our four-dimensional universe might be a three brane embedded in a higher-dimensional spacetime, which are assumed to arise as the fluctuations of branes in string theories. Therefore, if an extra space is a noncommutative fuzzy sphere, which can appear in the string/M theory [6–9], the Casimir energy evaluated in this Letter could give sufficiently repulsive force to stabilize the extra commutative compact space.

2. Quantized $\Phi^4$ theory on fuzzy sphere

2.1. Fuzzy sphere

In the presence of the RR 4-form background the configuration of the $N$ $D_0$ can be describe by the action

$$S = T_0 \text{Tr} \left( \frac{1}{2} \dot{X}_i^2 + \frac{1}{4} [X_i, X_j] [X_i, X_j] - \frac{i}{3} \lambda_N \epsilon_{ijk} X_i [X_j, X_k] \right).$$

(2.1)

where $X_i, i = 1, 2, 3$ are $N \times N$ matrices and $T_0 = \sqrt{2\pi}/g_s$ is the zero-brane tension. The constant $\lambda_N$ before the Chern–Simons term in the above equation is the constant RR 4-form potential in the D0-branes system [15]. The
static equations of motion of the above equation are
\[ [X_j, ([X_i, X_j] - i\lambda_N \epsilon_{ijk}X_k)] = 0, \] (2.2)
and the associated energy is
\[ E = -T_0 \text{Tr} \left( \frac{1}{4} [X_i, X_j][X_i, X_j] - i\frac{\lambda_N \epsilon_{ijk}X_i[X_j, X_k]]}{3} \right). \] (2.3)
Eq. (2.2) admits commutating solutions and static fuzzy sphere solutions [14,15].
The commutating solutions are known to represent \( N \) D0-branes and satisfy the relations
\[ [X_i, X_j] = 0, \] (2.4)
which have the energy
\[ E = 0. \] (2.5)
The noncommutative static fuzzy sphere solutions satisfy the equations
\[ [X_i, X_j] = i\lambda_N \epsilon_{ijk}X_k, \] (2.6)
and can be described by the relations
\[ X_i = \lambda_N J_i, \] (2.7)
\[ X_1^2 + X_2^2 + X_3^2 = r^2, \] (2.8)
where \( J_1, J_2, J_3 \) define, say, the \( N \)-dimensional irreducible representation of SU(2) and are labeled by the spin \( \alpha = N/2 \). The noncommutativity parameter \( \lambda_N \) is of dimension length, and can be taken positive. The radius \( r_N \) define in (2.8) is quantized in units of \( \lambda_N \) by
\[ \frac{r_N}{\lambda_N} = \sqrt{\frac{N}{2} \left( \frac{N}{2} + 1 \right)}, \quad N = 1, 2, \ldots. \] (2.9)
Besides the above solution \( X_i \) may be a direct sum of several irreducible representation of SU(2). Such a configuration could also solve the equation of motion and is described by
\[ X_i = \lambda_N \bigoplus_{r=1}^{\gamma} J_i^{(r)}, \quad \sum_{r=1}^{\gamma} (2J_r + 1) = N. \] (2.10)
The energy \( E \) of these static fuzzy sphere solutions is given by
\[ E = -T_0 \lambda_N^4 \frac{1}{6} \sum_{r=1}^{\gamma} J_r (J_r + 1)(2J_r + 1). \] (2.11)
From the above relation it is clear that the ground state is the \( N \)-dimensional fuzzy sphere [15].
In summary, when the \( N \) D0-branes are coupled to a constant RR 4-form potential, then the D0-branes will blow up into a fuzzy 2-sphere. And, furthermore, the fuzzy 2-sphere will evolve into in the largest fuzzy sphere to achieve minimum energy [15,20]. It is important to note that the radius of sphere has been fixed by the strength of the background as shown in (2.9).
2.2. Quantized scalar field on fuzzy sphere

We will consider the scalar field $\Phi$ living on the spacetimes of $\mathcal{M}_{FZ}^{D,d} = M^{D} \times T^{d} \times S_{FZ}$, where $M^{D}$ is the ordinary $D$-dimensional flat Minkowski spacetimes with coordinate $\vec{x}$, $T^{d}$ is the $d$-dimensional commutative tori with coordinate $\vec{y}$ and have same radius $R$ for convenience. $S_{FZ}$ is a noncommutative fuzzy two sphere with coordinate $\vec{X}$ and has a fixed quantized radius $r_N$. The action we considered is described by the equation

$$S = \int d^D\vec{x} \int d^d\vec{y} \int \frac{1}{2} \Phi \left( \Box \vec{x} + \Box \vec{y} + \Delta + m^2 \right) \Phi + \frac{g}{4!} \Phi^4,$$  \tag{2.12}

where $\Box_x = \sum_i \partial_x^2$ and $\Delta = \sum_i I_i^2$. Note that a field on the spacetimes $\mathcal{M}_ {FZ}^{D,d}$ is defined as an algebra $S_N^2$ generated by Hermitian operators $\vec{X} = (X_1, X_2, X_3)$ which are described in the Section 2.1. The integral of a function $F \in S_N^2$ over the spacetimes $\mathcal{M}_ {FZ}^{D,d}$ is given by

$$r_N^{-2} \int \frac{d^D\vec{x}}{\text{Tr}} \int d^d\vec{y} \int F = \frac{4\pi r_N^{-2}}{N + 1} \int d^D\vec{x} \int d^d\vec{y} \text{Tr} [F(\vec{X})],$$  \tag{2.13}

and the inner product can be defined by

$$(F_1, F_2) = \int \frac{d^D\vec{x}}{\text{Tr}} \int d^d\vec{y} \int F_1 \text{Tr} F_2.$$  \tag{2.14}

In this formula $\Phi$ is a Hermitian field. To quantize it we can expand $\Phi$ in terms of the modes,

$$\Phi(\vec{x}, \vec{y}, X_i) = \int_{-\infty}^{\infty} \frac{d^D\vec{p}}{(2\pi)^D} \sum_{L} \sum_{\hat{n}} \sum_{L_i} a_{L_i}^{\dagger}(\vec{p}, \vec{n}) e^{i\vec{p}\cdot\vec{x}} e^{i2\pi\vec{n}\cdot\vec{y}/R} Y_{L}^{L_i},$$  \tag{2.15}

in which the Fourier coefficient $a_{L_i}^{\dagger}(\vec{p}, \vec{n})$ are treated as the dynamical variables and $Y_{L}^{L_i}$ are the usual spherical harmonics.

In the path integral quantization [21,22] we shall integrate all the possible configuration of $a_{L_i}^{\dagger}(\vec{p}, \vec{n})$. Then the $k$-points Green’s functions can be computed by the relation

$$\langle a_{L_1}^{\dagger}(\vec{p}_1, \vec{n}) \ldots a_{L_k}^{\dagger}(\vec{p}_k, \vec{n}) \rangle = \frac{\int [D\Phi] e^{-S} a_{L_1}^{\dagger} \ldots a_{L_k}^{\dagger}}{\int [D\Phi] e^{-S}}.$$  \tag{2.16}

Note that the complete basis of functions on $S_N^2$ is given by the $(N + 1)^2$ spherical harmonics, $Y_{L}^{L_i}$ ($L = 0, 1, \ldots, N; -L \leq l \leq L$). They correspond to the usual spherical harmonics, however, the angular momentum has an upper bound $N$ here. This is a characteristic feature of fuzzy sphere.

The propagator so obtained is

$$D^{-1} = \langle a_{L_i}^{\dagger}(\vec{p}, \vec{n}) a_{L_i}(\vec{p}, \vec{n}) \rangle = \frac{1}{p^2 + (2\pi\vec{n}/R)^2 + L(L + 1) + m^2},$$  \tag{2.17}

in which $a_{L_i}^{\dagger}(\vec{p}, \vec{n}) = (-1)^l a_{-L_i}^{\dagger}(-\vec{p}, -\vec{n})$. We let $r_N = 1$ and let $M^D$ be the Euclidean space thereafter. The four-legs vertices so obtained is given by

$$V_4 = a_{L_1}^{\dagger} \ldots a_{L_4}^{\dagger} V_4(L_1, l_1; \ldots; L_4, l_4).$$  \tag{2.18}
where
\[ V_4(L_1, L_2, L_3, L_4) = \frac{g}{4!} \frac{N+1}{4\pi} (-1)^{L_1+L_2+L_3+L_4} \prod_{i=1}^{4} (2L_i + 1)^{1/2} \sum_{I, L} (-1)^I (2L + 1) \times \left( \begin{array}{ccc} L_1 & L_2 & L \\ l_1 & l_2 & l \end{array} \right) \left( \begin{array}{ccc} L_3 & L_4 & L \\ l_3 & l_4 & l \end{array} \right) \left\{ L \alpha \alpha \alpha \right\} \left\{ L_3 \alpha \alpha \alpha \right\} \left\{ L_4 \alpha \alpha \alpha \right\}. \] (2.19)

Here the first bracket is the Wigner 3j-symbol and the curly bracket is the 6j-symbol of SU(2), in the standard mathematical normalization [23]. Note that to derive the above Feynman rule we have used the following “fusion” algebra [23]
\[ \gamma^I \gamma^J = \sqrt{\frac{N+1}{4\pi}} \sum_{K, k} (-1)^{2\alpha + I + J + K + k} \sqrt{(2J + 1)(2J + 1)(2K + 1)} \left( \begin{array}{ccc} I & J & K \\ i & j & -k \end{array} \right) \left\{ I \alpha \alpha \alpha \right\} \left\{ J \alpha \alpha \alpha \right\} \left\{ K \alpha \alpha \alpha \right\} \gamma^K, \] (2.20)
where the sum is over 0 ≤ K ≤ N, -K ≤ k ≤ K, and \( \alpha = N/2 \). In next section, we will use the above formula to evaluate the one-loop correction to the spectrum and then used the mass-corrected term to compute the Casimir energy of the scalar field on the model spacetime in the Section 4.

3. Kaluza–Klein spectrum

Using the above Feynman rule the 1-PI two-point function at one loop is obtained by contracting 2 legs in (2.18) using the propagator in (2.17). The planar contribution so obtained is defined by contracting the neighboring legs:
\[ (I^{(2)}_{\text{planar}})^{L,L'}_{I,I'} = \frac{g^2}{4\pi} \frac{3}{3} \delta_{L,L'} \delta_{I,-I'} (-1)^I I^P, \] (3.1)
\[ I^P = \int \frac{d^D p}{(2\pi)^D} \sum_{n, J=0}^{N} \frac{2J + 1}{p^2 + (2\pi n/R)^2 + J(J + 1) + m^2}. \] (3.2)
All 8 contributions are identical. Similarly by contracting nonneighboring legs, we find the nonplanar contribution
\[ (I^{(2)}_{\text{nonplanar}})^{L,L'}_{I,I'} = \frac{g^2}{4\pi} \frac{6}{\delta_{L,L'} \delta_{I,-I'}} (-1)^I I^{NP}, \] (3.3)
\[ I^{NP} = \int \frac{d^D p}{(2\pi)^D} \sum_{n, J=0}^{N} (-1)^{L+J+2\alpha} \frac{(2J + 1)(2\alpha + 1)}{p^2 + (2\pi n/R)^2 + J(J + 1) + m^2} \left\{ I \alpha \alpha \alpha \right\} \left\{ J \alpha \alpha \alpha \right\}. \] (3.4)
Again the 4 possible contractions agree.
To derive the above relations we have use the following identities of the 3j and 6j symbols, which can be found in [23]: The 3j symbols satisfy the orthogonality relation
\[ \sum_{J, j} \left( \begin{array}{ccc} J & L & K \\ j & l & k \end{array} \right) \left( \begin{array}{ccc} J & L & K' \\ j & -l & -k' \end{array} \right) = \frac{(-1)^{K-L-J}}{2K+1} \delta_{K,K'} \delta_{k,k'}, \] (3.5)
assuming that (J, L, K) form a triangle. The 6j symbols satisfy the orthogonality relation
\[ \sum_{N}^{(2N+1)} \left\{ \begin{array}{ccc} A & B & N \\ C & D & P \end{array} \right\} \left\{ \begin{array}{ccc} A & B & N \\ C & D & Q \end{array} \right\} = \frac{1}{2P+1} \delta_{P,Q}. \] (3.6)
and the following sum rule
\[
\sum_{N} (-1)^{N+P+Q} (2N+1) \begin{vmatrix} A & B & N \\ C & D & P \end{vmatrix} \begin{vmatrix} A & B & N \\ C & D & Q \end{vmatrix} = \begin{vmatrix} A & C & Q \\ B & D & P \end{vmatrix},
\]
(3.7)
assuming that \((A, D, P)\) and \((B, C, P)\) form a triangle.

We now begin to evaluate \(I^P\) and \(I^{NP}\). After integrating the momenta \(\vec{p}\) we can evaluate \(I^P\) as follow:
\[
I^P = \pi^2 \frac{\Omega}{2} \Gamma\left(1 - \frac{D}{2}\right) \sum_{n} \sum_{J=0}^{N} \frac{2J+1}{(2\pi n/R)^2 + J(J+1) + m^2}^{1-\frac{D}{2}}
\]
\[
\approx \pi^2 \frac{\Omega}{2} \Gamma\left(1 - \frac{D}{2}\right) \sum_{n} (2J+1) \sum_{J=0}^{N} \frac{(R/2\pi)^{2-D}}{(n^2)^{1-\frac{D}{2}}}
\]
\[
= \pi^2 \frac{\Omega}{2} \Gamma\left(1 - \frac{D}{2}\right) (N^2 + 2N) Z_d(2-D)(R/2\pi)^{2-D}
\]
\[
= \pi^2 \frac{\Omega}{2} (N^2 + 2N) \Gamma\left(1 - \frac{D}{2}\right) \sum_{n} \frac{1}{(n^2)^{1-\frac{D}{2}}}
\]
\[
\approx \pi^2 \frac{\Omega}{2} (N^2 + 2N) \Gamma\left(1 - \frac{D}{2}\right) Z_d(D+2-D)(R/2\pi)^{2-D}.
\]
(3.8)

There we have used the following approximation
\[
\frac{1}{R^2} \gg N(N+1), \quad \frac{1}{R^2} \approx m^2.
\]
(3.9)

This means that we consider the case in which the torus radius \(R^2\) is smaller than the values of \(1/N(N+1)\) and \(1/m^2\). (Note that \(J \leq N\).) This is a reasonable approximation as the torus will be stabilized, if it can, at small radius. (Note that we use the scale \(r_N = 1\).) The Epstein function \(Z_d(s)\) used in the above equation is defined by [2]
\[
Z_d(s) = \sum_{n_1} \cdots \sum_{n_d} \left[n_1^2 + \cdots + n_d^2\right]^{-s/2},
\]
(3.10)
and the reflection formula of the Epstein function [2]
\[
\Gamma\left(\frac{s}{2}\right) Z_d(s) = \pi^{s-d/2} \Gamma\left(\frac{d-s}{2}\right) Z_d(d-s).
\]
(3.11)
has been used to obtain the final result in (3.8).

In the same way we have the relation
\[
I^{NP} = \pi^2 \frac{\Omega}{2} \Gamma\left(1 - \frac{D}{2}\right) \sum_{n} \sum_{J=0}^{N} (-1)^{L+J+2\alpha} (2J+1)(2\alpha+1) \begin{vmatrix} \alpha & \alpha & L \\ \alpha & \alpha & J \end{vmatrix}
\]
\[
\approx \pi^2 \frac{\Omega}{2} \Gamma\left(1 - \frac{D}{2}\right) \sum_{n} (-1)^{L+J+2\alpha} (2J+1)(2\alpha+1) \sum_{J=0}^{N} \frac{(R/2\pi)^{2-D}}{(n^2)^{1-\frac{D}{2}}}
\]
\[
= \pi^2 \frac{\Omega}{2} \Gamma\left(1 - \frac{D}{2}\right) (N+1)^2 Z_d(2-D)(R/2\pi)^{2-D}
\]
\[
= \pi^2 \frac{\Omega}{2} (N+1)^2 \Gamma\left(1 - \frac{D}{2}\right) \sum_{n} \frac{(R/2\pi)^{2-D}}{(n^2)^{1-\frac{D}{2}}}
\]
\[
\approx \pi^2 \frac{\Omega}{2} (N+1)^2 \Gamma\left(1 - \frac{D}{2}\right) Z_d(D+2-D)(R/2\pi)^{2-D}.
\]
(3.12)

To obtain the above results we have used the formula [23]
\[
\sum_{J} (-1)^{J+2\alpha} \begin{vmatrix} \alpha & \alpha & L \\ \alpha & \alpha & J \end{vmatrix} = \delta_{L,0}(2\alpha+1).
\]
(3.13)
Collecting the above results the leading correction to the spectrum of Kaluza–Klein system is

$$\Sigma_0 = AR^{2-D},$$

in which

$$A = \frac{g}{24}\pi^{-\frac{d}{2}}(3N^2 + 6N + 1)\Gamma\left(\frac{D}{2} + \frac{d}{2} - 1\right)Z_d(D + d - 2)(1/2\pi)^{2-D}.$$  \hfill (3.15)

The result is used in the next section to evaluate the Casimir energy to investigate the stability of the commutative torus due to the vacuum fluctuation in the model spacetimes which has an extra fuzzy space.

Note that the fuzzy space we considered is the fuzzy two-sphere with a radius fixed by (2.9). The radius of fuzzy is only dependent on the strength of the background RR 4-form potential. It is important to mention that as we use the approximation $\frac{1}{R^2} \gg N(N+1)$ in (3.9), we cannot consider the limit of $N \to \infty$. Therefore our one-loop result could not be used to discuss the case of commutative sphere.

4. Casimir energy

Follow [1] we can evaluate the Casimir energy by summing up the energy $\omega$ of all the modes:

$$u = \frac{1}{2} \int \frac{d^D\tilde{p}}{(2\pi)^D} \sum_{\tilde{n}} \sum_{J=0}^{N} \omega_{\tilde{n},\tilde{p}}^2$$

$$= \frac{1}{2} \int \frac{d^D\tilde{p}}{(2\pi)^D} \sum_{\tilde{n}} \sum_{J=0}^{N} \sqrt{\tilde{p}^2 + \frac{\tilde{n}^2}{R^2} + J(J+1) + m^2 + AR^{2-D}}$$

$$= \frac{1}{2} \int \frac{d^D\tilde{p}}{(2\pi)^D} \sum_{\tilde{n}} \sum_{J=0}^{N} \int_0^{\infty} \frac{dt}{t} t^{-1/2} e^{-t(\tilde{p}^2 + \frac{\tilde{n}^2}{R^2} + J(J+1) + m^2 + AR^{2-D})}$$

$$= -\frac{1}{4\sqrt{\pi}} \left(\frac{1}{4\pi}\right)^{D/2} \Gamma\left(-\frac{D+1}{2}\right) \sum_{\tilde{n}} \sum_{J=0}^{N} \left(\frac{\tilde{n}^2}{R^2} + J(J+1) + m^2 + AR^{2-D}\right)^{\frac{D+1}{2}}$$

$$\approx -\frac{1}{4\sqrt{\pi}} \left(\frac{1}{4\pi}\right)^{D/2} \Gamma\left(-\frac{D+1}{2}\right) \sum_{\tilde{n}} \sum_{J=0}^{N} \left(\frac{\tilde{n}^2}{R^2} + AR^{2-D}\right)^{\frac{D+1}{2}},$$

in which we have used the approximation (3.9). Note that to obtain the above result we first use the Schwinger’s proper time $t$ to handle the square root, then integrate the transverse momentum $\tilde{p}$ by doing a Gaussian integral, and finally integrate the Schwinger’s proper time by using the integral representation of Gamma function.

Without the one-loop correction, i.e., $A = 0$, we have the result of tree level

$$u_{\text{tree}} = -\frac{1}{4\sqrt{\pi}} \left(\frac{1}{4\pi}\right)^{D/2} \Gamma\left(-\frac{D+1}{2}\right) \sum_{\tilde{n}} \sum_{J=0}^{N} \left(\frac{\tilde{n}^2}{R^2}\right)^{\frac{D+1}{2}}$$

$$= -\frac{1}{4\sqrt{\pi}} \left(\frac{1}{4\pi}\right)^{D/2} \Gamma\left(-\frac{D+1}{2}\right)Z_D(-D+1) \frac{1}{R^{D+1}} = -u_0 \frac{1}{R^{D+1}},$$

4.2a
in which

\[ u_0 = \frac{(N + 1)}{\pi^{(3D+d-3)/2}} \Gamma \left( \frac{D + d + 1}{2} \right) Z_d[D + d + 1]. \]  

(4.2b)

Here we have used the reflection formula of the Epstein function in (3.11). The above result presents the well-known property that the Casimir force will become infinity attractive at small radius and it will render the extra spaces in the Kaluza–Klein unified theory to be very small.

We now consider the one-loop correction to the spectrum. If \( D \geq 3 \) then \( AR^{2-D} \) will become very large as \( R \ll 1 \). In this case we can use the formula of large \( Q \) expansion [2]

\[ \pi^{s/2} \Gamma \left( \frac{s}{2} \right) \sum_{n_1, \ldots, n_d} \left[ \left( \frac{Q}{\pi} \right)^2 + \left( \frac{n_1}{R} \right)^2 + \cdots + \left( \frac{n_d}{R} \right)^2 \right]^{s/2} = \frac{Q^{d-s} R^d \Gamma \left( \frac{s-d}{2} \right)}{\pi^{(d-s)/2}} \left[ 1 + O(1/Q) \right] \]  

(4.3)

to calculate (4.1). The Casimir energy can therefore be expressed as

\[ u_{\text{one-loop}} = \frac{1}{R^{(\frac{D}{2} - 1)(D+d+1) - d}}. \]  

(4.4a)

in which

\[ u_1 = \frac{-N \pi^{d+D+1}}{4\sqrt{\pi} (4\pi)^{D/2}} A^{d+D+1} \Gamma \left( -\frac{D+1}{2} \right) \Gamma \left( -\frac{D + d + 1}{2} \right). \]  

(4.4b)

Using the one-loop Casimir energy we now begin to analyze the Casimir effect. From (4.4b) we see that if \( D \) and \( d \) satisfy the relations

\[ D = 2n, \quad n \in \text{integral}, \quad d = 2\tilde{n}, \quad \tilde{n} \in \text{odd integral} \]  

(4.5)

then \( u_1 \) is positive. Also, comparing (4.2a) with (4.4a) we see that if

\[ \left( \frac{D}{2} - 1 \right)(D + d + 1) - d > D + 1 \quad \Rightarrow \quad D > 4 \]  

(4.6)

then at the small values of \( R \) the Casimir energy \( u_{\text{one-loop}} \) will dominate over \( u_{\text{tree}} \). Thus the Casimir energy due to vacuum fluctuation in the our model spacetime, which has an extra fuzzy two sphere space, could give rise a repulsive force to stabilize the commutative torus. The stable radius of torus so obtained is

\[ R_{\text{stable}} = \left( \frac{u_1}{u_0} \left( \frac{4}{D^2 - 1} \right)(D + d + 1) - d \right)^{\frac{1}{(D^2 - 2)(D+d+1) - d}}. \]  

(4.7)

This is our final result.

5. Conclusions

Let us make some comments to conclude this Letter:

(1) We have studied the Casimir effect on a Kaluza–Klein spacetime in which the higher dimensions are commutative \( d \) tori and an extra two dimensions are noncommutative fuzzy sphere. We have used the \( \zeta \) function regularization method to evaluate the vacuum fluctuation in our model spacetime. When the radius of torus is small we can find the analyzed formula of the Casimir energy. From the result we have found the possibility of the stabilization of the torus in presence of the extra fuzzy space. This thus suggests a possible stabilization mechanism for a scenario in Kaluza–Klein theory, where some of the extra dimensions are commutative compact space and fuzzy sphere.
(2) During our evaluation we have used the approximation of (3.9): $\frac{1}{R^2} \gg N(N+1)$, $\frac{1}{R^2} \gg m^2$. This means that the Casimir energy obtained in (4.2) and (4.4) only useful for the case in which the torus radius $R$ is smaller than the values of $1/N(N+1)$ and $1/m^2$. Therefore the stable radius of torus obtained in (4.7) could be reliable on if $R_{\text{stable}} \ll 1$. (Note that we have used the scale $r_N = 1$ and, as shown in (2.9), $r_N$ is a quantized value which only depend on the $N$ and strength of the background RR three-form.) Under this approximation our result cannot be used to discuss the limit of $N \to \infty$ in which the fuzzy sphere becomes a commutative sphere.

(3) Note that when $D$ or $d$ is an odd number, then the divergence of $\Gamma$ function in (4.4b) will lead the Casimir energy to be infinite. This may mean that our regularization method is broken down in these cases. The reason may be traced to the lack of a reflection formula to regularize (4.1). (Note that the reflection formula (3.1) has been used to regularize $u_0$ to obtain the finite result (4.2).) This problem is deserved to be investigated furthermore.

(4) The Casimir effect is null in the supersymmetry system, as the contribution of boson field is just canceled by that of the fermion field. However, some mechanisms are proposed to break the supersymmetry to describe the physical phenomena. Thus the remaining Casimir effect may be used to render the compact space stable. An interesting mechanism to break the supersymmetry is the temperature effect, which is the scenario in the early epoch of the universe. Therefore it is useful to investigate the finite-temperature Casimir effect on our model spacetimes. It also remains to see the Casimir effect in the more general system including the Fermion and vector field in our model spacetimes. These investigations will be presented in elsewhere.

References

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On the dual equivalence of the self-dual and topologically massive
$B \wedge F$ models coupled to dynamical fermionic matter

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Abstract

We study the equivalence between the $B \wedge F$ self-dual (SD $B \wedge F$) and the $B \wedge F$ topologically massive (TM $B \wedge F$) models including the coupling to dynamical, $U(1)$ charged fermionic matter. This is done through an iterative procedure of gauge embedding that produces the dual mapping. In the interactive cases, the minimal coupling adopted for both vector and tensor fields in the self-dual representation is transformed into a non-minimal magnetic like coupling in the topologically massive representation but with the currents swapped. It is known that to establish this equivalence a current–current interaction term is needed to render the matter sector unchanged. We show that both terms arise naturally from the embedding procedure.

1. Preliminaries

This work is devoted to the study of duality symmetry in the context of the $B \wedge F$ theory, viz., in models presenting a topological, first-order derivative coupling between forms of different ranks that is a dimensional extension of the duality between the self-dual (SD) [1] and Maxwell–Chern–Simons models (MCS) [2], shown by Deser and Jackiw [3] long time ago. To this end we investigate the existence of a constraint of self duality in the massive, non-invariant model (SD $B \wedge F$) and adopt a new dynamical embedding formalism [4,5], that is alternative to the master Lagrangian approach, to obtain the gauge invariant $B \wedge F$ model. Our study also includes the case of dynamical fermionic matter minimally coupled to the self-dual sector.

Duality is a fascinating symmetry concept allowing the connection of two opposite regimes for the same dynamics. It plays an important role in nowadays physics, both in the original contexts of condensed matter and Maxwell electromagnetism, as well as in the recent research of extended objects. The existence of such a symmetry within a model has important consequences—it can be used to derive (exact) non-perturbative results since swapping opposite regimes allows a perturbative investigation of theories with large coupling constants.

The study of this symmetry has received renewed interest in recent research in diverse areas in field theory such as, supersymmetric gauge theories [6], sine-
Gordon model [7], statistical systems [8] and, in the context of condensed matter models, applied for instance to planar high-$T_C$ materials, Josephson junction arrays [9] and Quantum Hall Effect [10]. In particular, the duality mapping has been of great significance in order to extend the bosonization program from two to three dimensions with important phenomenological consequences [11]. It also plays preponderant role in the ADS/CFT correspondence [12] that illustrates the holographic principle [13].

The idea of duality has also been used in recent developments of string theory [14], where different vacua are shown to be related by duality [15]. In this context a general procedure for constructing dual models was proposed by Busher [16] and generalized by Rocek and Verlind [17] that consists in lifting the global symmetry of the tensor fields with a new gauge field, whose field strength is then constrained to zero by the use of a Lagrange multiplier. Integrating, sequentially, the multiplier and the gauge field yields the original action while the dual action is obtained if one integrates the gauge field together with the original tensor field, keeping the Lagrange multiplier that then plays the role of dual field to the original tensor field. This line of research was used in the investigation of bosonization as duality by Burgess and Quevedo [18] and to discuss S-duality, the relation between strong and weak couplings in gauge theories [19]. This procedure has also been shown to be related to canonical transformations [20]. Recently, this line of research has been applied in the context of the topologically massive $B \wedge F$ theory, which is related to our interest here, to study its equivalence with the Stuckelberg construction of gauge invariant massive excitations [21].

The duality we are treating in this work deals with the equivalence between models describing the same physical phenomenon involving the presence of a topological term in a four-dimensional spacetime. It is closely related to the odd-dimensional duality involving the Chern–Simons term (CST) [22], whose paradigm is the equivalence between SD [1] and MCS [2,3] theories in $(2 + 1)$ dimensions. As shown in [3], in three dimensions there are two different ways to describe the dynamics of a single, freely propagating spin one massive mode, using either the SD theory [1] or the MCS theory. They also established the identification that relates the basic field of the SD model with the dual of the MCS field [3]. This correspondence displays the way the gauge symmetry of the MCS representation, gets hidden in the SD representation [3]. It is well understood by now that it is the presence of the topological and gauge invariant Chern–Simons term the responsible for the essential features manifested by the three-dimensional field theories, while in the four-dimensional context this role is played by the $B \wedge F$ term. To extend this duality symmetry relation and study its consequences in the context of four-dimensional field theories with $B \wedge F$ term, is the main purpose of this work.

The study of gauge theories with a topological term, in $3 + 1$ dimensions, has received considerable attention recently. Among other possibilities the $B \wedge F$ term is interesting for providing a gauge invariant mechanism to give mass to the gauge field and to produce statistical transmutation in $3 + 1$ dimensions. Here $B$ is a Kalb–Ramond field, i.e., a totally antisymmetric tensor potential (a potential 2-form) while $F = dA$ is the field strength of the one-form potential $A$. An Abelian antisymmetric tensor potential was probably first used in the context of the particle theory to describe a massless particle of zero-helicity [23, 24]. It reappeared later on in the context of fundamental strings [25,26], has been used to study cosmic strings [27–29] and to put topological charge (hair) on black holes [30–32]. The free theory of a rank-2 antisymmetric tensor has also been intensively studied both classically [33] and quantically [34,35] and has been shown to be dynamically dual (under the Hodge mapping) to a massless scalar field (zero-form).

In this work we study duality in field theories involving the $B \wedge F$ term. To this end, in the next section, we investigate the gauge non-invariant SD$_{B \wedge F}$ model, define a new, non-Hodge, (derivative) duality operation and show the existence of self-duality. Next, in Section 3, we apply an iterative dynamical embedding procedure to construct an invariant theory out of the self-dual $B \wedge F$ model—the topologically massive $B \wedge F$ model (TM$_{B \wedge F}$). This is a gauge embedding procedure that is done with the inclusion of counter terms in the non-invariant action, built with powers of the Euler vectors and tensors (whose kernels give the field equations for the potentials $A$ and $B$) to warrant the dynamical equivalence. Such construction discloses hidden gauge symmetries in such systems. One can then consider the non-invariant model as the gauge
fixed version of a gauge theory. A deeper and more illuminating interpretation of these systems is then obtained. The advantage in having a gauge theory lies in the fact that the underlying gauge symmetry allows us to establish a chain of equivalence among different models by choosing different gauge fixing conditions. In Section 4 we consider the minimal coupling with fermionic matter. Our results are discussed in the final section of the Letter.

2. Self dual $B \wedge F$ theory

The model with a built-in SD constraint in $2+1$ dimensions was proposed in [1] as an alternative to the concept of topologically massive modes proposed in [2]. The former is a non-gauge invariant, first order model, while the later is a second order gauge invariant formulation, both making use of the topological Chern–Simons term. In this section we want to formulate and study a first order, non-gauge invariant model, making use of the topological $B \wedge F$ term and prove the existence of self duality property as a consequence of a built in SD constraint.

The model in question shows the coupling of a vector field potential $A_\mu$ with a tensor field potential $B_{\mu\nu}$ [31] as,

$$L_{SD}(0) = \frac{1}{2} m^2 A_\mu A^\mu - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

$$+ \frac{\chi \theta}{4} \epsilon^{\mu\nu\lambda\rho} B_{\mu\nu} F_{\lambda\rho},$$

where the superscript index in the Lagrangian is the counter of the iterative algorithm to be implemented in the sequel, $\chi = \pm 1$ will be shown to display the self or antiself duality, $\theta$ is the coupling constant and the field strength of the basic potentials are,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}. $$

(1)

The coefficients of the mass terms are so chosen to give mass dimension one and two, respectively, to the potentials $A_\mu$ and $B_{\mu\nu}$ which, consequently, keeps dimensionless the coupling constant $\theta$ in the $B \wedge F$ term. Here the potentials play an active role in the duality transformations. This shall be in contrast with the dynamical matter field, to be considered latter on that, although coupled to the potentials, are passive fields (spectators) in the duality mapping. It is immediate to work out the equations of motion of the basic potentials $A_\mu$ and $B_{\mu\nu}$ to obtain, respectively,

$$A_\mu = - \frac{\chi \theta}{2m^2} \epsilon^{\mu\nu\lambda\rho} \partial_\nu B_{\lambda\rho},$$

$$B^{\mu\nu} = \chi \theta \epsilon^{\mu\nu\lambda\rho} \partial_\lambda A_\rho, $$

(3)

satisfying the constraints

$$\partial_\mu A_\mu = 0,$$

$$\partial_\mu B^{\mu\nu} = 0 $$

(4)

identically. Eq. (3) constitute a set of first-order coupled equations that can be combined into a decoupled second-order, massive, wave equations as

$$\left( \partial_\mu \partial^\mu + \frac{m^2}{\theta^2} \right) F = 0, \quad F = \{ A_\mu; B_{\mu\nu} \},$$

(6)

whose mass depends crucially on the value of the coupling constant.

Next, we discuss the self-duality inherent to the above theory. To this end we define a new derivative duality operation by means of a set of star-variables as

$$^* A_\mu \equiv - \frac{\theta}{2m^2} \epsilon_{\mu\nu\lambda\rho} \partial^\nu B^{\lambda\rho},$$

$$^* B_{\mu\nu} \equiv \theta \epsilon_{\mu\nu\lambda\rho} \partial^\lambda A^\rho. $$

(7)

With this definition we obtain, for the double duality operation, the relations

$$^* (^* F) = F, \quad F = \{ A_\mu; B_{\mu\nu} \} $$

(8)

after use of the equations of motion (6). This is important because it validates the notion of self (or antiself) duality

$$^* F = \chi F, \quad F = \{ A_\mu; B_{\mu\nu} \} $$

(9)

as a solution for the field equations, very much like the three-dimensional SD model. However, this conceptualization of duality operation and self-duality in four dimensions is new.

Before we start the iterative procedure for the transformation of the SD $B \wedge F$ model into a topological $B \wedge F$ model let us digress on the consequences of the self-duality relation (9). Notice first that under the usual gauge transformations of the potentials

$$A_\mu \rightarrow A_\mu + \partial_\mu A, $$

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu A_\nu - \partial_\nu A_\mu $$

(10)
the fields strengths $F_{\mu\nu}$ and $H_{\mu\nu\lambda}$ are left invariant. Therefore, although the basic potentials are gauge dependent their duals, defined in (7), are not. This situation parallels the three-dimensional case involving the Chern–Simons term which is the origin for the presence of a hidden (gauge) symmetry in the SD model of [1] while it is explicit in the topologically massive model of [2]. Here too the $SD_{B\wedge F}$ model hides the gauge symmetry (10) that is explicit in the $TM_{B\wedge F}$ model. Let us next consider the direct automorphism

$$\mathcal{L}(A, B) \rightarrow \ast \mathcal{L}(\ast A, \ast B).$$

Since it is constructed with the dual fields it is automatically gauge invariant and reads

$$\ast \mathcal{L}(\ast A, \ast B) = \frac{1}{12m^2} H_{a\beta\lambda} H^{a\beta\lambda} - \frac{1}{8} F_{a\beta} F^{a\beta} + \frac{1}{2} \epsilon^{\alpha\beta\rho\lambda} \partial_{\beta} B_{\rho\lambda}.$$

It is a simple algebra to check that the equations of motion for the SD action (1) are also solutions of the field equation of (12). This exercise clearly shows an intimate connection between the $SD_{B\wedge F}$ with a gauge invariant version through a dual transformation. However, although establishing the dual connection, the result obtained in (12) produces a set of field equations involving higher derivatives that will produce more solutions than the original set. Besides they are not the usual $TM_{B\wedge F}$ model. In the next section we shall discuss a dynamical gauge embedding procedure that will clearly produce an equivalent gauge invariant model.

3. The gauge invariant $B \wedge F$ theory

In previous works [4,5] we have used the dynamical gauge embedding formalism to study dual equivalence in $2 + 1$ dimensions in diverse situations with models involving the presence of the topological Chern–Simons term. In this section we extend that technique to study duality symmetry among four-dimensional models involving the presence of a topological $B\wedge F$ term.

Our basic goal is to transform the symmetry (10) that is hidden in the Lagrangian (1) into a local gauge symmetry by lifting the global parameter $\Lambda$ into its local form, i.e., $\Lambda \rightarrow \Lambda(x^\mu)$. The method works by looking for an (weakly) equivalent description of the original theory which may be obtained by adding a function $f(K_\mu, M_{\mu\nu})$ to the Lagrangian (1). Here $K_\mu$ and $M_{\mu\nu}$ are the Euler tensors, defined by the variation

$$\delta \mathcal{L}^{(0)}_{SD} = K_\mu \delta A^\mu + M_{\mu\nu} \delta B^{\mu\nu}$$

whose kernels give the equations of motion for the $A_\mu$ and $B_{\mu\nu}$ fields, respectively. The minimal requirement for $f(K_\mu, M_{\mu\nu})$ is that it must be chosen such that it vanishes on the space of solutions of (1), viz., $f(0,0) = 0$, so that the effective Lagrangian $\mathcal{L}^{(0)}_{eff} \rightarrow \mathcal{L}^{(0)}_{eff} + f(K_\mu, M_{\mu\nu})$

is dynamically equivalent to $\mathcal{L}^{(0)}_{SD}$. To find the specific form of this function that also induces a gauge symmetry into $\mathcal{L}^{(0)}_{SD}$ we work iteratively. To this end we compute the variations (13) of $\mathcal{L}^{(0)}_{SD}$ to find the Euler tensors as

$$K_\mu = m^2 A_\mu - \frac{\chi}{2} \epsilon_{\mu\nu\rho\lambda} \partial^\nu B^{\rho\lambda},$$

$$M_{\mu\nu} = -\frac{1}{2} B_{\mu\nu} + \frac{\chi}{2} \epsilon_{\mu\nu\rho\lambda} \partial^\rho A^{\lambda},$$

and define the first-iterated Lagrangian as,

$$\mathcal{L}^{(1)}_{SD} = \mathcal{L}^{(0)}_{SD} - a_\mu K^\mu - b_{\mu\nu} M^{\mu\nu}$$

with the Euler tensors being imposed as constraints and the new fields, $a_\mu$ and $b_{\mu\nu}$, to be identified with ancillary gauge fields, acting as a Lagrange multipliers.

The transformation properties of the auxiliary fields $a_\mu$ and $b_{\mu\nu}$ accompanying the basic field transformations (10) is chosen so as to cancel the variation of $\mathcal{L}^{(0)}_{SD}$, which gives

$$\delta a_\mu = \delta A_\mu, \quad \delta b_{\mu\nu} = \delta B_{\mu\nu}. $$

A simple algebra then shows

$$\delta \mathcal{L}^{(1)}_{SD} = -a_\mu \delta K^\mu - b_{\mu\nu} \delta M^{\mu\nu}$$

where we have used (10) and (17). Because of (18), the second iterated Lagrangian is unambiguously defined as

$$\mathcal{L}^{(2)}_{SD} = \mathcal{L}^{(1)}_{SD} + \frac{m^2}{2} a_\mu a^\mu - \frac{1}{4} b_{\mu\nu} b^{\mu\nu}.$$
that is automatically gauge invariant under the combined local transformation of the original set of fields \((A_\mu, B_{\mu\nu})\) and the auxiliary fields \((a_\mu, b_{\mu\nu})\).

We have therefore succeed in transforming the global SD\(_{B^+F}\) theory into a locally invariant gauge theory. We may now take advantage of the Gaussian character displayed by the auxiliary field to rewrite (19) as an effective action depending only on the original variables \((A_\mu, B_{\mu\nu})\). To this end we use (19) to solve for the fields \(a_\mu\) and \(b_{\mu\nu}\) (call the solutions \(\tilde{a}_\mu\) and \(\tilde{b}_{\mu\nu}\)) collectively by \(\tilde{h}(\mu)\), and replace it back into (19) to find

\[
\mathcal{L}_{\text{eff}}^{(2)} = \mathcal{L}_{\text{SD}}(\mu) = \mathcal{L}_{\text{SD}}^{(0)} - \frac{1}{2m^2} K_\mu K^\mu + M_{\mu\nu} M^{\mu\nu} \tag{20}
\]

from which we identify the function \(f(K_\mu, M_{\mu\nu})\) in (14). This dynamically modified action can be rewritten to give the TM\(_{B^+F}\) theory,

\[
\mathcal{L}_{\text{eff}} = \frac{1}{12m^2} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{21}
\]

after the scaling \(\theta A_\mu \rightarrow A_\mu\) and \(\theta B_{\mu\nu} \rightarrow B_{\mu\nu}\) is performed. Notice the inversion of the coupling constant \(\theta \rightarrow 1/\theta\) resulting from the duality mapping. It becomes clear from the above derivation that the difference between these two models is given by a function of the Euler tensors of the SD\(_{B^+F}\) model that vanishes over its space of solutions. This establishes the dynamical equivalence between the SD\(_{B^+F}\) and the TM\(_{B^+F}\) theory.

4. The minimal coupling with fermionic matter

Once the duality mapping between the free theories has been established one is ready to consider the requirements for the existence of duality when the coupling with dynamical matter is included. In this section we consider the case of \(U(1)\) charged fermionic matter. For clarity, we consider first the situation where only the vector field, in the self-dual representation, is minimally coupled to the fermionic current. This seems appropriate since it illustrates the main features of the duality via gauge embedding. The results for the full coupling are then quickly presented in the following subsection.

4.1. Single coupling

Let us consider first a Dirac field minimally coupled to the vector field \(A_\mu\) specified by the SD\(_{B^+F}\) model. To this end let us introduce the fermionic current

\[
J^\mu = \bar{\psi} i \gamma^\mu \psi \tag{22}
\]

and define the rank-2 charge \(J^{\mu\nu}\) as the (derivative) dual of the fermionic current \(J_\mu\),

\[
J^{\mu\nu} = -\frac{\theta}{2m^2} \epsilon^{\mu\nu\alpha\beta} \partial_\alpha J_\beta \tag{23}
\]

so that the interacting Lagrangian becomes

\[
\mathcal{L}_{\text{eff}}^{(0)} = \mathcal{L}_{\text{SD}}^{(0)} - e A_\mu J^\mu + \mathcal{L}_D, \tag{24}
\]

where \(M\) is the fermion mass. Here the Dirac Lagrangian is,

\[
\mathcal{L}_D = \bar{\psi} (i \partial - M) \psi. \tag{25}
\]

The fermionic field is treated as an spectator in the dual transformation between the gauge fields from the SD to the TM sectors. But to remain as a bystander field the coupling to the gauge fields and to itself has to be readjusted in the TM, as shown below.

As before, our basic strategy is to transform the hidden symmetry of the Lagrangian (24) into a local gauge symmetry. A variation of the Lagrangian (24) gives the Euler vectors as,

\[
K_\mu \rightarrow K_\mu^D = K_\mu - e J_\mu, \tag{26}
\]

\[
M_{\mu\nu} \rightarrow M_{\mu\nu}^D = M_{\mu\nu}.
\]

From now on we follow the same steps as in the free case just making the replacement \(K_\mu \rightarrow K_\mu^D\) and \(M_{\mu\nu} \rightarrow M_{\mu\nu}^D\) to obtain an effective action as,

\[
\mathcal{L}_{\text{eff}}^{(0)} = \mathcal{L}_{\text{eff}}^{(0)} - \frac{1}{2m^2} K^D_\mu K^D_\mu - M^D_{\mu\nu} M^D_{\mu\nu}. \tag{27}
\]

A further manipulation shows the presence of a new term, compared to the free case and of a non-minimal coupling

\[
\mathcal{L}_{\text{eff}} = \frac{1}{12m^2} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
\]
\[ -\frac{\chi}{2m^2} J^\mu J^\mu + \frac{\chi}{2} e J^\mu B^\mu\nu. \] 

(28)

The presence of these two terms involving the matter current are worth discussing since they play an important role in preserving the structure of the fermionic sector upon application of the dual mapping upon the gauge sector. They are the Thirring-like self-interaction and the non-minimal magnetic-like interaction of the dual of the fermionic current with \( B^\mu\nu \). It is interesting to observe that the minimal coupling involving the \( A^\mu \) field became, through the dualization process, a non-minimal coupling for the \( B^\mu\nu \) field. This swapping of the coupling is hardly a surprise. It is the manifestation, in the latent sector of duality, of the traditional duality inversion and, as far as we know, this phenomenon has not been reported before. Notice that both terms appear naturally as a consequence of the embedding algorithm. The presence of these terms, as we show in the sequel, are important to maintain the dynamical structure of the fermionic matter in both representations of the dual pair [36]. This can be seen by carefully examining the dynamics of the fermionic sector for both theories. To see this we compute the fermionic equation for the SD\( B\wedge F \) sector, that reads

\[ (i/\partial - M)\psi = e A^\mu \gamma_\mu \psi. \] 

(29)

To eliminate the bosonic field \( A^\mu \) in favor of the fermionic one we rewrite the equations of motion for the SD\( B\wedge F \) sector as,

\[ A^\mu = \frac{\chi}{2m^2} e^{\mu\nu\rho\sigma} \partial_{\nu} B_{\rho\sigma} + e J^\mu, \] 

(30)

\[ B^{\mu\nu} = \frac{\chi}{2m^2} e^{\mu\nu\rho\sigma} \partial_{\rho} A_{\sigma}, \] 

(31)

which can be combined to give, after using the constraint (4),

\[ R^{-1} A_\mu = e J_\mu, \] 

(32)

where \( R \) is a differential operator such that its inverse is the wave-operator, defined as

\[ R^{-1} = \partial_\mu \partial^\mu + \frac{m^2}{\theta^2}. \] 

(33)

Substituting this equation back in the matter equation gives,

\[ (i\partial - M)\psi = e^2 R J^\mu \gamma_\mu \psi. \] 

(34)

that is a non-linear, integro-differential equation, now written completely in terms of the fermionic fields.

Next we compare this result with the equations of motion for the fermionic matter from the TM\( B\wedge F \) sector, that reads

\[ (i/\partial - M)\psi = \chi e^{\ast A^\mu} \gamma_\mu \psi + \frac{e^2}{m^2} J^\mu \gamma_\mu \psi. \] 

(35)

where we have to obtain the bosonic functional \( ^\ast A^\mu \rightarrow ^\ast A^\mu (\psi) \) in terms of the fermions from the equations of motion for the gauge fields in the TM\( B\wedge F \) sector,

\[ ^\ast A^\mu = -\frac{\chi}{2m^2} e^{\mu\nu\rho\sigma} \partial_{\nu} B_{\rho\sigma}, \] 

\[ ^\ast B^{\mu\nu} = \frac{\chi}{2m^2} e^{\mu\nu\rho\sigma} \partial_{\rho} A_{\sigma} - 2eG^{\mu\nu} \] 

(36)

that can be combined to give,

\[ ^\ast A^\mu (\psi) = \chi e^{\left( \frac{\theta^2}{m^2} - R \right) J^\mu.} \] 

(37)

When inserted into (35) gives

\[ (i\partial - M)\psi = e^2 RJ^\mu \gamma_\mu \psi, \] 

(38)

which agrees with the dynamical equation for the fermions in the SD side. We just stress the importance of the Thirring-like self-interaction for the fermions and the non-minimal interaction with the tensor field to keep the dynamics of the latent, fermionic sector unaltered.

### 4.2. Double coupling

It is now a simple task to consider the full coupling of the bosonic fields \( A^\mu \) and \( B^{\mu\nu} \) to fermionic matter. To this end we introduce the rank-2 current

\[ G^{\mu\nu} = C \bar{\psi} \gamma^\mu \gamma^\nu \psi, \] 

(39)

where \( C \) is a complex normalization constant and its dual

\[ G^\mu = \theta e^{\mu\lambda\sigma\rho} \partial_\lambda G_{\sigma\rho}. \] 

(40)

The interacting Lagrangian now takes the form

\[ L^{(0)}_{\text{min}} = L^{(0)}_{\text{SD}} - e A^\mu J^\mu + g B^{\mu\nu} G^{\mu\nu} + L_D, \] 

(41)

with \( e \) and \( g \) being the strengths of the coupling with \( A^\mu \) and \( B^{\mu\nu} \), respectively. The effective, gauge invariant action is obtained directly from (20) just
operating the replacement
\[ K_\mu \to K^D_\mu = K_\mu - eJ_\mu, \]
\[ M_{\mu\nu} \to M^D_{\mu\nu} = M_{\mu\nu} + gG_{\mu\nu} \]
(42)
to produce
\[ \mathcal{L}_{\text{eff}} = \mathcal{L}^{(0)}_{\text{min}} - \frac{1}{2m^2} K^D_\mu K^D_\mu - M^D_{\mu\nu} M^D_{\mu\nu} \]
(43)
which, after some algebraic manipulation, gives
\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_D + \frac{1}{12m^2} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]
\[ - \frac{\chi}{20} \mu\nu \rho \lambda A_\mu \partial_\rho B_\lambda - \frac{e^2}{2m^2} J_\mu J^\mu \]
\[ + g^2 G^{\mu\nu} G_{\mu\nu} - \frac{\chi e}{\theta} J^{\mu\nu} B_{\mu\nu} \]
\[ + \frac{\chi g}{\theta} A_\mu G^\mu. \]
(44)

From this result it becomes clear the full action of the
dual mapping over the active and passive fields involved
in the transformation. Notice the exchange of
the minimal coupling adopted in the SD sector into a
non-minimal, magnetic like interaction in the TM sec-
tor, including a swapping between the fields and cur-
rents and the presence of the current–current interac-
tion for the fermionic sector which is needed to main-
tain the dynamics of the spectator field unmodified.
This is easily checked by just computing the equations
of motion for the Dirac fields in both representations

to obtain,
\[ (i\gamma - M)\psi \]
\[ = \frac{R}{\theta^2} \gamma_\mu \left[ (e^2 J_\mu - \chi g e G^\mu) \right. \]
\[ + 2m^2 C \left( \chi g e J^{\mu\nu} + g^2 G^{\mu\nu} \right) \gamma_\nu \left] \right. \psi. \]
(45)

5. Conclusions

In this work we studied dual equivalence in four-
dimensional topological models, namely, between the
\( B \wedge F \) self-dual (SD\( B \wedge F \)) and the \( B \wedge F \) topologi-
cally massive (TM\( B \wedge F \)) models using an iterative pro-
cedure of gauge embedding that produces the dual
mapping. We defined a new derivative type of duality
mapping, very much like the one adopted in the three-
dimensional case and proved the self and antiself-
duality property of the SD\( B \wedge F \) model, according to
the relative sign of the topological term. Working out
the free case firstly, where the \( A \) and \( B \) fields participate actively in the dual transformation we observed, as expected, the traditional inversion in the coupling constant. The coupling to dynamical fermionic matter,
which acts as a spectator field in the dual transforma-
tion, brought into the scene some new features. The appearance of a Thirring like self-interaction term in the dualized theory, that had already been observed in the 2 + 1 case, as well as the shift from minimal to non-minimal coupling. However, in this case we observed a swapping of the couplings from a tensor to another. This is a new result due to the presence of tensors of distinct ranks participating actively in the dual transformation. We proved that the presence of these terms are demanded to maintain the equivalent dynamics in the fermionic sector in either representations of the duality. The cases where the active tensors appear non-linearly and the coupling with bosonic matter are postponed for a forthcoming publication.

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references therein.
Vortices in de Sitter spacetimes

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\section{Abstract}

We investigate vortex solutions to the Abelian Higgs field equations in a four-dimensional de Sitter spacetime background. We obtain both static and dynamic solutions with axial symmetry that are generalizations of the Nielsen–Olesen gauge vortices in flat spacetime. The static solution is located in the static patch of de Sitter space. We numerically solve the field equations in an inflationary (big bang) patch and find a time dependent vortex solution, whose effect to create a deficit angle in the spacetime. We show that this solution can be interpreted in terms of a renormalization group flow in accord with a generalized $\epsilon$-theorem, providing evidence in favour of a dS/CFT correspondence. © 2002 Elsevier Science B.V. All rights reserved.

\section{1. Introduction}

De Sitter spacetimes are becoming of increasing interest in theoretical physics for a variety of reasons. They provide interesting arenas in which to study the classical no-hair conjecture first proposed by Ruffini and Wheeler \cite{1}, which states that after a given distribution of matter undergoes complete gravitational collapse, the only long range information of the resultant black hole is its electromagnetic charge, mass and angular momentum. The verification of this conjecture for a scalar field minimally coupled to gravity in asymptotically flat black-hole spacetimes has been extended to its de Sitter counterpart \cite{2,3}. While it is tempting to extend the no-hair theorem claim to all forms of matter, it is known that some long range Yang–Mills and/or quantum hair can be “painted” on a black hole \cite{4}. Explicit calculations have been carried out which verify the existence of a long range Nielsen–Olesen vortex solution as a single stable hair for a Schwarzschild black hole in four dimensions \cite{5}, although it might be argued that this situation falls outside the scope of the classical no-hair theorem due to the non trivial topology of the string configuration.

Another motivation for studying de Sitter (dS) spacetimes is connected with the recently proposed holographic duality between quantum gravity on dS spacetime and a quantum field theory living on the past boundary of dS spacetime \cite{6}. This proposed correspondence is undergoing extensive study in various directions \cite{7–17}. So far this work suggests that the conjectured dS/CFT correspondence has a lot of similarity with the AdS/CFT correspondence, although some interpretive issues remain \cite{13}.

Motivated by these considerations, we pursue in this Letter a study of vortex solutions in de Sitter spacetimes. We have already shown that the $U(1)$
Higgs field equations have a vortex solution in both a four-dimensional AdS spacetime [18] and in a four-dimensional Schwarzschild–AdS background [19]. Employing the well known AdS/CFT correspondence conjecture, the boundary conformal field theory can detect the presence of the vortex in the four-dimensional AdS spacetime: the mass density of the vortex solution is encoded in the discontinuity of the two-point correlation function of the dual conformal operator [18]. We have also shown that vortex solutions exist in the background of rotating Kerr–AdS and charged Reissner–Nordstrom–AdS black holes [20], even in the extreme case. Concurrently, it has recently been shown that in an asymptotically AdS spacetime that a black hole can have scalar hair [21]. Indeed, insofar as the no-hair theorem is concerned it has been shown that there exists a solution to the SU(2) Einstein–Yang–Mills equations which describes a stable Yang–Mills hairy black hole that is asymptotically AdS [4].

We therefore seek to learn if an analog of vortex holography discussed in [18] holds true for dS spacetime. In this Letter we take the first steps toward consideration of such a holographic phenomenon by searching for possible solutions of the Abelian Higgs field equations in a four-dimensional dS background. Although an analytic solution to these equations appears to be intractable, we confirm by numerical calculation that vortex solutions do exist in dS spacetime. We find both static and time-dependent vortex solutions. The static solution corresponds to a core of vortex field energy located within the cosmological horizon in a static patch of dS spacetime. In the time-dependent solution, the energy of the vortex core is dilated by the cosmological expansion in an inflationary patch. To our knowledge this is the first construction of vortex solutions in asymptotically dS spacetimes.

We also consider the implications of our solutions for the recently conjectured dS/CFT correspondence [6]. We compute the renormalization group flow associated with the time-dependent vortex solution in the inflationary patch and find that the generalized c-function monotonically increases, in accord with the generalized c-theorem [17].

In Section 2, we solve numerically the Abelian Higgs equations in the static dS background for different values of the cosmological constant. In Section 3, we compute the effect of the vortex solution on the dS spacetime. In Section 4, we solve numerically the same equations in a big bang patch of dS background. This is the first investigation of time dependent vortices in curved spacetime. In Section 5, we obtain the behaviour of the dS c-function of this solution and find that it increases (decreases) in an expanding (contracting) dS patch. We give some closing remarks in Section 6.

2. Abelian Higgs vortex in static de Sitter spacetime

In this section, we consider the Abelian Higgs Lagrangian in the background of de Sitter spacetime

\[ \mathcal{L}(\Phi, A_\mu) = -\frac{1}{2} (D_\mu \Phi)^* D^\mu \Phi - \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \xi (\Phi^\dagger \Phi - \eta^2)^2, \]

where \( \Phi \) is a complex scalar Klein–Gordon field, \( F_{\mu\nu} \) is the field strength of the electromagnetic field \( A_\mu \) and \( D_\mu = \nabla_\mu + i e A_\mu \) in which \( \nabla_\mu \) is the covariant derivative. We employ Planck units \( G = \hbar = c = 1 \) which implies that the Planck mass is equal to unity. We use the following four-dimensional static dS spacetime background

\[ ds^2 = -\left(1 - \frac{r^2}{l^2}\right) dt^2 + \frac{1}{\left(1 - r^2/l^2\right)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]

where the cosmological constant \( \Lambda \) is equal to \( 3/l^2 \). The horizon is at \( r = l \) and so the range of the coordinate \( r \) is bounded to \( 0 \leq r \leq l \). In this coordinate system \( \partial / \partial t \) is a future-pointing timelike Killing vector in only one diamond of the Penrose diagram [22] which generates the symmetry \( t \to t + t_0 \) for any constant \( t_0 \). In other diamonds of the Penrose diagram, this Killing vector is spacelike or else past-pointing timelike. After redefining the new fields \( X(x^\mu), P_\mu(x^\nu) \) by

\[ \Phi(x^\mu) = \eta X(x^\mu) e^{i \omega(x^\mu)}, \]

\[ A_\mu(x^\nu) = \frac{1}{e} \left( P_\mu(x^\nu) - \nabla_\mu \omega(x^\mu) \right) \]

and employing a suitable gauge, the equations of motion for a string with winding number \( N \) are

\[ \left(1 - \frac{\rho^2}{l^2}\right) \frac{d^2 X}{d\rho^2} + \left(\frac{1}{\rho} - \frac{4 \omega}{l^2}\right) \frac{dX}{d\rho} - \frac{1}{2} \omega (X^2 - 1) \]
\[- \frac{N^2}{\rho^2} XP^2 = 0, \tag{4}\]
\[
\left(1 - \frac{\rho^2}{l^2}\right) \frac{d^2P}{d\rho^2} - \frac{dP}{d\rho} \left(\frac{1}{\rho} + \frac{2\rho}{l^2}\right) - \alpha PX^2 = 0, \tag{5}\]

where \(\rho = r \sin \theta\) and \(\alpha = \frac{4\pi e^2}{\xi}\). Note that by changing \(l\) to \(il\), Eqs. (4) and (5) change to the equations of motion in an AdS background [18], or to the \(m = 0\) equations discussed in [19]. We have solved the above equations numerically for dS spacetimes with \(l = 3, 5, 10\) and unit winding number using the over-relaxation method [23].

Fig. 1 shows the results for our calculation and also compares them with the vortex fields in the AdS spacetime [18]. For fixed \(\rho \leq 3\), the \(X\) field of the dS spacetime decreases with decreasing \(l\), in contrast to the AdS spacetime which \(X\) field increases with decreasing \(l\).

In general the value of the \(X\) field for AdS spacetime is always greater than its value for any dS spacetime for a given \(\rho\). In other words, by increasing the cosmological constant \(\Lambda\) from \(-\infty\) to \(+\infty\), the \(X\) field decreases for fixed \(\rho\). The \(X\) field of flat spacetime is located between the dS and AdS cases where \(|l| = \infty\). Over this same range of \(\Lambda\) the value of the \(P\) field increases. Physically the negative cosmological constant exerts additional pressure on the vortex, causing it to become thinner; a positive cosmological constant has the opposite effect, causing the core of the vortex to expand beyond the thickness it would have in flat spacetime. Moreover, we expect that the effect of the vortex on dS spacetime is to create a deficit angle in the metric (2) by replacing \(\varphi \to a \varphi\), which \(a\) is a constant. In the next section, we verify this point.

For \(r \geq l\), the coefficients of the \(d^2X/d\rho^2\) and \(d^2P/d\rho^2\) terms in Eqs. (4) and (5) change sign and the numerical solution of the corresponding equations outside the cosmological horizon shows that the values of the \(X\) and \(P\) fields remain at their respective constant values of 1 and 0. Consequently the vortex is confined totally behind the cosmological horizon. In Section 4 we will find that in a different patch of dS spacetime the vortex solution exists everywhere in space and changes with time.

### 3. Vortex self gravity in static de Sitter spacetime

In this section, we consider the effect of the vortex on dS spacetime described by the static metric (2). This requires finding the solutions of the coupled Einstein–Abelian Higgs differential equations in dS. This is a formidable problem—even for flat spacetime no exact solutions have yet been found. Using a thin-core approximation and numerical methods, we obtained in [19] the effect of the vortex on AdS. Here, we use the same method to obtain the effect of the vortex on dS spacetime.

To obtain physical results, we make some approximations. First, we again assume the thin-core approximation, namely, that the thickness of the vortex is much smaller than all other relevant length scales. Second, we assume that the gravitational effects of the string are weak enough so that the linearized Einstein–Abelian Higgs differential equations are applicable. For convenience, in this section we use the following...
form of the metric of $dS_4$:

$$ds^2 = -\mathcal{A}(r, \theta)^2 dt^2 + \mathcal{B}(r, \theta)^2 d\varphi^2 + \mathcal{C}(r, \theta)\left(\frac{dr^2}{1 - r^2/l^2} + r^2 d\theta^2\right).$$  \hspace{1cm} (6)

In the absence of the vortex, we must have

$$\mathcal{A}(r, \theta) = \sqrt{1 - \frac{r^2}{l^2}} = \mathcal{A}_0(r, \theta),$$

$$\mathcal{B}(r, \theta) = r \sin \theta = \mathcal{B}_0(r, \theta),$$

$$\mathcal{C}(r, \theta) = 1 = \mathcal{C}_0(r, \theta),$$

yielding the well known metric (2) of pure $dS_4$. Employing the two assumptions concerning the thickness of the vortex core and its weak gravitational field, we solve numerically the Einstein field equations,

$$G_{\mu\nu} + \frac{3}{l^2}g_{\mu\nu} = -8\pi G T_{\mu\nu}$$ \hspace{1cm} (7)

to first order in $\varepsilon = -8\pi G$, where $T_{\mu\nu}$ is the energy–momentum tensor of the Abelian Higgs field in the $dS$ background. By taking $g_{\mu\nu} \simeq g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(1)}$, where $g_{\mu\nu}^{(0)}$ is the usual $dS_4$ metric and $g_{\mu\nu}^{(1)}$ is its first order correction, and writing

$$\mathcal{A}(r, \theta) = \mathcal{A}_0(r, \theta)\left(1 + \varepsilon \mathcal{A}(r, \theta)\right),$$

$$\mathcal{B}(r, \theta) = \mathcal{B}_0(r, \theta)\left(1 + \varepsilon \mathcal{B}(r, \theta)\right),$$

$$\mathcal{C}(r, \theta) = \mathcal{C}_0(r, \theta)\left(1 + \varepsilon \mathcal{C}(r, \theta)\right),$$ \hspace{1cm} (8)

we obtain first-order corrections to the three functions $\mathcal{A}_0(r, \theta)$, $\mathcal{B}_0(r, \theta)$ and $\mathcal{C}_0(r, \theta)$ in (8). Hence in the first approximation Eq. (7) become

$$G^{(1)}_{\mu\nu} + \frac{3}{l^2}g^{(1)}_{\mu\nu} = T^{(0)}_{\mu\nu}.$$ \hspace{1cm} (9)

where $T^{(0)}_{\mu\nu}$ is the energy–momentum tensor of the vortex field in the $dS_4$ background metric, and $G^{(1)}_{\mu\nu}$ is the correction to the Einstein tensor due to $g^{(1)}_{\mu\nu}$. The rescaled components of the energy–momentum tensor of string in the background of $dS_4$ are given by

$$T_{\varphi\varphi}^{(0)}(\rho) = -\frac{1}{2} \left(\frac{dX}{d\rho}\right)^2 \left(1 - \frac{\rho^2}{l^2}\right) - \frac{1}{2} \left(\frac{dP}{d\rho}\right)^2 \left(1 - \frac{\rho^2}{l^2}\right) - \frac{1}{2} \frac{P^2 X^2}{\rho^2} - (X^2 - 1)^2,$$

$$T_{t\varphi}^{(0)}(\rho) = -\frac{1}{2} \left(\frac{dX}{d\rho}\right) \left(1 - \frac{\rho^2}{l^2}\right) + \frac{1}{2} \left(\frac{dP}{d\rho}\right) \left(1 - \frac{\rho^2}{l^2}\right) + \frac{1}{2} \frac{P^2 X^2}{\rho^2} - (X^2 - 1)^2,$$

$$T_{t\varphi}^{(0)}(\rho) = -\frac{1}{2} \left(\frac{dX}{d\rho}\right)^2 \left(1 - \frac{\rho^2}{l^2}\right) + \frac{1}{2} \left(\frac{dP}{d\rho}\right)^2 \left(1 - \frac{\rho^2}{l^2}\right) + \frac{1}{2} \frac{P^2 X^2}{\rho^2} - (X^2 - 1)^2,$$

$$T_{rr}^{(0)}(\rho) = -\frac{1}{2} \left(\frac{dX}{d\rho}\right)^2 \left(1 - \frac{\rho^2}{l^2}\right) + \frac{1}{2} \left(\frac{dP}{d\rho}\right)^2 \left(1 - \frac{\rho^2}{l^2}\right) + \frac{1}{2} \frac{P^2 X^2}{\rho^2} - (X^2 - 1)^2,$$

$$T_{rr}^{(0)}(\rho) = -\frac{1}{2} \left(\frac{dX}{d\rho}\right)^2 \left(1 - \frac{\rho^2}{l^2}\right) + \frac{1}{2} \left(\frac{dP}{d\rho}\right)^2 \left(1 - \frac{\rho^2}{l^2}\right) + \frac{1}{2} \frac{P^2 X^2}{\rho^2} - (X^2 - 1)^2,$$

We consider the case $l = 3$, for which the behaviour of $X$ and $P$ fields are given in Fig. 1. By using Eqs. (10), we obtain Fig. 2 showing the behaviour of the stress tensor components inside the cosmological horizon as a function of $\rho$, whose minimum value is 0.1.

![Fig. 2. $T_{\varphi\varphi}^{(0)}$ (solid), $T_{t\varphi}^{(0)}$ (dash), $T_{rr}^{(0)} + T_{t\varphi}^{(0)}$ (dash-dotted) curves of a vortex in $dS$ spacetime with $l = 3$, versus $\rho$.](image-url)
We note that outside the cosmological horizon, the stress tensor vanishes due to the constant values of the vortex fields. With this knowledge, we solve the coupled differential equations (11), inside and outside the cosmological horizon which gives the behaviour of the functions \(A(\rho), B(\rho)\) and \(C(\rho)\). The results are plotted in Fig. 3. These results emphasize that the functions \(A(\rho), B(\rho)\) are due to the numerical calculation.

Hence by a redefinition of the time coordinate in (6) the metric can be rewritten as

\[
\begin{align*}
\text{ds}^2 &= \left(1 - \frac{r^2}{l^2}\right)dt^2 + \frac{1}{\left(1 - \frac{r^2}{l^2}\right)}d\tau^2 \\
&\quad + r^2(d\theta^2 + \alpha^2 \sin^2 \theta \, d\phi^2)
\end{align*}
\] (12)

which is the metric of dS space with a deficit angle. So, the effect of the vortex on dS spacetime is to create a deficit angle in the metric (2) by replacing \(\varphi \rightarrow \alpha \varphi\), where \(\alpha \simeq 1 + 2\varepsilon\) is a constant, since \(\varepsilon < 0\). The above calculation for the effect of the vortex on dS spacetimes is not restricted to the special case \(l = 3\); with other values of cosmological parameter \(l\), the final result is the metric (12), with some other deficit angle \(\alpha\).

4. Abelian Higgs vortex in a big bang patch of de Sitter spacetime

In this section, we consider the Abelian Higgs Lagrangian (1) in the following coordinate system

\[
\text{ds}^2 = -d\tau^2 + e^{2\tau/l}(dx^2 + dy^2 + dz^2),
\] (13)

where the coordinate \(\tau \in (-\infty, +\infty)\). After using the new fields \(X(\tau, r, \theta), P_\mu(\tau, r, \theta)\) given by (3), and applying the radial variable \(R = \sqrt{x^2 + y^2} = r \sin \theta\), the equations of motion for a string with winding number \(N\) become

\[
\begin{align*}
&\frac{e^{-2\tau/l}}{R} \frac{\partial^2 X}{\partial R^2} + \frac{e^{-2\tau/l}}{R} \frac{\partial X}{\partial R} \frac{\partial^2 X}{\partial \tau^2} - \frac{3}{l} \frac{\partial X}{\partial \tau} \\
&- \frac{N^2}{R^2} X + \frac{1}{2} X(X^2 - 1) = 0,
\end{align*}
\] (14)

\[
\begin{align*}
&\frac{e^{-2\tau/l}}{R} \frac{\partial^2 P}{\partial R^2} - \frac{e^{-2\tau/l}}{R} \frac{\partial P}{\partial R} \frac{\partial^2 P}{\partial \tau^2} - \frac{1}{l} \frac{\partial P}{\partial \tau} \\
&- \alpha XP^2 = 0.
\end{align*}
\] (15)

By numerically solving equations (14), (15), we are able to show that a vortex solution exists on a dS spacetime background for different values of the winding number \(N\) and cosmological parameter \(l\). As in the pure AdS case [18] and asymptotically AdS spacetimes [19,20] the results indicate that increasing the winding number yields a greater vortex thickness. Furthermore, for a vortex with definite winding number, the string core increases with decreasing \(l\) at constant positive time. The \(X\) and \(P\) fields less rapidly approach their respective maximum and minimum values at larger distances as \(l\) decreases.

To obtain numerical solutions of (14) and (15), we must first select appropriate boundary conditions. At large distances physical considerations motivate a clear choice. Since in the limit \(l \rightarrow \infty\), the results must be in agreement with the results of flat spacetime, we demand that the solutions approach the solutions of the vortex equations in the flat spacetime given in Ref. [19]. This means that we demand \(X \rightarrow 1\) and \(P \rightarrow 0\) as \(R\) goes to infinity. On the symmetry axis

![Fig. 3. A (dotted), B (solid), C (dashed) versus \(\rho\), inside and outside the cosmological horizon of the dS spacetime with \(l = 3\).](image-url)
In the following, we consider the case \( N = 0 \) on the symmetry axis) at the initial time \( \tau = -\infty \). In the following, we consider the case \( N = 1 \). We can straightforwardly obtain similar results for other values of the winding number \( N \).

We then employ a polar grid of points \((R_i, \tau_j)\), where \( R \) goes from 0 to some large value \( R_\infty \) which is much greater than \( l \) and \( \tau \) runs from a large negative number \( -\tau_\infty \) to a large positive number \( \tau_\infty \). Using finite difference methods, we rewrite the non-linear partial differential equation (14) and (15) as

\[
A_{ij} X_{i+1,j} + B_{ij} X_{i-1,j} + C_{ij} X_{i,j+1} + D_{ij} X_{i,j-1} + E_{ij} X_{i,j} = F_{ij},
\]

where \( A_{ij} X_{i+1,j} + B_{ij} X_{i-1,j} + C_{ij} X_{i,j+1} + D_{ij} X_{i,j-1} + E_{ij} X_{i,j} = F_{ij} \) and \( P_{ij} = P(R_i, \tau_j) \). For \( i \) and \( j \) the interior grid points, the coefficients \( A_{ij}, \ldots, F_{ij} \) can be straightforwardly determined from the corresponding continued differential equations (14) and (15). Using the well known successive overrelaxation method [23] for the above mentioned finite difference equations, we obtain the values of \( X \) and \( P \) fields inside the grid. Initially, inside the grid points, we set the value of \( X \) and \( P \) fields 0 and 1, respectively, which we denote them by \( X(0) \) and \( P(0) \). Then these values of \( X \) and \( P \) fields are used in the next step to obtain the values of \( X \) and \( P \) fields inside the grid which could be denoted by \( X(1) \) and \( P(1) \). Repeating this procedure, the value of the each field in the \( n \)th iteration is related to the \((n-1)\)th iteration by

\[
X_{ij}^{(n)} = X_{ij}^{(n-1)} - \omega \frac{\xi_{ij}^{(n-1)}}{F_{ij}^{(n-1)}},
\]

\[
P_{ij}^{(n)} = P_{ij}^{(n-1)} - \omega \frac{\eta_{ij}^{(n-1)}}{F_{ij}^{(n-1)}},
\]

where the residual matrices \( \xi_{ij}^{(n)} \) and \( \eta_{ij}^{(n)} \) are the differences between the left- and right-hand sides of Eqs. (16) and (17), respectively, evaluated in the \( n \)th iteration. \( \omega \) is the overrelaxation parameter. The iteration is performed many times to some value \( n = K \), such that

\[
\sum_{i,j} |X_{ij}^{(K)} - X_{ij}^{(K-1)}| < \varepsilon \quad \text{and} \quad \sum_{i,j} |P_{ij}^{(K)} - P_{ij}^{(K-1)}| < \varepsilon
\]

for a given error \( \varepsilon \). It is a matter of trial and error to find the value of \( \omega \) that yields the most rapid convergence. Some typical results of this calculation are displayed in Figs. 4 and 5 for value of \( l = 3 \).

In Fig. 4, time is running from \(-\tau_\infty \) to \( 0 \) whereas in Fig. 5 the time runs from \( 0 \) to \( \tau_\infty \). We notice that by increasing the time from \(-\tau_\infty \) to \( 0 \), the string core increases rapidly to a well defined value at time \( \tau = 0 \). Then increasing the time to more positive values, the string core increases more and more such that at time \( \tau = \tau_\infty \), the \( X \) and \( P \) fields approach the constant values 0 and 1, respectively. So, we conclude that by increasing the time from \(-\tau_\infty \) to \( \tau_\infty \), for which the constant time slices of

![Fig. 4. X(R) and P(R) fields of a vortex in dS spacetime with l = 3, for the different values of time. Time goes from -\tau_\infty to 0, from left to right on the curves.](image)
Fig. 5. $X(R)$ and $P(R)$ fields of a vortex in dS spacetime with $l = 3$, for the different values of time. Time goes from $0$ to $\tau_\infty$, from left to right on the curves.

Fig. 6. $X(R)$ and $P(R)$ fields of a vortex in dS spacetime with $l = 10$, for the different values of time. Time goes from $-\tau_\infty$ to $0$, from left to right on the curves.

Fig. 7. $X(R)$ and $P(R)$ fields of a vortex in dS spacetime with $l = 10$, for the different values of time. Time goes from $0$ to $\tau_\infty$, from left to right on the curves.
the spacetime (13) become bigger and bigger, the string thickness becomes wider and wider. Moreover the energy density of the string spreads to bigger and bigger distances as time increases due to the larger size of the constant time slices.

The preceding calculation of vortex fields on dS spacetimes is not restricted to the special case of \( l = 3 \); with other values of the cosmological parameter \( l \), we obtain similar results. For example, another typical results of our calculation are displayed in Figs. 6 and 7 for value of \( l = 10 \).

For time slices near \( \tau = 0 \), the behaviour of the vortex fields does not change by changing the cosmological parameter \( l \). The physical reason is that at this special time the spacetime is flat and independent of the cosmological constant. However, if we consider a constant positive time slice, then by increasing the cosmological parameter \( l \), the string thickness decreases; for a constant negative time slice, by increasing the cosmological parameter \( l \), it increases. The asymmetric behaviour of string thickness for positive and negative times is due to the inflation function \( e^{2\tau/\ell} \) in (13). Hence by increasing \( l \), the density of vortex field time-constant contours mainly near the time constant \( \tau = 0 \) slice.

To obtain the effect of the vortex on the big bang patch of dS spacetime, we use the results of the preceding section, in which we found that vortex induces a deficit angle on the static spacetime metric given in (12). By using the following transformations,

\[
\tau = - t + \frac{l}{2} \ln \left| 1 - \frac{r^2}{l^2} \right|,
\]

\[
X = \frac{r}{\sqrt{1 - r^2/l^2}} \exp(t/l) \sin \theta \cos(\alpha \varphi),
\]

\[
Y = \frac{-r}{\sqrt{1 - r^2/l^2}} \exp(t/l) \sin \theta \sin(\alpha \varphi),
\]

\[
Z = \frac{r}{\sqrt{1 - r^2/l^2}} \exp(t/l) \cos \theta,
\]

the following metric

\[
ds^2 = -d\tau^2 + e^{2\tau/l}(dX^2 + dY^2 + dZ^2)
\]

can be written in the well known static dS spacetime with deficit angle \( \alpha \) given by (12). Hence the effect of the vortex on a big bang patch of dS spacetime is to create a deficit angle in the \( X-Y \) plane that is constant as the (locally) flat spatial slice evolves in time.

Similar calculations for larger winding numbers show that increasing the winding number yields a greater vortex thickness in each constant time slice relative to winding number \( N = 1 \). This tendency runs counter to that of increasing \( l \), for which the vortex thickness decreases on a constant positive time slice. As \( l \to \infty \) we can increase the winding number to larger and larger values, and we find that the resultant thickness increases. Hence the effect of increasing winding number to thicken the vortex dominates over the thinning of the vortex due to decreasing cosmological constant.

What we have found about the behaviour of the vortex in the big bang patch of dS spacetime can be straightforwardly generalized to the big crunch patch of the dS spacetime. The big crunch patch is given by replacing \( \tau \to - \tau \) in the metric (13). Increasing the time from \( -\tau_{\infty} \) to \( \tau_{\infty} \), for which the constant time slices of the big crunch spacetime become smaller and smaller, the string thickness becomes narrower and narrower. In fact, increasing the time, the energy density of the string concentrates to smaller and smaller distances due to the smaller size of the constant time slices.

5. De Sitter c-function

Obtaining evidence in support of (or against) a conjectured dS/CFT correspondence is somewhat harder to come by than its AdS/CFT counterpart. Although it is tempting to think of the former as a ‘Wick-rotation’ of the latter, a number of subtleties arise whose physical interpretation is not always straightforward [13].

One way of making progress in this area is via consideration of the UV/IR correspondence. In both the AdS and dS cases there is a natural correspondence between phenomena occurring near the boundary (or in the deep interior) of either spacetime and UV (IR) physics in the dual CFT. Solutions that are asymptotically dS lead to an interpretation in terms of renormalization group flows and an associated generalized dS c-theorem. This theorem states that in a contracting patch of dS spacetime, the renormalization group flows toward the infrared and in an expanding spacetime, it flows toward the ultraviolet [17]. More precisely, the c-function in \((n+1)\)-dimensional inflation-
The energy density of the vortex is given by:

\[ \rho = \frac{1}{2} e^{2\tau} \left( \frac{\beta X}{\partial R} \right)^2 - \frac{1}{2R^2 e^{4\tau}} \left( \frac{\beta P}{\partial R} \right)^2 - \left( X^2 - 1 \right)^2 - \frac{X^2 P^2}{2R^2 e^{2\tau}}. \]

Using the curves of the vortex fields we obtained for different time slices in Figs. 4 and 5, we get Fig. 8 for the \( c \)-function in terms of time at a fixed point \( R = 2 \) of space.

We find that by increasing the time from \(-\tau_\infty\) to \( \tau_\infty \), the \( c \)-function monotonically increases as the universe expands. We emphasize that this monotonic increase is not restricted to this special value of \( R \): by changing the value of \( R \) we find the same increasing behaviour of the \( c \)-function as time evolves.

Similar calculations show that in the big-crunch patch of dS spacetime, the \( c \)-function decreases monotonically as time evolves from \(-\tau_\infty\) to \( \tau_\infty \).

6. Conclusion

We have solved the Nielsen–Olesen equations in a static dS\(_4 \) background, and found that the Higgs and gauge fields are axially symmetric, with non-zero winding number. Our solution in the limit of large \( l \) (small cosmological constant) reduces to the well known flat-space solution. The solution (to leading order in the gravitational coupling) induces a deficit angle in dS\(_4 \). We find in the static patch that an increasingly positive cosmological constant tends to make a thicker vortex solution due to cosmological expansion.

We solved the same equations in the big bang patch of the dS\(_4 \) background, and found that by increasing the time from early times at \( \tau \to -\infty \) to \( \tau = 0 \), the string thickness increases from an infinitesimally small value to a finite size. Increasing the time to vary large values \( \tau \to +\infty \), the string thickness grows more and more, so that at future infinity, the string fields approach constant values everywhere. At past infinity, which is in fact the dS\(_4 \) horizon, although the magnitude of the magnetic field goes to infinity near the string axis, the spatial dimensions shrink relative to spatial dimensions at other times, yielding a constant value for the magnetic flux of the vortex. In contrast...
to past infinity, at future infinity the magnetic field is very tiny over a big range of spatial coordinate, giving rise to the same value of the magnetic flux of the vortex at past infinity. The presence of a vortex in asymptotically dS spacetime technically violates the cosmic no-hair theorem in that the spacetime is not pure de Sitter; rather it is only locally dS due to the deficit angle induced from the non-trivial topology of the vortex solution. Violation of the cosmic no hair theorem has also been observed in Einstein–Maxwell–Dilaton theory with a positive cosmological constant [24].

Our results are in accord with the generalized dS c-theorem, providing further evidence in favor of a conjectured dS/CFT correspondence. An interesting future challenge is that of obtaining a holographic description of a vortex solution in dS spacetimes. The boundary field theory will have to be on a sphere with the same deficit angle as that induced by the vortex. In the AdS case, the conformal two-point correlation function is obtained by evaluating the bulk propagator of a scalar field between two points on the boundary by integrating over all spacelike geodesics paths between the points, and the presence of a vortex is signified by a discontinuity in this correlation function [18]. However, in the dS case the geodesic paths will be timelike, and will necessarily have to penetrate the cosmological horizon to detect a vortex in the static patch. Not all of $I^+$ ($I^-$) is causally connected to the vortex in the big bang (big crunch) patch, raising issues of causality reminiscent of those considered in the case of particles forming black holes in $2+1$ dimensions [25], and their resolution is far from clear.

We close by commenting on the physical relevance of our solutions. Since we employ Planck units $G = \hbar = c = 1$, the constant $l$ is in units of the Planck length. We have found that only for $l \lesssim 10$ do our solutions appreciably differ from the flat space case. In this regime the radius of curvature of the de Sitter spacetime becomes comparable to the Planck length, equaling it at $l = 1$, and one might be concerned that our results will be substantively modified due to quantum gravitational effects. Nevertheless we maintain that our results have both utility and validity for the following reasons. First, the qualitative behaviour of our solutions for large $l$ is the same as for $l \lesssim 10$; for example, Fig. 8 will retain the same qualitative features for all finite values of $l$, and we have numerically checked this for $l = 100$. However, smaller values of $l$ more clearly highlight the significance of the features of these solutions, which is why we have presented them here. Second, it is at the very least useful to know what the classical physics is that underlies any quantum effects, and our solutions furnish will such knowledge when it becomes available. Finally, quantum gravitational effects will typically induce quantitative corrections to our results of order $l^{-2}$, which is still only about 10% even for $l = 3$. Hence we expect the qualitative features of our solutions to remain even when quantum gravity is taken into account.

Inclusion of quantum gravitational effects is, of course, important. Other problems include the generalization of our solutions to asymptotically dS spaces with black holes and the possible dS/CFT correspondence of these solutions. Work on these problems is in progress.

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References

Noncommutativity in open string: new results in a gauge-independent analysis

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Abstract

Noncommutativity in an open string moving in a background Neveu–Schwarz field is investigated in a gauge-independent Hamiltonian approach, leading to new results. The noncommutativity is shown to be a direct consequence of the nontrivial boundary conditions, which, contrary to several approaches, are not treated as constraints. We find that the noncommutativity persists for all string points. In the conformal gauge our results reduce to the usual noncommutativity at the boundaries only.

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Keywords: Hamiltonian analysis; Noncommutativity; Strings

1. Introduction

The study of open string, in the presence of a background Neveu–Schwarz two-form field $B_{\mu\nu}$, leading to a noncommutative structure has recently evoked considerable interest [1,2]. This structure manifests in the noncommutativity in the space–time coordinates of D-branes, where the end points of the string are attached. Different approaches have been adopted to obtain this result. A Hamiltonian operator treatment was provided in [3] and a world sheet approach in [4]. Also, an alternative Hamiltonian (Dirac [5]) approach based on regarding the Boundary Conditions (BC) as constraints was given in [6]; the corresponding Lagrangian (symplectic) version being done in [7]. The interpretation of the BC as primary constraints usually led to an infinite tower of second class constraints [8], in contrast to the usual Dirac formulation of constrained systems [5,9]. Some other approaches to this problem have been discussed in [10,11]. As has been stressed in [1], it is very important to understand this noncommutativity from different perspectives.

In the present Letter, we provide an exhaustive analysis of the noncommutativity in open string theory moving in the presence of a constant Neveu–Schwarz field, in the conventional Hamiltonian framework. In contrast to the usual studies, our model of string theory is very general in the sense that no gauge is fixed.
at the beginning. Let us recall that all computations of noncommutativity, mentioned before, were done in the conformal gauge. Our gauge-independent analysis yields a new noncommutative structure, which correctly reduces to the usual one in conformal gauge. This shows the compatibility of the present analysis with the existing literature. In the general case, the noncommutativity is manifested at all points of the string, in contrast to conformal gauge results where it appears only at the boundaries. Indeed, in this gauge-independent scheme, one finds a noncommutative algebra among the coordinates, even for a free string, a fact that was not observed before. Expectedly, this noncommutativity vanishes in the conformal gauge. Note however, that there is no gauge for which noncommutativity vanishes in the interacting gauge. To gain further insight, both the Polyakov and Nambu–Goto (NG) formalisms of string theory have been studied.

At the outset, let us point out the crucial difference between existing Hamiltonian analysis [6] and our approach. This is precisely in the interpretation of the BC arising in the string theory. The general consensus has been to consider the BCs as primary constraints of the theory and attempt a conventional Dirac constraint analysis [5]. The aim is to induce the noncommutativity in the form of Dirac brackets between coordinates. The subsequent analysis turns out to be ambiguous since it involves the presence of $\delta(0)$-like factors (see Chu and Ho in [6]). Different results are obtained depending on the interpretation of these factors.

We, on the other hand, do not treat the BCs as constraints, but show that they can be systematically implemented by modifying the canonical Poisson bracket (PB) structure. In this sense our approach is quite similar in spirit to that of Hanson, Regge and Teitelboim [9], where modified PBs were obtained for the free NG string, in the orthonormal gauge, which is the counterpart of the conformal gauge in the free Polyakov string.

The Letter is organized as follows. In Section 2, the gauge-independent analysis of free Polyakov string is discussed. This also helps to fix the notations. The free NG string is developed in Section 3 for a comparison. A new structure, in the form of an interpolating action is presented in Section 4, which connects the Polyakov and NG actions in a smooth way. It also highlights the role of the boundary conditions in the present context. The noncommutativity is revealed in a gauge-independent analysis, in free Polyakov model in Section 5, which incidentally is a new result. Section 6 discusses the noncommutativity in the interacting theory in the Polyakov formulation and Section 7 does the same in the NG formalism. The Letter ends with a conclusion in Section 8.

2. The free Polyakov string

In order to study the various ramifications of different formulations of string theory, let us first consider the free Polyakov string action

$$S_P = -\frac{1}{2} \int_{-\infty}^{+\infty} d\tau \int_{0}^{\pi} d\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X_\mu,$$

(1)

where $\tau$ and $\sigma$ are the usual world-sheet parameters and $g_{ab}$, up to a Weyl factor, is the induced metric on the world-sheet. $X^\mu(\xi)$ are the string coordinates in the $D$-dimensional Minkowskian target space with metric $G_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$. This action has the usual Poincaré, Weyl and diffeomorphism invariances. Contrary to the usual approach of working in the reduced space by choosing the conformal gauge at the very beginning, we prefer to carry out the analysis in the complete space by regarding both $X^\mu$ and $g_{ab}$ as independent dynamical variables [12]. The canonical momenta are

$$\Pi_\mu = \frac{\delta S_P}{\delta (\partial_\sigma X^\mu)} = -\sqrt{-g} \partial^\sigma X_\mu,$$

$$\pi_{ab} = \frac{\delta S_P}{\delta (\partial_\tau g^{ab})} = 0.$$

(2)

It is clear that while $\Pi_\mu$ is a genuine momenta, $\pi_{ab} \approx 0$ are the primary constraints of the theory. To determine the secondary constraints, one can either follow the traditional Dirac’s Hamiltonian approach, or just read it off from the equation obtained by varying $g_{ab}$ since this is basically a Lagrange multiplier. This imposes the vanishing of the symmetric energy–momentum tensor,

$$T_{ab} = \frac{2}{\sqrt{-g}} \frac{\delta S_P}{\delta g^{ab}}$$

$$= -\partial_a X^\mu \partial_b X_\mu + \frac{1}{2} g_{ab} g^{cd} \partial_c X^\mu \partial_d X_\mu = 0.$$

(3)
Because of the Weyl invariance, the energy–momentum tensor is traceless,
\[ T_{a}^{\mu} = g^{\mu b} T_{ab} = 0, \]  
so that only two components of \( T_{ab} \) are independent. These components, which are the constraints of the theory, are given by
\[
\begin{align*}
\chi_1 & = gT^{00} = -T_{11} = \frac{1}{2}(\Pi^2 + (\partial_1 X)^2) = 0, \\
\chi_2 & = \sqrt{-g} T^0_1 = \Pi. \partial_1 X = 0. 
\end{align*}
\]
(5)

The canonical Hamiltonian obtained from (1) by a Legendre transformation is given by
\[
H = \int d\sigma \sqrt{-g} \ T^{00} = \int d\sigma \sqrt{-g} \left( \frac{1}{2} g^{01} \chi_1 + \frac{g^{01}}{\sqrt{-g} 11} \chi_2 \right). 
\]
(6)

Expectedly, the Hamiltonian turns out to be a linear combination of the constraints.

Just as variation of \( g_{ab} \) yields the constraints, variation of \( X^\mu \) gives the equation of motion
\[
\partial_a (\sqrt{-g} g^{ab} \partial_b X^\mu) = 0. 
\]
(7)

Finally, there is a mixed BC, \[ \partial^1 X^\mu (\tau, \sigma) \big|_{\sigma = 0, \pi} = 0, \]  
(8a)

where the string parameters are in the region \(-\infty \leq \tau \leq +\infty, 0 \leq \sigma \leq \pi\). In the covariant form involving phase space variables, this is given by
\[
\left( \partial_1 X^\mu + \sqrt{-g} g^{01} \Pi^\mu \right) \big|_{\sigma = 0, \pi} = 0. 
\]
(8b)

It is quite clear that the above boundary conditions are incompatible with the first of the basic Poisson brackets (PB),
\[
\begin{align*}
\{ X^\mu (\tau, \sigma), \Pi_a (\tau, \sigma') \} & = \delta^\mu_a \delta (\sigma - \sigma'), \\
\{ g_{ab}(\tau, \sigma), \pi^{cd} (\tau, \sigma') \} & = \frac{1}{2} (\delta^a_c \delta^d_b + \delta^b_d \delta^c_a) \delta (\sigma - \sigma'). 
\end{align*}
\]
(9)

where \( \delta (\sigma - \sigma') \) is the usual one-dimensional Dirac delta function. We would also like to mention that there is an apparent contradiction of the constraint \( \pi_{ab} \approx 0 \) with the PB (9). However, this equality is valid in Dirac’s “weak” sense only, so that it can be set equal to zero only after the relevant brackets have been computed. These weak equalities will be designated by \( \approx \), rather than an equality, which is reserved only for a strong equality. In this sense, therefore, there is no clash between this constraint and the relevant PB. Indeed, we can even ignore the canonical pair \( (g_{ab}, \pi^{cd}) \). From the basic PB, it is easy to generate a first class (involutive) algebra
\[
\begin{align*}
\{ \chi_1 (\sigma), \chi_1 (\sigma') \} & = 4 (\chi_2 (\sigma) + \chi_2 (\sigma')) \partial_\sigma \delta (\sigma - \sigma'), \\
\{ \chi_2 (\sigma), \chi_1 (\sigma') \} & = (\chi_1 (\sigma) + \chi_1 (\sigma')) \partial_\sigma \delta (\sigma - \sigma'), \\
\{ \chi_2 (\sigma), \chi_2 (\sigma') \} & = (\chi_2 (\sigma) + \chi_2 (\sigma')) \partial_\sigma \delta (\sigma - \sigma'). 
\end{align*}
\]
(10)

The situation is quite similar to usual electrodynamics. There the Lagrange multiplier is \( A_0 \), which corresponds to \( g_{ab} \) in the string theory. The multiplier \( A_0 \) enforces the Gauss constraint just as \( g_{ab} \) enforces the constraints \( \chi_1 \) and \( \chi_2 \). Furthermore, the Gauss constraint generates the time-independent gauge transformations, while \( \chi_1 \), \( \chi_2 \) generate the diffeomorphism transformations.

The BC (8a), (8b), on the other hand, is not a constraint in the Dirac sense [5], since it is applicable only at the boundary.\(^2\) Thus, there has to be an appropriate modification in the PB, to incorporate this condition. This is not unexpected and occurs, for instance, in the example of a free scalar field \( \phi(x) \) in 1 + 1 dimension, subjected to periodic BC of period, say, \( 2\pi \) \( \phi(t, x + 2\pi) = \phi(t, x) \). There the PB between the field \( \phi(t, x) \) and its conjugate momentum \( \pi(t, x) \) are given by
\[
\{ \phi(t, x), \pi(t, y) \} = \delta_p (x - y), 
\]
(11a)

where
\[
\delta_p (x - y) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{i n (x - y)} 
\]
(11b)
is the periodic delta function of period \( 2\pi \) and occurs in the closure properties of the basis functions \( e^{inx} \) for the space of square integrable functions, defined on the unit circle \( S^1 \). In fact, one can easily show that this PB

\(^1\) It is a mixed boundary condition in the sense that \( \partial^1 X^\mu = g^{11} \partial_1 X^\mu + g^{01} \partial_0 X^\mu \) will consist of both \( \tau \) and \( \sigma \) derivatives.

\(^2\) We are, therefore, differing from recent approaches [6] which regard the BCS as Dirac constraint. Our views are similar to those of [9], who discuss the free NG string.
algebra is obtained automatically if one starts with the canonical harmonic oscillator algebra for each mode in the Fourier space.

Before actually computing the modifications in the usual PB, let us take a look at the free NG action.

3. The free Nambu–Goto action

The NG action is given by

$$S_{\text{NG}} = -\int d\tau d\sigma \left[ (\dot{X}^\mu)^2 - \dot{X}^2 \dot{X}^2 \right]^{1/2}, \quad \text{(12)}$$

where $X^\mu = \frac{\partial X^\mu}{\partial \sigma} = \partial_1 X^\mu$ and $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} = \partial_0 X^\mu$ have been introduced for notational convenience. Note that, here the induced metric on the world-sheet has not been introduced, as we are exclusively working with $\tau$, $\sigma$ variables. A systematic constrained analysis of this action has already been carried out in [9] and here we just give the results. This will also put the analysis of the Polyakov string formulation in a proper perspective. The Euler equations are

$$\partial_0 \Pi^\mu + \partial_1 K^\mu = 0, \quad \text{(13a)}$$

where

$$\Pi^\mu = \frac{\partial L_{\text{NG}}}{\partial \dot{X}^\mu} = \frac{(X^\prime \dot{X}) X^\mu - X^\prime X^\mu}{[(X^\prime X)^2 - X^\prime \dot{X}^2 X^\prime X^2]^{1/2},}$$

and

$$K^\mu = \frac{\partial L_{\text{NG}}}{\partial X^\mu} = \frac{(X^\prime \dot{X}) X^\mu - X^\prime X^\mu}{[(X^\prime X)^2 - X^\prime \dot{X}^2 X^\prime X^2]^{1/2}}. \quad \text{(13b)}$$

The definition of the momenta $\Pi^\mu$ immediately leads to two primary constraints,

$$\Pi^2 + X^\prime^2 \approx 0, \quad \text{(14a)}$$

$$\Pi \cdot X^\prime \approx 0. \quad \text{(14b)}$$

And the BCs are

$$K^\mu(\tau, 0) = K^\mu(\tau, \pi) = 0. \quad \text{(15)}$$

A simple comparison shows that although the constraints in the Polyakov (5) and NG formulations (14a), (14b) have the same functional form, the BCs do not share this property (8a), (8b), (15).

If one wants to match the BCs also, it is necessary to choose a particular gauge. In the NG formulation one can take the orthonormal gauge conditions [9]

$$\lambda_\mu \left( X^\mu (\tau, \sigma) - \frac{\tau}{\pi} \mathcal{P}^\mu \right) \approx 0, \quad \text{(16)}$$

where $\lambda_\mu$ is an arbitrary constant $D$-vector and $\mathcal{P}^\mu = \int_0^\pi d\sigma \Pi^\mu$ denotes the conserved momentum, following from the equations of motion.

With these conditions the NG action weakly (i.e., on the constraint surface) reduces to

$$S_{\text{NG}} \approx \frac{1}{2} \int d\tau d\sigma \left( \dot{X}^2 - X^\prime^2 \right), \quad \text{(17)}$$

while the BCs become the usual Neumann type:

$$X^\mu|_{\sigma = 0, \pi} \approx 0. \quad \text{(18)}$$

The orthonormal gauge corresponds to the conformal gauge in the Polyakov formulation, so that the induced metric $g_{ab} = \eta_{ab} = \text{diag}(-1, 1)$. Then the Polyakov action (1) and the BC (8a), (8b) exactly match with the corresponding expressions for the NG case.

4. The interpolating free string action

From our analysis in the previous sections, we saw that the NG and Polyakov actions, along with their BCs, agreed in the orthonormal and conformal gauge, respectively. Here we discuss a new form of the action that interpolates between the two forms, without the need of any gauge fixing.

The starting point is to rewrite the free NG action in a first order form [12], incorporating the constraints

$$\mathcal{L}_I = \Pi^\mu \dot{X}^\mu - \mathcal{H} = \Pi^\mu \dot{X}^\mu + \frac{1}{2} \lambda (\Pi^2 + X^\prime^2) + \rho \Pi X^\mu. \quad \text{(19)}$$

Note that there is no contribution from the canonical Hamiltonian, obtained by a Legendre transformation, as it vanishes identically—a typical feature of a reparametrisation-invariant theory like this. So the expression of the Hamiltonian $\mathcal{H}$ appearing here is just a linear combination of the constraints, with $\lambda$ and $\rho$ playing the roles of Lagrange multipliers enforcing the respective constraints. Consequently, the time evolution of the system here is given by a gauge transformation.
Coming back to (19), we observe that $\Pi_\mu$ appears here as an auxiliary variable. It is thus possible to eliminate it using its equation of motion. We find
\[
\mathcal{L}_1 = -\frac{1}{2\lambda} \left( \dot{X}_\mu^2 + 2\rho \ddot{X}_\mu + (\rho^2 - \lambda^2) X_\mu^{(2)} \right). \tag{20}
\]
This is the cherished form of our interpolating Lagrangian.

If $\rho$ and $\lambda$ are eliminated by their respective equations of motion,
\[
\rho = \frac{-\dot{X}_\mu X_\mu^{(\mu)}}{X_\mu^{(2)}},
\]
\[
\lambda^2 = -\frac{\hbar}{X_\mu^{(2)} X_\mu^{(2)}}, \tag{21}
\]
then the above Lagrangian (20) reduces to the NG form (12).

If, on the other hand, we identify $\rho$ and $\lambda$ with the following contravariant components of the world-sheet metric,
\[
g^{ab} = (-g)^{-1/2} \left( \frac{\partial^a \sigma}{\partial^b \tau} - \frac{\partial^b \sigma}{\partial^a \tau} \right), \tag{22}
\]
then the action reduces to the Polyakov form (1). In this sense, therefore, the Lagrangian in (20) is referred to as an interpolating Lagrangian [13]. Also, note that with this mapping, the Hamiltonian read-off from (19) just reproduces the result (6).

Next, the BC is analysed. In general, the BC of an open string is given by
\[
K^\mu = \frac{\partial L}{\partial X_\mu} \bigg|_{\tau=0,\pi} = 0.
\]
From the interpolating Lagrangian (20), we find
\[
K_\mu = \left( \frac{\rho}{\lambda} \dot{X}_\mu + \frac{\rho^2 - \lambda^2}{\lambda} X_\mu^{(\mu)} \right) \bigg|_{\tau=0,\pi} = 0 \tag{23}
\]
at $\tau = 0, \pi$. Now using the expressions (21) for $\rho$ and $\lambda$, we recover the usual BC (15) for NG string.

To get the BC for Polyakov string, it is useful to rewrite (23) in terms of phase space variables, $X^\mu$ and $\Pi_\mu$, as
\[
K_\mu = \left( \rho \Pi_\mu + \lambda X_\mu^{(\mu)} \right) |_{\tau=0,\pi} = 0, \tag{24}
\]
where
\[
\Pi_\mu = \frac{\partial L}{\partial \dot{X}_\mu} = -\frac{1}{\lambda} \left( \dot{X}_\mu + \rho X_\mu^{(\mu)} \right). \tag{25}
\]
Now identifying $\rho$ and $\lambda$ with the metric components, it is easy to check that the Polyakov form of BC (8a), (8b) is reproduced. Hence it is possible to interpret either of (23) or (24) as an interpolating BC.

It is noteworthy that although the Polyakov BC can be expressed in terms of pure phase space variables, the Nambu–Goto BC cannot be done so, because of the presence of velocities in $\rho$ (see (21)). This is an important distinction when it comes to the study of the modification in the basic algebra, as will become evident in the next section.

5. Boundary conditions and modified brackets for a free theory

Before discussing the mixed type condition, that emerged in a completely gauge-independent formulation of the Polyakov action, consider the simpler Neumann type condition (8a), (8b) that leads to $(\partial_1 X^\mu) |_{\tau=0,\pi} = 0$ in an orthonormal (conformal) gauge.

Since the string coordinates $X^\mu(\tau, \sigma)$ transform as a world-sheet scalar under its reparametrisation, it will be even more convenient to get back to our scalar field $\phi(t, x)$ defined on $(1 + 1)$-dimensional space–time, but with the periodic BC of $2\pi$ replaced by Neumann BC
\[
\partial_\tau \phi |_{\tau=0,\pi} = 0 \tag{26}
\]
at the end points of a 1-dimensional box of compact size, i.e., of length $\pi$. Correspondingly, the $\delta_P(x)$ appearing there in the PB (11a), (11b)—consistent with periodic BC—have to be replaced now with a suitable “delta function” incorporating Neumann BC, rather than periodic BC. Interestingly, such a “delta function” is not difficult to construct from purely algebraic arguments.

One starts by noting that the usual properties of a delta function is also satisfied by $\delta_P(x)$:
\[
\int_{-\pi}^{+\pi} dx' \delta_P(x' - x) f(x') = f(x) \tag{27}
\]
for any periodic function $f(x) = f(x + 2\pi)$ defined in the interval $[-\pi, +\pi]$. Let us now restrict to the case of even (odd) functions $f_{\pm}(x) = \pm f_{\pm}(x)$. Then it can be easily seen that the above integral (27) reduces
\[
\int_0^\pi dx' \Delta_\pm(x', x) f_\pm(x') = f_\pm(x),
\]
where
\[
\Delta_\pm(x', x) = \delta_P(x' - x) \pm \delta_P(x' + x).
\]
(29a)

Using (11b), the explicit form of \(\Delta_+(\sigma', \sigma)\), in particular, can be given as
\[
\Delta_+(\sigma, \sigma') = \frac{1}{\pi} + \frac{1}{\pi} \sum_{n \neq 0} \cos(n\sigma') \cos(n\sigma).
\]
(29b)

We will not have to deal with \(\Delta_-(\sigma', \sigma)\) henceforth in our Letter, for reasons explained below.

Since any function \(\phi(x)\) defined in the interval \([0, \pi]\) can be regarded as a part of an even/odd function \(f_\pm(x)\) defined in the interval \([-\pi, \pi]\), both \(\Delta_+(\sigma', \sigma)\) act as delta functions defined in half of the interval at the right, i.e., \([0, \pi]\) (28). It is still not clear which of these \(\Delta(x', x)\) functions should replace \(\delta_P(x' - x)\) in the PB relation. We can invoke the Neumann BC at this stage, to fix the matter. To see this, consider the Fourier decomposition of an arbitrary function \(f(x)\) satisfying periodic BC, \((f(x) = f(x + 2\pi))\)
\[
f(x) = \sum_{n \in \mathbb{Z}} f_n e^{inx}.
\]
(30)

Clearly,
\[
\begin{align*}
\phi'(0) &= i \sum_{n > 0} n(f_n - f_{-n}), \\
\pi'(\pi) &= i \sum_{n > 0} (-1)^n n(f_n - f_{-n}).
\end{align*}
\]
(31)

Now for even (odd) functions, the Fourier coefficients are related as
\[
f_{-n} = \pm f_n
\]
(32)

so that Neumann’s BC
\[
\phi'(0) = \pi'(\pi) = 0
\]
(33)

are satisfied if and only if \(f(x)\) is even. Therefore, one has to regard the scalar field \(\phi(x)\) defined in the interval \([0, \pi]\) and subjected to Neumann BC (26) as a part of an even periodic function \(f_+(x)\) defined in the extended interval \([-\pi, +\pi]\). It thus follows that the appropriate PB for the scalar theory is given by
\[
\{\phi(t, x), \pi(t, x')\} = \Delta_+(\sigma, \sigma').
\]

It is clearly consistent with Neumann BC as
\[
\partial_\sigma \Delta_+(\sigma, \sigma')|_{\sigma = 0, \pi} = \partial_\sigma' \Delta_+(\sigma, \sigma')|_{\sigma = 0, \pi} = 0
\]
(34a)

is automatically satisfied. It is straightforward to generalise it to the string case, where it is given by
\[
\{X^\mu(\tau, \sigma), \Pi_\nu(\tau, \sigma')\} = \delta^\mu_\nu \Delta_+(\sigma, \sigma'),
\]
(34b)

and the Lorentz indices are playing the role of “iso-spin” indices, as viewed from the world-sheet. This form first appeared in [9]. Observe also that the other brackets
\[
\{X^{\mu}(\tau, \sigma), X^\nu(\tau, \sigma')\} = 0
\]
(34c)

are consistent with the BCs and hence remain unchanged.

For a gauge-independent analysis, the Nambu–Goto BC poses problems since it cannot be expressed in phase space variables. To overcome this problem it is necessary to fix a gauge and this was elaborated in Section 3. The generalisation of this in the interacting NG string will be given later in Section 7. Here we take recourse to the mixed condition (8a), (8b) that occurs in the Polyakov string. A simple inspection shows that this is also compatible with the modified brackets (34a), (34c), but not with (34b). Hence the bracket among the coordinates should be altered suitably. We therefore make an ansatz,
\[
\{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} = C^{\mu\nu}(\sigma, \sigma'),
\]
(35a)

where
\[
C^{\mu\nu}(\sigma, \sigma') = -C^{\nu\mu}(\sigma', \sigma).
\]
(35b)

Imposing the BC (8a), (8b) on this algebra, we get
\[
\partial_\sigma C^{\mu\nu}(\sigma, \sigma')|_{\sigma' = 0, \pi} = \partial_\sigma' C^{\mu\nu}(\sigma, \sigma')|_{\sigma = 0, \pi}
\]
\[
= -\sqrt{-g} g^{01} \{\Pi^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\}
\]
\[
= \sqrt{-g} g^{01} G^{\mu\nu} \Delta_+(\sigma, \sigma').
\]
(36)

For an arbitrary form of the metric tensor, it might be technically problematic to find a solution for
δ(σ algebra (10) is still preserved, only that it is possible to give a quick solution of Cµν(σ, σ’) as

\[ C^{\mu\nu}(\sigma, \sigma’) = \sqrt{-g} g^{01} G^{\mu\nu} \left[ \Theta(\sigma, \sigma’) - \Theta(\sigma’, \sigma) \right] \]

(38)

where the generalised step function Θ(σ, σ’) satisfies

\[ \partial_\sigma \Theta(\sigma, \sigma’) = \Delta_+(\sigma, \sigma’) \]

(39)

An explicit form of Θ(σ, σ’) is given by

\[ \Theta(\sigma, \sigma’) = \frac{\sigma}{\pi} + \frac{1}{\pi} \sum_{n \neq 0} \frac{1}{n} \sin(n\sigma) \cos(n\sigma’), \]

(40a)

having the properties

\[ \Theta(\sigma, \sigma’) = 1 \quad \text{for} \quad \sigma > \sigma’, \]

\[ \Theta(\sigma, \sigma’) = 0 \quad \text{for} \quad \sigma < \sigma’. \]

(40b)

Using these relations, the simplified structure of noncommutative algebra follows:

\[ \{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma’)\} = 0 \quad \text{for} \quad \sigma = \sigma’, \]

\[ \{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma’)\} = \pm \sqrt{-g} g^{01} G^{\mu\nu} \]

for \( \sigma > \sigma’ \) and \( \sigma < \sigma’ \),

(41)

respectively. Thus a noncommutative algebra for distinct coordinates \( \sigma \neq \sigma’ \) of the string emerges automatically in a free string theory if a gauge-independent analysis is carried out like this. But this noncommutativity can be made to vanish in gauges like conformal gauge) right at this stage. A usual canonical analysis leads to the following set of primary first class constraints

\[ g T^{00} = \frac{1}{2} \left[ (\Pi_\mu + e B_{\mu\nu} \partial_1 X^\nu)^2 + (\partial_1 X)^2 \right] \approx 0, \]

(44)

\[ \sqrt{-g} T^0_1 = \Pi_1 X^0 \approx 0, \]

(45)

where

\[ \Pi_\mu = -\sqrt{-g} g^{0\nu} \partial_\nu X^\mu + e B_{\mu\nu} \partial_1 X^\nu \]

(46)

is the momentum conjugate to \( X^\mu \).

Likewise, the BC is given by

\[ \left( \partial_1 X^\mu + \frac{1}{\sqrt{-g}} e B^{\mu\nu} \partial_\nu X^\nu \right) \bigg|_{\sigma = 0, \pi} = 0. \]

(47)

Using phase space variables, this can be written in a completely covariant form as

\[ \left( \partial_1 X^\mu M^\rho_{\mu} + \Pi^\nu N_{\nu\mu} \right) \bigg|_{\sigma = 0, \pi} = 0. \]

(48)
where
\[ M^{\rho\mu} = \frac{1}{g^{01}} \left[ \delta^{\rho\mu} - \frac{2e}{\sqrt{-g}} g^{01} B^{\rho\mu} + e^2 B^{\rho\mu} B_{\nu\mu} \right], \]  
(49a)

\[ N_{\nu\mu} = -\frac{g^{01}}{g^{00} \sqrt{-g}} G_{\nu\mu} - \frac{1}{g^{11}} e B_{\nu\mu}, \]  
(49b)

are two matrices. This nontrivial BC leads to a modification in the original (naive) canonical PBs.

Now the BC (48) is recast as
\[ \left( \partial_1 X_\mu + \Pi^\rho (N M^{-1})_{\rho\mu} \right) |_{\sigma=0,\pi} = 0. \]  
(50)

The \( \{X^\mu(\sigma), \Pi_{\nu}(\sigma')\} \) PB is the same as that of the free string (34a). Considering the general structure (35a), (35b), we obtain
\[ \{ \partial_\sigma X^\mu(\sigma), X^\nu(\sigma') \} = \partial_\sigma C^{\mu\nu}(\sigma, \sigma'). \]  
(51)

Putting the BC and exploiting (34a), we get
\[ \partial_\sigma C^{\mu\nu}(\sigma, \sigma') |_{\sigma=0,\pi} = (NM^{-1})_{\nu\mu} \Delta_+ (\sigma, \sigma') |_{\sigma=0,\pi}. \]  
(52)

As we did in the free case, we restrict to the class of metrics defined by (37). Taking a cue from the free theory, the solution for \( C^{\mu\nu}(\sigma, \sigma') \) must involve the generalised \( \Theta \) function, introduced in (40a), (40b). Splitting \( (NM^{-1})_{\nu\mu} \) into its symmetric \( (NM^{-1})_{\nu\mu} \) and antisymmetric \( (NM^{-1})_{[\nu\mu]} \) components, a general solution for \( C^{\mu\nu} \) is given by
\[ C^{\mu\nu}(\sigma, \sigma') = \frac{1}{2} (NM^{-1})_{\nu\mu} |_{\nu\mu} \left[ \Theta(\sigma, \sigma') - \Theta(\sigma', \sigma) \right] + \frac{1}{2} (NM^{-1})_{[\nu\mu]} |_{\nu\mu} \left[ \Theta(\sigma, \sigma') + \Theta(\sigma', \sigma) - 1 \right]. \]  
(53)

Observe that, by demanding (35b), \( (NM^{-1})_{[\nu\mu]} \) must be multiplied by an antisymmetric combination of \( \Theta \)'s, which is precisely \( \left[ \Theta(\sigma, \sigma') - \Theta(\sigma', \sigma) \right] \). Likewise, the other factor \( (NM^{-1})_{\nu\mu} \) must be multiplied by a symmetric combination \( \left[ \Theta(\sigma, \sigma') + \Theta(\sigma', \sigma) \right] \), plus an undetermined constant. We fix this constant to \( -1 \) by requiring that the vanishing result (41) in the free case is retained for all \( \sigma = \sigma' \) away from the boundary (using \( \Theta(\sigma, \sigma) = \frac{1}{2} \)). An advantage of this normalisation is that by passing to the conformal gauge, where \( g = -1 \) and \( g^{01} = 0 \), one obtains
\[ C^{\mu\nu}(\sigma, \sigma') = B^{\mu\nu} \left[ \Theta(\sigma, \sigma') + \Theta(\sigma', \sigma) - 1 \right], \]  
(54a)

where
\[ B^{\mu\nu} = -e \left[ B_0^2 (1 + e^2 B_0^2)^{-1} \right]_{\mu\nu}, \]  
(54b)

which reproduces the standard noncommutative algebra in the presence of a background field [1-4,6,7].

It is evident that the modified algebra is gauge-dependent, depending on the choice of the metric. However, there is no choice, for which the noncommutativity vanishes. To show this, note that the origin of the noncommutativity is the presence of non-vanishing \( \Pi^\mu \) term in the BC (48). If this can be eliminated, then the usual commutative algebra is obtained. This requires \( N_{\nu\mu} = 0 \). From (49b) this implies \( B^{\mu\nu} \) and \( G_{\nu\mu} \) have to be proportional which obviously cannot happen, as the former is an antisymmetric and the latter is a symmetric tensor. Hence noncommutativity will persist for any choice of world-sheet metric \( g_{ab} \).

Specially interesting are the expressions for noncommutativity (53) at the boundaries,
\[ C^{\mu\nu}(0, 0) = -C^{\mu\nu}(\sigma, \pi) = \frac{1}{2} (NM^{-1})_{[\nu\mu]} |_{\nu\mu}, \]  
\[ C^{\mu\nu}(0, \pi) = -C^{\mu\nu}(\sigma, 0) = -\frac{1}{2} (NM^{-1})_{[\nu\mu]} |_{\nu\mu}. \]  
(55)

It should be pointed out that in the conformal gauge, \( (NM^{-1}) \) does not have a symmetric component, so that
\[ C^{\mu\nu}(0, \pi) = C^{\mu\nu}(\pi, 0) = 0. \]

7. The interacting theory: Nambu–Goto formulation

Although the Polyakov and NG formulations for free strings are regarded to be classically equivalent, there are some subtle issues. Indeed, the structures of BCs in the two formulations are different as was also illuminated by our interpolating action. Also more complications are expected in the presence of interactions. Since the occurrence of noncommutativity is directly connected with the BCs, it is, therefore, useful to study this feature in the NG formulation. This motivates us to carry out an exhaustive analysis of the classical relativistic string interacting with a constant, second rank, antisymmetric tensor \( B_{\mu\nu} \) in the NG formulation in this subsection. Here we present a generalisation of the analysis of Hanson, Regge and Teitelboim.
The analysis for the free theory [9] has already been reproduced briefly in Section 3.

We start with the action

\[ S = \int_{-\infty}^{+\infty} d\tau \int d\sigma \left[ \mathcal{L}_0 + eB_{\mu\nu}\dot{X}^\mu X'^\nu \right]. \]  \hspace{1cm} (56)

Here \( \mathcal{L}_0 \) denotes the free string Lagrangian density appearing in (12). From the variation of the action, we obtain the following equations of motion and the BCs,

\[ \Pi^\mu + K'^\mu = 0, \]  \hspace{1cm} (57)

\[ K^\mu|_{\sigma=0,\pi} = 0, \]  \hspace{1cm} (58)

where

\[ \Pi^\mu = \frac{\partial L}{\partial \dot{X}_\mu} = \mathcal{L}_0^{-1} \left( -X'^2 \dot{X}^\mu + (\dot{X} X') X'^\mu \right) \]

\[ + eB^{\mu\nu} X'_\nu, \] \hspace{1cm} (59)

\[ K^\mu = \frac{\partial L}{\partial \dot{X}_\mu} = \mathcal{L}_0^{-1} \left( -X'^2 \dot{X}^\mu + (\dot{X} X') \dot{X}^\mu \right) 
\]

\[ - eB^{\mu\nu} \dot{X}'_\nu. \] \hspace{1cm} (60)

The Primary constraints of the theory are

\[ \chi_1 = (\Pi^\mu - eB^{\mu\nu} X'_\nu)^2 + X'^2 \approx 0, \]

\[ \chi_2 = \Pi \cdot X' \approx 0 \] \hspace{1cm} (61)

which are similar to those obtained in the Polyakov version (see (44), (45)). Using the standard canonical PB, it is straightforward to verify the diffeomorphism algebra (10).

A gauge-independent analysis, as was done for the Polyakov formulation, is not feasible here, since the BCs involve time derivatives that are not eliminatable in terms of the momenta. To properly account for the BCs, a gauge choice becomes necessary. This is equally valid for a free theory. As was shown in [9] and discussed in Section 3, the free theory becomes most tractable in the orthonormal gauge. Inspired by their choice, we consider the following gauge conditions,

\[ \lambda_{\mu} \left( X^\mu - \frac{P^\mu}{\pi} \right) \approx 0, \]

\[ \lambda_{\mu} \left( \Pi^\mu - \frac{P^\mu}{\pi} \right) \approx 0, \] \hspace{1cm} (62)

which for \( e = 0 \) reduce to the orthonormal gauge in free theory. Here \( \lambda_{\mu} \) is a constant \( D \)-vector. For our present analysis there arises no need to fix \( \lambda_{\mu} \). Same notations as in Section 3 are used here.

Let us study the consequences of the gauge choice. From the gauge conditions (62), we obtain \( \partial_0 (\lambda \cdot \Pi) = \frac{\partial_0 \lambda_{\mu} P^\mu}{\pi} = 0 \) and together with the equations of motion (57) this leads to \( \partial_1 (\lambda \cdot K) = 0 \). Compatibility with the BC (58) then ensures that \( \lambda \cdot K = 0 \) for all \( \sigma \). Again, from the gauge choice (62), we find \( \lambda \cdot X' = 0 \) and \( \lambda \cdot \dot{X} = (\lambda \cdot \Pi) \). In short, the following three exact relations are valid:

\[ \lambda \cdot K = 0, \quad \lambda \cdot X' = 0, \quad \lambda \cdot \dot{X} = (\lambda \cdot \Pi). \] \hspace{1cm} (63)

From the defining equations (59), (60) and (62), we find

\[ (\lambda \cdot \Pi) = -\mathcal{L}_0^{-1} X'^2 (\lambda \cdot \dot{X}) + eB^{\mu\nu} \lambda_{\mu} X'_\nu, \]

\[ \lambda \cdot K = \mathcal{L}_0^{-1} (\lambda \cdot X') (\lambda \cdot \dot{X}) - eB^{\mu\nu} \lambda_{\mu} \dot{X}'_\nu = 0. \] \hspace{1cm} (64)

Using (62) once again we obtain

\[ \mathcal{L}_0^{-1} X'^2 = -1 + eA, \quad \mathcal{L}_0^{-1} (\lambda \cdot X' \cdot \Pi) = eB, \] \hspace{1cm} (66a)

where

\[ A = \frac{B^{\mu\nu} \lambda_{\mu} X'_\nu}{\lambda \cdot \Pi}, \quad B = \frac{B^{\mu\nu} \lambda_{\mu} \dot{X}'_\nu}{\lambda \cdot \Pi}. \] \hspace{1cm} (66b)

From now on we will work in the lowest nontrivial order in the coupling \( e \). The explicit expressions for \( A \) and \( B \) are not needed for the \( O(e) \) results presented here. Recalling the explicit form of \( \mathcal{L}_0 \) from (56), we find,

\[ \mathcal{L}_0 \approx -(-X'^2 X'^2)^{1/2}. \] \hspace{1cm} (67)

Using (66a), (66b) we obtain

\[ X'^2 = -\dot{X}^2(1 - 2eA) \rightarrow \dot{X}^2 + X'^2 \approx 2eA\dot{X}^2. \]

The \( O(e) \) correction to the orthonormal vanishes in the free theory \( (e = 0) \), where \( \dot{X}, X' = 0 \), and \( \dot{X}^2 + X'^2 = 0 \) [9]. Now \( \mathcal{L}_0 \) is simplified to,

\[ \mathcal{L}_0 \approx -(-X'^2 X'^2)^{1/2} \approx \dot{X}^2(1 - eA) \]

\[ \approx \left[ \frac{1}{2} \left( \dot{X}^2 - X'^2 \right) - eA(\dot{X}^2 + X'^2) \right] \]

\[ \approx \frac{1}{2} (\dot{X}^2 - X'^2). \] \hspace{1cm} (68)

Finally, we recover the Lagrangian of the string coupled to \( B_{\mu\nu} \), in this particular gauge, to lowest
order in the coupling $e$ as
\begin{equation}
\mathcal{L} = \frac{1}{2} (\dot{X}^2 - X') + eB_{\mu\nu} \dot{X}^\mu X'^\nu + O(e^2). \tag{69}
\end{equation}

The equation of motion in this gauge is that of a free theory,
\begin{equation}
(\dot{a}_0^2 - \dot{a}_1^2)X^\mu = 0, \tag{70}
\end{equation}
but crucial modifications have appeared in the BC,
\begin{equation}
(\dot{X}^\mu + \frac{e}{N} B_{\mu\nu} \dot{X}^\nu)\bigg|_{\sigma = 0, \pi} = 0.
\tag{71}
\end{equation}

In fact, the interaction has changed the BC from Neumann type in free theory to a mixed one. Elimination of $\dot{X}^\mu$ from (58) reproduces the BC in phase space,
\begin{equation}
X'^\mu + e (M^{-1} B)^{\mu\nu} \Pi^\nu = 0, \tag{72}
\end{equation}
where $M^{\mu\lambda} = G^{\mu\lambda} - \frac{e}{2} B^{\mu\nu} B_\nu^{\lambda}$. It is amusing to note that this BC is identical to the one used in the Polyakov model in the conformal gauge [3,6] but in our case we should consider $M^{\mu\lambda} \approx G^{\mu\lambda}$, since our results are of $O(e)$ only.

It is worthwhile to make a comparison with Polyakov formulation at this stage. The Lagrangian (69) is identical to the Polyakov one (43) in the conformal gauge. There is a similar mapping between BCs (72) and (48) again in the conformal gauge. Consequently, we shall be reproducing the same set of modified brackets (34a) and (54a), (54b), displaying noncommutativity among various coordinates. It should be emphasised, however, that this agreement is only upto $O(e)$ in the coupling parameter in the specific gauge (62).

8. Conclusion

In this Letter we have derived expressions for a noncommutative algebra that are more general than the standard results found in the conformal gauge. Indeed, our results reproduce the standard ones, once the conformal gauge is implemented.

The origin of any modification in the usual Poisson algebra is the presence of boundary conditions. This phenomenon is quite well known for a free scalar field subjected to periodic boundary conditions. We showed that its exact analogue is the conformal gauge fixed free string, where the boundary condition is of Neumann-type. This led to a modification only in the $\{X^\mu(\sigma), \Pi_\nu(\sigma')\}$ algebra, where the usual Dirac delta function got replaced by $\Delta_+(\sigma, \sigma')$. A more general type of boundary condition occurs in the gauge-independent formulation of a free Polyakov string. Using certain algebraic consistency requirements, we showed that the boundary conditions in the free theory naturally led to a noncommutative structure among the coordinates. This noncommutativity vanishes in the conformal gauge, as expected.

The same technique was adopted for the interacting string. A more involved boundary condition led to a more general type of noncommutativity than has been observed before. Contrary to the standard conformal gauge expressions, this noncommutative algebra survives at all points of the string and not just at the boundaries. Furthermore, in contrast to the free theory, this noncommutativity cannot be removed in any gauge. We have also shown that the noncommutativity does not affect the usual diffeomorphism algebra among the gauge generators. In the conformal gauge, our results reduce to the standard noncommutativity found only at the string end points.

A perturbative analysis of the noncommutativity has also been performed in the interacting Nambu–Goto string. Surprisingly, the conformal gauge result in the Polyakov formulation is reproduced in the Nambu–Goto scheme in the lowest nontrivial order in the Neveu–Schwarz coupling, in an orthonormal-like gauge. It would be interesting to see if there is an alternative gauge condition in which the above equivalence can be shown exactly.

References

Erratum to: “Next-to-leading order QCD corrections to $bg \rightarrow tW^-$ at the CERN Large Hadron Collider”

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Due to a numerical program bug, one of the conclusions of the original Letter “the NLO QCD corrections do not reduce the scales dependence compared to that at tree level” is not correct. As shown in Fig. 2, the NLO QCD corrections do reduce the scales dependence compared to that at tree level. At the same time, as shown in Fig. 3, for

![Graph 1: Cross sections of NLO (dashed) and LO (solid) for $pp \rightarrow bg \rightarrow tW^-$ as functions of $\mu/\mu_0$ at the LHC with $\sqrt{s} = 14$ TeV, where $\mu_0 = m_t + m_W$.](image1)

![Graph 2: K-factor for $pp \rightarrow bg \rightarrow tW^-$ as functions of $\mu/\mu_0$ at the LHC.](image2)

Due to a numerical program bug, one of the conclusions of the original Letter “the NLO QCD corrections do not reduce the scales dependence compared to that at tree level” is not correct. As shown in Fig. 2, the NLO QCD corrections do reduce the scales dependence compared to that at tree level. At the same time, as shown in Fig. 3, for
the renormalization and factorization scales $\mu = \mu_0$ ($\mu_0 = m_t + m_W$), the NLO hadronic cross section is $\sim 37$ pb, while $\sim 25$ pb for tree level. The $K$-factor varies from 1.33 to 1.66 for $\frac{\mu_0}{\mu} < \mu < 2\mu_0$. It should be noted that the different input of $\alpha_s$ ($A^{(5)} = 226$ (146) MeV for two-loop (one-loop) evolution) has been used, which is consistent to that in PDF.
1. Omit the last line in the abstract.

2. In the third paragraph in the introduction, in the sentence “We shall show here that the conclusions of [8] remain valid and that supersymmetric Einstein domain wall solutions in presence of non-trivial matter, within the framework of [10], are not allowed” replace “are not allowed” with “may be allowed under stringent conditions”.

3. Replace the sentence before last in the introduction with “We demonstrate that the constraints derived from the equations of motion when combined with the integrability conditions (coming from the requirement of unbroken supersymmetry) give strong constraints”.

4. Insert after Eq. (3.25) the following: “where

\[ c(r) = c(s) \cdot O_{sr}, \]

\[ O_{sr} \] are constants and \( O_{sr} \) is an orthogonal matrix

\[\begin{align*}
O_{sr} & = \cos 2\alpha \tilde{\delta}^{sr} + 2 \sin^2 \alpha \frac{\alpha^t \alpha^r}{\alpha^2} - \epsilon^{sr} \sin 2\alpha \frac{\alpha^t}{\alpha}, \\
\alpha^t & = ce^{-U} \epsilon^{rst} Q^t N^t - \frac{i}{2} q' X \omega_X; i (\sigma^r)^j, \\
\alpha^2 & = \alpha^t \alpha^r.
\end{align*}\]

Notice that \( c(r) c(s) = c(r)^2 \) and \( O_{sr} O_{tr} = \delta_{st} \).

5. Change Eq. (3.29) to

\[\partial_x (BW) = \frac{c_0}{c} W Q^{(r)} \partial_x O^{st}. \quad (3.29)\]

6. Omit the paragraph after Eq. (3.29).

7. Replace the last sentence in the first paragraph in the Discussion (Section 4) with “It turns out that these conditions can be made compatible with integrability conditions coming from the Killing spinor equations provided certain constraints are satisfied.”

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